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MAXIMAL ELEMENTS IN BANACH SPACES

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The existence of maximal elements is proved in closed, bounded and convex, but not necessarily compact, subsets of Banach spaces. The theorems proved in the paper are general enough to apply to the standard infinite-dimensional commodity spaces used in economic analysis.

1. INTRODUCTION

Suppose that K is a subset of a Hausdorff topological vector space E. Then each binary relation P on K gives rise to a multivalued map $T : K \to 2^K$ as follows: if $x \in K$, then $T(x) = \{y \in k : (x, y) \in P\}$. Conversely, if $T: K \to 2^K$ is a multivalued map, then a binary relation P on K is defined as follows: $(x, y) \in P$ if and only if $y \in T(x)$. A point x_0 of K is said to be a maximal element of the map $T : K \to 2^K$, with respect to the binary relation defined above, if $T(x_0) = \phi$.

Theorems on the existence of maximal elements have important applications in mathematical economics. For example, in recent work in general equilibrium theory without ordered preferences, the existence of an equilibrium in an abstract economy or qualitative game is often proved by constructing a map P, which may be construed as a 'preference map', on a subset K of a Hausdorff topological vector space and then by showing that there exists a point x_0 such that $P(x_0) = \phi$. See, for example, Aliprantis and Brown[1], Gale and Mas-Colell[10], Sonnenschein[18], Borglin and Keiding[5], Bergstrom[2], Schofield[16], Walker[21] and Toussaint[20]. In this work the assumption is made that the preference map is defined on a compact and convex subset of a finite-dimensional Cartesian space, or more generally, a Hausdorff topological vector space.

Some of these results have been generalized by Yannelis and Prabhakar[23] (Theorem 5.3) who have proved the existence of maximal elements in paracompact convex spaces and by Border[4] (Chapter 7) who has proved the existence of maximal elements in σ-compact convex spaces by using the idea of an escaping sequence.

For an extension of some of the results in the literature to the manifold setting the reader is referred to the interesting paper of Schofield[17]. Schofield's line of argument is somewhat removed from our main theme and will not be considered further here.

The object of this paper is to remove altogether the compactness assumptions on the domain and codomain of the 'preference map'. This is achieved by strengthening the assumptions on the 'preference map'. More precisely, we prove that a contracting 'preference map' (definitions follow) which satisfies the other usual conditions, has a maximal element in any closed bounded and convex but not necessarily compact subset of a Banach space. The theorms that we prove are general enough to apply to the standard infinite-dimensional commodity spaces used in economic analysis such as the sequence spaces l_p ($1 \leqslant p \leqslant \infty$), the Lebesgue spaces L_p ($1 \leqslant p \leqslant \infty$) and the space $M(K)$ of finite signed Baire measures on a compact metric space K, since these are all Banach spaces.

The economic motivation for the approach used in this paper will be made clearer by the following somewhat informal remarks. In R^n the existence of maximal elements is easily obtained as a consequence of natural economic assumptions (see Debru[7]). The argument goes as follows. Assume that income and all prices are strictly positive. Under these conditions the budget set is easily seen to be closed and bounded. Now in R^n, a closed and bounded set is compact. Hence if preferences are continuous (or even upper semicontinuous) in any of the equivalent topologies on R^n, there is a maximal element in the budget set and the demand correspondence is nonempty valued.

This argument fails in infinite-dimensional spaces because of their greater complexity. In a general topological vector space it is not always true that a closed and bounded set is compact. In fact, this property holds only for a very special class of spaces called semi-Montel spaces (see Wilansky[22], p. 90). Since every semi-Montel space is semi-reflecxive and a locally convex metric space is semi-reflexive if and only if it is reflexive, it follows that every non reflexive Banach space is not a semi-Montel space (see Wilansky[22], p. 153 and p. 265 and Duffie[8] for a discussion of semi-reflexive and semi-Montel spaces).

Let us consider the untoward implication of this fact for the space L_∞ which has been widely used in the economics literature. See, for example Bewley[3], Brown and Lewis[6] and Toussaint[20]. Since L_∞ is not reflexive (see Royden[15], p. 191) it is not a semi-Montel space. Hence the natural argument which shows that a closed and bounded set is compact does not extend to it and the existence of maximal elements is more difficult to prove (cf. Jones[12]).

The upshot of this discussion is that in infinite-dimensional spaces there is no 'natural' way in which one can prove the existence of maximal elements.

