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ON THE ENDL-TYPE GENERALIZATION OF CERTAIN SUMMABILITY METHODS

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It is shown that if A is a regular Euler, Taylor or Meyer-König matrix and $r > 0$, then A and its Endl-type generalization A^r are absolutely equivalent for all sequences $s_n = o(n^{1/2})$.

Endl[4] and Jakimovski[6] introduced the "Endl-type" generalization of Hausdorff and quasi-Hausdorff matrices and, more recently, Kuttner and Parameswaran[7] introduced the generalization of the same type of the Meyer-König — Ramanujan matrix. The Hausdorff, quasi-Hausdorff and Meyer-König—Ramanujan matrices, when they are conservative, may be considered as "built" around the Euler, Taylor and Meyer-König methods $A = E_\alpha$, T_α and S_α respectively and reflect some of the properties of these methods. A similar statement holds good when we consider the Endl-type generalizations of all these methods. It is the object of this note to study some properties of the Endl-type generalization A^r, where $A = E_\alpha$, T_α or S_α. It is shown that A and A^r $(r > 0)$ are absolutely equivalent for all sequences $s_n = o(n^{1/2})$ (see below for the definitions).

Definitions—(1) Let $0 < \alpha < 1$ and $r \geqslant 0$. The generalized Euler matrix $E^r_\alpha = \left(a^r_{nk}\right)$ is defined by

$$a^r_{nk} = \begin{cases} \binom{n+r}{n-k} \alpha^{k+r}(1-\alpha)^{n-k} & (k \leqslant n) \\ 0 & (k > n) \end{cases} \qquad \ldots(1)$$

$n, k = 0, 1, \ldots, \ldots$ when $r = 0$ this gives the Euler matrix E_α. We write a_{nk} for a^0_{nk}

Let $0 < \alpha < 1$ and $r > 0$. The generalized Taylor matrix $T^r_\alpha = \left(b^r_{mn}\right)$ is defined by

$$b^r_{mn} = \begin{cases} \binom{n+r}{n-m} \alpha^{n-m}(1-\alpha)^{m+r+1} & (n \geqslant m \geqslant 0) \\ 0 & (n < m). \end{cases} \qquad \ldots(2)$$

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When $r = 0$, this gives the Taylor matrix T_α. We write b_{mn} for b^0_{mn}.

(3) Let $0 < \alpha < 1$ and $r \geqslant 0$. The generalized Meyer-König matrix $S^r_\alpha = \left(c^r_{mn}\right)$ is defined by

$$c^r_{mn} = c_n(m + r) = \binom{m + r + n}{n}(1 - \alpha)^{m+r+1}\,\alpha^n \quad (m, n \geqslant 0). \qquad \ldots(3)$$

When $r = 0$, this gives the Meyer-König matrix S_α. We write c_{mn} for c^0_{mn}. {For the definitions of T_α, S_α given above and further properties of these matrices, see Meyer-König[10]}.

(4) Following Cooke[2,3] we shall say that two given matrices A and B are absolutely equivalent for sequences s of a class X if $As - Bs \in (c_0)$ whenever $s \in X$. {For the matrices and sequences we shall discuss in this note, this notion coincides with "volläquivalenz" as defined in Zeller and Beekmann[15]}. Absolute equivalence with respect to fixed class X of sequences is obviously an equivalence relation and is in particular transitive.

Theorem 1—Let $0 < \alpha < 1$ and $r > 0$. Then E_α and E^r_α are absolutely equivalent for all sequences $s_n = o(n^{1/2})$.

PROOF : Let $s_n = o(n^{1/2})$, $E_\alpha\, s = t = \{t_n\}$ and $E^r_\alpha\, s = u = \{u_n\}$.

Now

$$|\, t_n - u_n \,| = |\, \Sigma\, a_{nk}\, s_k - \Sigma\, a^r_{nk}\, s_k \,| \leqslant \Sigma\, |\, a_{nk} - a^r_{nk} \,||\, s_k \,|. \qquad \ldots(4)$$

(Throughout the proof of Theorem 1, the symbol Σ stands for $\sum_{k=0}^{n}$.) It is enough to prove that the sum in (4) tends to 0 as $n \to \infty$.

Case I : $r = 1$—Since $a^1_{nk}\ a_{n+1,k+1}$, the sum in (4) is

$$\Sigma\, |\, a_{nk} - a_{n+1,k+1} \,||\, s_k \,| \leqslant \Sigma\, |\, a_{nk} - a_{n,k+1} \,||\, s_k \,| + \Sigma\, |\, a_{n,k+1} - a_{n+1,k+1} \,||\, s_k \,|$$

$= A_n + B_n$ (say). But $s_k = o(k^{1/2})$ and hence (i) $A_n \to 0$ (see Lorentz[9], Theorem 2 and §5.2 or Parameswaran[11], Theorem 3) and (ii) $B_n \to 0$ (Parameswaran[12], Lemma 5). Hence the theorem is true when $r = 1$.