In certain situations the following approach has been used to solve the problem Suppose that the commodity space E is the dual of some Banach space G. An example is the space L_∞ which is the dual of L_1. Under these conditions, the Alaoglu-Bourbaki theorem (see Wilansky[22], p. 130) asserts that a norm bounded set in E is relatively compact in the weak * topology. This means that a norm bouded and closed set in E is weak * compact. Hence, if preferences are continuous, every closed and bounded attainable set in E has a maximal elemnent. This technique of obtaining the compactness of closed and bounded sets by the use of a suitable topology and then proving the existence of maximal elements has been used by Florenzano[9].

Observe, however, that this approach only works in the case where the Banach commodity space has a predual. It does not work for general commodity spaces. Conditions for a Banach space to have a predual are given in Holmes[11] (pp. 211-14).

The object of this paper is to suggest and another way in which the existence of maximal elements can be obtained. This approach is more general since it does not require the Banach space to have a predual. In view of the difficulty of obtaining compact sets in infinite-dimensional spaces, it serves a heuristic purpose by suggesting that compactness may be needed to prove the existence of maximal elements in budget sets or attainable sets if preferences satisfy a contraction condition.

2. Preliminaries

We shall use the following notation. If K is a subset of a Banach space, then int K denotes the topological interior of K, co K denotes the convex haul of K and $\overline{\text{co}}\, K$ denotes the closed convex hull of K.

Let X be a Banach space and S a bounded subset of X. Then the Kuratowski 'measure of non-compactness' of S, $\alpha(S)$ is defined by

$$\alpha(S) = \inf \{\epsilon > 0 : S \text{ can be covered by a finite number of sets with diameter no larger than } \epsilon\}.$$

Observe that if S is compact, $\alpha(S) = 0$. The proofs of the following two theorms can be found in Lloyd[13], [Chapter 6].

Theorem (Kuratowski)—Suppose that (A_n) is a decreasing sequeuce of non-empty closed sets in a Banach space such that $\alpha_n = \alpha(A_n)$ tends to zero an n tends to infinity. Then $A = \bigcap_{n=1}^{\infty} A_n$ is non-empty and compact.

Theorem (Darbo)—If S is a bounded subset of a Banach space, then $\alpha(S) = \alpha(\overline{\text{co}}\, S)$.

Let Y_1, Y_2 be metric spaces. Then a multivalued map $T : Y_1 \to 2^{Y_2}$ is said to be a k-set contraction if for all bounded subsets S of Y_1, $T(S)$ is bounded and $\alpha(T(S)) \leqq k\, \alpha(S)$. The map $T : Y_1 \to 2^{Y_2}$ is said to be a strict set contraction if it is a k-set contraction with $k < 1$.

Let K be a subset of a topological vector space and $T : K \to 2^K$ a multivalued map. We say that T is a preference map if it is generated from a binary relation P on

K so that $y \in T(x)$ if and only if $(x, y) \in P$. A point x in K is said to be a maximal element of the preference map T if $T(x) = \phi$.

3. Maximal Elements

We now prove the following theorem on the existence of maximal elements.

Theorem 1—Let E be a Banach space and D a non-empty closed, bounded and convex subset of E. Let the preference map $P : D \to 2^D$ satisfy the following conditions:

(i) for each $x \in D$, $P(x)$ is convex;

(ii) P is irreflexive, i.e. for each $x \in D$, $x \notin P(x)$;

(iii) for each $x \in D$ such that $P(x) \neq \phi$, there exists $y \in D$ such that $x \in \text{int}\, P^{-1}(y)$;

(iv) the preference map P is a strict set contraction, i.e. $\alpha(P(S)) \leqslant t\,\alpha(S)$ for all bounded S, where $0 \leqslant t < 1$.

Then there exists a maximal element, i e. a point $x^* \in D$ such that $P(x^*) = \phi$.

Proof : Suppose that the theorem is false. Then $P(x) \neq \phi$ for every $x \in D$. As in Martin[14], [Chapter 4] the proof consists of two parts. In the first part, we show that condition (iv) implies the existence of a compact convex subset K of D such that P is a map from K into 2^K. In the second part of the proof, a fixed point theorem is applied to the map P.

Let $K_0 = D$ and for $n = 1, 2, \ldots$ $K_n = \overline{\text{co}}\,(P(K_{n-1}))$. We claim that for $n = 1, 2, \ldots$

$$K_n \subset K_{n-1} \text{ and } \alpha(K_n) \leqslant t^n \alpha(K_0) \quad (1)$$

Now $K_1 \subset K_0$ since D is closed and convex. Furthermore,

$$\alpha(K_1) = \alpha(\overline{\text{co}}\, P(K_0)) = \alpha(P(K_0)) \leqslant t\,\alpha(K_0)$$

where the second equality holds because of Darbo's theorem and the last inequality is a consequence of the fact that P is a strict set contraction. Hence, (1) hold for $n = 1$.

Assume that (1) holds for some $n \geqslant 1$. Then

$$K_{n+1} = \overline{\text{co}}\,(P(K_n)) \subset \overline{\text{co}}\,(P(K_{n-1})) = K_n$$

and

$$\alpha(K_{n+1}) = \alpha(\overline{\text{co}}\, P(K_n)) = \alpha(P(K_n) \leqslant t\,\alpha(K_n) \leqslant t^{n+1}\,\alpha(K_0)$$

so that (1) holds for all n and the proof of the claim is finished. Since K_n is a decreasing sequence of closed sets such that $\alpha(K_n) \to 0$ Kuratowski's theorems implies that the set $K = \bigcap_{n=0}^{\infty} K_n$ is nonempty and compact. K is also convex as an intersection of convex sets.

Observe that $P(K_n) \subset P(K_{n-1}) \subset P(\overline{\text{co}}(K_{n-1})) = P(K_n)$ so that P maps K into 2^K, and the first part of the proof has been completed.

It only remains to check that the map $P : K \to 2^K$ satisfies all the conditions of the fixed-point theorm of Tarafdar[19]. Now $P(x) \neq \phi$ for every $x \in K \subset D$, by hypothesis, so that P is nonempty valued on K. Condition (i) implies that P is convex-valued on K. It only remains to prove that for each $x \in K$ there exists $y \in K$ such that $x \in \text{int } P^{-1}(y)$,in the relative topology of K. So suppose $x \in K \subset D$. Then condition (iii) implies that there exists $y \in D$ such that $x \in \text{int } P^{-1}(y)$. This means that $y \in P(x) \subset K$ so that $y \in K$. Hence, for each $x \in K$ there exists $y \in K$ such that $x \in \text{int } P^{-1}(y)$ where the interior is in D. Therefore, there exists an open neighbourhood N in D such that $x \in N \subset \text{int } P^{-1}(y)$. Since N is open in D, $N \cap K$ is open in K. Consequently, $x \in N \cap K \subset$ relative interior of $P^{-1}(y)$ in K. This proves that for each $x \in K$ there exists $y \in K$ such that $x \in \text{int } P^{-1}(y)$ in the relative topology of K.

Hence, from the fixed-point theorem of Tarafdar[19] we conclude that there exists a point $x_0 \in K$ such that $x \in P(x_0)$, contradicting condition (ii). The contradiction proves the theorem.

Corollary—Let D be a closed, bounded and convex subset of a Banach space. Suppose that the preference map $P : D \to 2^D$ satisfies the following conditions:

(i) for each $x \in D$, $P(x)$ is convex;

(ii) for each $x \in D$, $x \notin P(x)$;

(iii) for each $y \in D$, $P^{-1}(y)$ is open in D;

(iv) P is a strict set contraction.

Then there exists a maximal element.

PROOF : It is easily verified that if $P^{-1}(y)$ is open in D for $y \in D$, then condition (iii) of the the theorem is satisfied.

A map satisfying conditions (i), (ii) and (iii) of the corollary is a B-map. The following generalization of a B-map is due to Yannelis and Prabhakar[23].

Definition—A map $T : K \to 2^K$ where K is a subset of topological vector space E is said to be a BS map if the following conditions hold:

(i) for each $x \in K$, $x \notin \text{co } T(x)$;

(ii) $T^{-1}(y)$ is open in K for each $y \in K$.

Clearly every B-map is a BS map. The following theorem is due to Yannelis and Prabhakar[23], (Corollary 5.1).

Theorem 2—Let K be a compact convex subset of a Hausdorff topological vector space and $P : K \to 2^K$ a map that is locally BS-majorized (i.e. locally P is a sub-map of a BS map). Then P has a maximal element.

The assumption of the compactness of the domain can be weakened in Banach spaces if P is a strict set contraction. More precisely, we have the following theorem:

Theorem 3—Let D be a closed, bounded and convex subset of a Banach space E. Let the preference map $P : D \to 2^D$ satisfy the following conditions:

(i) P is a locally BS-majorised.

(ii) P is a strict set contraction.

Then there exists a maximal element.