Case II: *r is a positive integer* > 1—Now the sum in (4) is

$$\Sigma \mid a_{nk} - a_{n+r,k+r} \| s_k \mid \leqslant \sum_{t=0}^{r-1} \Sigma \mid a_{n+t,k+t} - a_{n+t+1,k+t+1} \| s_k \mid$$

$\to 0$ an $n \to \infty$ by repeated applications of Case I to $\{s_n\}$ and its translates.

Case III: $0 < r < 1$—For arbitrary fixed integers $n > k \geqslant 0$, let

$$a_{nk}(x) = \binom{n+x}{n-k} \alpha^{k+x} (1-\alpha)^{n-k} \tag{5}$$

$$= \frac{(n+x)(n-1+x)\ldots(k+1+x)\,\alpha^{k+x}(1-\alpha)^{n-k}}{(n-)!}.$$

Then

$$\log a_{nk}(x) = \sum_{\nu=k+1}^{n} \log(x+\nu) - \log(n-k)!$$

$$+ (k+x)\log\alpha + (n-k)\log(1-\alpha)$$

and

$$\frac{a'_{nk}(x)}{a_{nk}(x)} = \sum_{\nu=k+1}^{n} \frac{1}{x+\nu} + \log\alpha. \quad \text{.. (6)}$$

Since $\log \alpha < 0$ ang $\displaystyle\sum_{\nu=k+1}^{n} \frac{1}{x+\nu}$ is a positive decreasing function of x which tends to $+\infty$ as $x \to -(k+1)$ and to 0 as $x \to +\infty$, there exists $x_0 = x_0(n, k)$ such that the right-hand side of (6) is positive for $-(k+1) < x < x_0$ and negative for $x > x_0$. Thus

the function $a_{nk}(x)$ is strictly increasing for $-(k+1) < x < x_0$ and strictly decreasing for $x > x_0$. (*)

Now, for any $t > -k$, $\dfrac{a_{nk}(t)}{a_{nk}(t-1)} = \dfrac{\alpha(n+t)}{k+t}$

and this equals 1 exactly when $t = (\alpha n - k)/(1-\alpha)$. It follows from (*) that $t - 1 < x_0 = x_0(n, k) < t$; that is,

$$\frac{\alpha n - k}{1-\alpha} - 1 < x_0(n, k) < \frac{\alpha n - k}{1-\alpha}. \quad \text{...(7)}$$

Hence

$$a_{nk} = a_{nk}(0) < a_{nk}(x) < a_{nk}(1) = a_{n+1,k+1} \text{ for all } x \in (0, 1) \text{ if } 1 < x_0$$

and hence if $1 \leqslant (1-\alpha)^{-1}(\alpha n - k) - 1$. Thus

$$0 < a_{nk}(x) - a_{nk} < a_{n+1,k+1} - a_{nk}$$

if

$$\frac{\alpha n - k}{1 - \alpha} \geqslant 2. \qquad \ldots(8)$$

Similarly $a_{nk} = a_{nk}(0) > a_{nk}(x) > a_{nk}(1) = a_{n+1,k+1}$ for all $x \in (0, 1)$ if $0 > x_0$, and hence if

$$\frac{\alpha n - k}{1 - \alpha} \leqslant 0. \qquad \ldots(9)$$

Thus if n, k satisfy (8) or (9), then

$$\left.\begin{array}{c} | a_{nk}(x) - a_{nk} | < | a_{n+1,k+1} - a_{nk} | \\ \text{for all } x \in (0, 1). \end{array}\right\} \qquad \ldots(10)$$

The inequality (10) may fail to hold only for those n, k such that neither (8) nor (9) holds; thus (10) holds except possibly when

$$0 < \frac{\alpha n - k}{1 - \alpha} < 2. \qquad \ldots(11)$$

We note that for each n, there are at most two the values of k for which (11) holds.