PROOF : Suppose that the theorem is false. Then $P(x) \neq \phi$ for each $x \in D$. Proceeding as Theorem 1, we get a compact convex subset K of D such that $P : K \to 2^K$. Hence, $P(x) \neq \phi$ for every $x \in K$. Since P is locally BS-majorized, for each x, there exists a BS-map T_x ; $D \to 2^D$ such that $z \notin \text{co}\, T_x(z)$ for all $z \in D$ and an open neighbourhood U_x such that $z \in U_x$ implies $P(z) \subset T_x(z)$. Define $T'_x(z) = \text{co}\, T_x(z) \cap K$ and $U'_x = U_x \cap K$. Then $T'_x : K \to 2^K$ is convex-valued and for every $z \in U'_x$, $P(z) \subset T'_x$. Clearly, $z \notin T'_x(z)$ for all $z \in K$. Hence, the map $P : K \to 2^K$ is locally BS-majorized and Theorem 2 implies that $P(x^*) = \phi$ for some x^*. The contradiction proves the theorem.

Remark : Since every locally B-majorized map is locally BS-majorized, it follows from Theorem 3 that every locally B-majorized preference map that is a strict set contraction has a maximal element in any closed, bounded and convex subset of a Banach space. This enables us to weaken the compactness condition of a Theorem 2.2 of Toussaint[20].

We turn now to the consideration of acyclic preference maps. First, we prove a generalization of a theorem of Bergstrom[2] and Walker[21].

Theorem 4—Let K be a compact subset of a topological space. Suppose the preference map $P : K \to 2^K$ satisfies the following conditions:

(i) P is acyclic; i.e. if

$$x_{i+1} \in P(x_i) \text{ for } i = 1, 2, \ldots, n \text{ then } x_1 \notin P(x_{n+1});$$

(ii) for each $x \in K$ such that $P(x) \neq \phi$, there exists $y \in K$ such $x \in \text{int}\, P^{-1}(y)$ in K.

Then there exists a maximal element.

PROOF : Suppose that the theorem is false. Then $P(x) \neq \phi$ for each $x \in K$. This implies that for each $x \in K$ there exists $y \in K$ such that $y \in P(x)$. Hence, condition (ii) implies that for each $x \in K$ there exists $y \in K$ such that $x \in \text{int } P^{-1}(y) = O_y$. The relatively open sets O_y cover K i.e. $K \cup O_y$. Since K is compact, there exists a finite subset $\{y_1, y_2, \ldots, y_n\}$ such that $K \bigcup_{i=1}^{n} O_{y_i}$.

It is easily verified that since P is acyclic, the set $\{y_1, y_2, \ldots, y_n\}$ has a maximal element $y' \in \{y_1, y_2, \ldots, y_n\}$, i.e. $y_i \notin P(y')$ for $i = 1, 2, \ldots, n$. Hence $y' \notin P^{-1}(y_i)$ (a fortiori, $y' \notin O_{y_i}$) for $i = 1, 2, \ldots, n$. This is a contradiction since $K = \bigcup_{i=1}^{n} O_{y_i}$, $y' \in K$ and $y' \notin O_{y_i}$ for $i = 1, 2, \ldots, n$. The contradiction proves the theorem.

Remark : If we assume that $P^{-1}(y)$ is open for each $y \in K$, we get the theorem of Bergstrom and Walker which is special case of Theorem 4 above.

For contracting preference maps the assumption of the compactness of the domain can be weakened. More precisely we have the following theorem.

Theorem 5—Let D be a closed bounded and convex subset of a Banach space E. Suppose that the preference map $P : D \to 2^D$ satisfies the following conditions:

(i) P is acyclic;

(ii) for each $x \in D$ such $P(x) \neq \phi$, there exists

$y \in D$ such that $x \in \text{int } P^{-1}(y)$ in D;

(iii) P is a strict set contraction.

Then there exists a maximal element.

PROOF : Suppose that the theorem is false. Then $P(x) \neq \phi$ for every $x \in D$. Proceeding as in the first part of Theorem 1 we get a compact convex set K such that $P : K \to 2^K$. Arguing as in the second part of Theorem 1 we can prove that for each $x \in K$ there exists $y \in K$ such that $x \in \text{int } P^{-1}(y)$ where the interior is in the relative topology of K. Hence, condition (ii) of Theorem 4 is satisfied. The restriction of P to K is clearly acyclic. This implies that condition (i) of Theorem 4 is satisfied. Consequently, Theorem 4 implies that there exists a point $x_0 \in K$ such that $P(x_0) = \phi$. The contradiction proves the theorem.

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