We write the sum in (4) as

$$\Sigma \mid a_{nk}(r) - a_{nk} \| s_k \mid \leqslant \Sigma_1 + \Sigma_2$$

where Σ_1 denotes the sum taken over those k which satisfy (10) and Σ_2 denotes the sum over those k which satisfy (11). Then

$\Sigma_1 \leqslant \Sigma \mid a_{nk} - a_{n+1,k+1} \| s_k \mid \to 0$ (as $n \to \infty$) as seen in Case I above; and Σ_2 is the sum of at most two terms which are $o(1)$ since $s_n = o(n^{1/2})$ and $\max\limits_k a_{nk}(x) = O(n^{-1/2})$ for each fixed $x > 0$ (see e. g. Hardy[5], Theorem 138 (2) or Lorentz[9], p. 313). Hence $\Sigma_2 \to 0$ as $n \to \infty$, and the theorem is proved for the case $0 < r < 1$ also.

Case IV—$r = [r] + q$, where $1 \leqslant [r]$ = the largest integer less than equal to r and $0 < q < 1$. Then by (1)

$$\Sigma \mid a^{r}_{nk} - a^{[r]}_{nk} \| s_k \mid$$

$$= \Sigma \mid a^{q}_{n+[r],k+[r]} - a_{n+[r],k+[r]} \| s_k \mid$$

$\to 0$ by Case III applied to the sequence $\{s_{k-[r]}\}$ (where, as usual, s_j is defined to be 0 when j is a negative integer). Thus E^{r}_{α} and $E^{[r]}_{\alpha}$ are absolutely equivalent for s and since E_{α} and $E^{[r]}_{\alpha}$ are absolutely equivalent for s by Case II, we see that the theorem is true for the Case IV also. This completes the proof of the theorem.

Definition 5—We say that a matrix method A is absolutely regular for a sequence

s if $As - As^* \in (c_0)$ where $s_n^* = s_{n-1}$ $(n > 0)$, or equivalently if

$$\sum_{k=0}^{\infty} a_{nk}\, s_k - \sum_{k=0}^{\infty} a_{n,k+1}\, s_k \to 0 \text{ as } n \to \infty \text{ (Cooke[2,3]).}$$

Theorem 2—The method E_α^r, where $0 < \alpha < 1$ and $r > 0$, is absolutely regular for all $s_n = o\,(n^{1/2})$

PROOF : It is enough to prove that

$$\rho_n \equiv \Sigma \mid a_{nk}^r - a_{n,k+1}^r \mid\mid s_k \mid \to 0 \text{ if } s_n = o\,(n^{1/2}). \qquad \text{...(12)}$$

This is true when $r = 0$ (by Theorems 2 and 9 (ii) of Lorentz[9]. Since

$$0 \leqslant \rho_n \leqslant \Sigma \,\{ \mid a_{nk}^r - a_{nk} \mid + \mid a_{nk} - a_{n,k+1} \mid + \mid a_{n,k+1} - a_{n,k+1}^r \mid \} \mid s_k \mid$$

the relation (12) follows from Theorem 1.

Lemma 1—For $0 < t < 1$ and $r \geqslant 0$, let $\{E_n^r\ (t; s)\}$ denote the E_t^r-transform of $s = \{s_n\}$. Then

$$E_n^r\ (t; s) - E_{n-1}^r\ (t; s) = t\, \Sigma_{n-1}^r\ (t; \bar{s} - s)$$

where $\bar{s} = \{s_{n+1}\}$.

The proof for the case $r = 0$ has been given in Parameswarn[12]; the proof for the case $r > 0$ is similar.

Theorem 3— Let $A = (H^r, g)$ be the generalized Hausdorff matrix defined by

$$a_{nk} = \binom{n+r}{n-k} \int_0^1 t^{+r}\,(1-t)^{n-k}\, dg\,(t)$$

where $g \in BV\,[0, 1]$ and $r > 0$, and let $s = \{s_n\} = \{\sum_0^n a_1\}$ satisfy the conditions $s_n = o\,(n^{1/2})$ and $a_n = O\,(1)$. Then the set of limit points of $u = As$ will be connected if $g\,(1) = g\,(1) = g\,(1-0)$.

Remark : Theorem 3 for the case $r = 0$ was proved by the author in Parameswaran[13]; Liu and Rhoades[8] proved the theorem for bounded $\{s_n\}$ and $r > 0$; (see also

Kuttner and Parameswaran[7]). Proofs in the more special case $s_n = O(1)$ and $r = 0$ have been given by various authors; Liu and Rhoades[8] or Parameswaran[13] for detailed references.

PROOF OF THEOREM 3 : Since $u_n = \int_0^1 E_n^r\ (t: s)\, dg\,(t)$ we have for $n \geqslant 1$,

$$u_n - u_{n-1} = \int_0^1 t\, E_{n-1}^r\,(t, a)\, dg\,(t) = \int_0^{1-0} + \int_{1-0}^1 \qquad \ldots(13)$$
$$= o\,(1) + [g\,(1) - g\,(1-0)]\, a_n$$

using Lemma 1 and the facts thnt $a_n = O\,(1)$ implies that $t\, E_{n-1}^r\,(t,\ a)$ is uniformly bounded in $[0, 1]$ and tends to 0 as $n \to \infty$ (since $s_n = o\,(n^{1/2})$ implies that $\{a_n\}$ is E_t-summable to 0 (Hardy[5], p. 213) and hence is E_t^r -summable by Theorem 1). It follows from (13) that $u_n - u_{n-1} = o\,(1)$ if $g\,(1) = g\,(1-0)$ and hence the set of limit points of the bounded sequence $u = As$ is connected, by a theorem due to Barone[1].

Taeorim 4—Let $0 < \alpha < 1$ and $r > 0$. Then S_α and S^r_σ are absolutely equivalent for all sequences $s_n = o\,(n^{1/2})$.

PROOF : The proof is similar to that of Theorem 1 and we omit some of the details. Let $s_n = o\,(n^{1/2})$, $t = S_\alpha\, s$, $u = S_\alpha^r\, s$, $\bar{s} = \{s_{n+1}\}$, $\bar{t} = S_\alpha\, \bar{s}$ and $a_n = s_n - s_{n-1}$.

Case I : *r is a positivc integer*—Then $u_m = t_{m+r}$ by (3) and hence $u_m - t_m = \sum_{n=m+1}^{m+r} (t_n - t_{n-1})$. Now, as noted by (Meyer-König[10], p. 275),

$$t_n - t_{n-1} = \alpha\,(1 - \alpha^n \sum_{\nu=0}^{\infty} \left(\frac{n+\nu}{\nu}\right) \alpha^\nu\ a_{\nu+1}$$

$$= \left(\frac{\alpha}{1-\alpha}\right)(1-\alpha)^{n+1} \sum_{\nu=0}^{\infty} \binom{n+\nu}{\nu} \alpha^\nu\,(s_{\nu+1} - s_\nu)$$

$$= \left(\frac{\alpha}{1-\alpha}\right)\ (\bar{t}_n - t_n).$$ But $t_n - t_n). = o\,(1)$ by the the absolute regularity of sequences $s_n = o\,(^{1/2})$ (Parmeswaran[11], Theorem 3). Since r is a fixed positive integer, it follows that $u_m - t_m = o\,(1)$.

Case II : $0 < r < 1$—It is enough to prove that

$$\sum_{n=0}^{\infty} | c_n (m + r) - c_n (m)||s_n | = o (1). \quad ...(14)$$

We follow Kuttner's argument used by Sitaraman[14], namely that

$$| c_n (m + r) - c_n (m) | < | c_n (m) - c_n (m + 1) | \quad .. (15)$$

except possibly for those values of n for which

$$m\alpha (1 - \alpha)^{-1} < n < (m + 2) \alpha (1 - \alpha)^{-1}. \quad .. (16)$$

We write the sum in (14) as $\Sigma_1, + \Sigma_2$, where Σ_1 is the contribution to the sum by those n which satisfy (16) and Σ_2 is the contribution by those n which do not satisfy (16). Now the number $g (m)$ of integers n satisfying (16) is a bounded function of m and hence it follows from the well known fact that $\max\limits_n c_n (x) = O (x^{-1/2})$ as $x \to \infty$, that

$$\Sigma_1 \to 0 \text{ as } m \to \infty. \quad ...(17)$$

Now $| \Sigma_2 | \leqslant \sum\limits_{n=0}^{\infty} | c_n (m) - c_n (m + 1)||s_n | = \sum\limits_{n=0}^{\infty} | c_{mn} - c_{m+1,n}||s_n | = \rho_m$ (say),

and

$$\sum_{n=0}^{\infty} c_{mn} - c_{m+1, n}) s_n = o (1) \text{ whenever } s_n = o (n^{1/2}) \quad .. (18)$$

(as we observed in the proof of Case I). But (18) holds if and only $\rho_m = o(1)$ Cooke[2], Theorem 6 or Cooke[3], Theorem 5.5I). It follows that $\Sigma_2 = o(1)$. The conclusion (14) now follows from (15) and (17).

Case III : $r > 1$—The desired result follows from Cases I and II since the mth term of $S_\alpha^r s$ is the $(m + [r])$th term of $S_\alpha^{r-[r]}$ s, where $[r]$ = the integral part of r.

Theorem 5—Let $0 < \alpha < 1$ and $r > 0$. Then T_α and T_α^r are absolutely equivalent for sequences $s_n = o (n^{1/2})$.

PROOF : We note that the relations (2) and (3) imply that

$$b_{mn}^r = 0 \text{ for } n < m \text{ and } b_{m,m+n}^r = c_{mn}^r \text{ for all } m, n.$$

Hence

$$\sum_{n=0}^{\infty} (b_{mn}^r - b_{mn})s_n = \sum_{n=0}^{\infty} (b_{m,m+n}^r - b_{m,m+n}) s_{m+n}$$

$$= \sum_{n=0}^{\infty} (c_{mn}^r - c_{mn}) s_{m+n}. \quad .. (21)$$

We may now prove the theorem either by using arguments similar to those used in the proof of Theorem 4, or, as suggested by the referee, we can derive the result from Theorem 4 as follows.

We note that if (α_{mn}), (β_{mn}) are regular matrices and if $\{\lambda_n\}$ is a given sequence of positive numbers then, in order that (α_{mn}). (β_{mn}) should be absolutely equivalent for sequences satisfying $s_n = o(\lambda_n)$, it is necessary and sufficient that

$$\sum_{n=0}^{\infty} | \alpha_{mn} - \beta_{mn} | \lambda_n = O(1). \qquad ...(22)$$

For we require that the transformation

$t_m = \sum_{n=0}^{\infty} (\alpha_{mn} - \beta_{mn}) \lambda_n (s_n/\lambda_n)$ regarded as a transformation from $\{u_n\} = \{s_n/\lambda_n\}$ to $\{t_m\}$, should transform null sequences into null sequences. The "standard" necessary and sufficient conditions for this are that (22) should hold and that, for fixed n,

$$\alpha_{mn} - \beta_{mn} \to 0 \text{ as } m \to \infty. \qquad ...(23)$$

But if (α_{mn}), (β_{mn}) are regular, then (23) is necessarily satisfied. Thus, Theorem 4 is equivalent to the assertion that

$$\sum_{n=0}^{\infty} | c^{r}_{mn} - c_{mn} | n^{1/2} = O(1). \qquad ...(24)$$

and, by equation (21), Theorem 5 is equivalent to the assertion that

$$\sum_{n=0}^{\infty} | c^{r}_{mn} - c_{mn} | (m+n)^{1/2} = O(1). \qquad ...(25)$$

Since Theorem 4 has been proved, we know that (24) holds. We have to prove that(25) also holds.

Now choose a constant A with $0 < A < \alpha/(1-\alpha)$. Then

$$\sum_{n<Am} c_{mn} (m+n)^{1/2} = O(e^{-\gamma m}) \qquad ...(26)$$

and

$$\sum_{n<Am} c^{r}_{mn} (m+n)^{1/2} = O(e^{-\gamma m}) \qquad ...(27)$$

where γ is a positive constant. The first of these results is given by Theorem 139 (2) of Hardy[5] the second may be proved in a similar way. But $(m+n)^{1/2} = O(n^{1/2})$ uniformly in $n \geqslant Am$ and hence, by (24),

$$\sum_{n \geq Am} | c^r_{mn} - c_{mn} | (m + n)^{1/2} = O(1). \quad \ldots(28)$$

The required result (25) now follows from (26), (27) and (28).

References

1. H. G. Barone, *Duke Math. J.* **5** (1939), 740–52.
2. R. G. Cooke, *Proc. London Math. Soc.* (2) **41** (1936), 113–25.
3. R. G. Cooke, *Infinite Matrices and Sequence Spaces*. Macmillan and Co. London, 1950.
4. K. Endl, *Math. Annalen* **139** (1960), 403–32.
5. G. H. Hardy, *Divergent Series,* Oxford Univ. Press Oxford, 1949.
6. A. Jakimovski, *Technical Report No. 8, Jerusalem,* 1959.
7. B. Kuttner and M. R. Parameswaran, *Houston. J. Math.* 4 (1978), 569–76.
8. M. Liu and B. E. Rhoades, *Houston J. Math.* 2 (1976), 239–50.
9. G. G. Lorentz, *Canadian J. Math.* **1** (1949), 305–19.
10. W. Meyer-König, *Math. Z.* **52** (1949), 257–304.
11. M. R. Parameswaran, *Proc. Nat. Inst. Sci. India* **A25** (1959), 171–75.
12. M. R. Parameswaran, *J. Indian Math. Soc.* **(NS) 23** (1959), 46–64.
13. M. R. Parameswaran, *Canadian Math. Bull.* **23** (1980), 137–40
14. Y. Sitaraman, *Math. Z.* **106** (1968), 153–57.
15. K. Zeller and W. Beekmann, *Theorie der Limitierungsverfahren.* Springer-Verlag Berlin 1970.