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MATRIX TRANSFORMATIONS IN SOME SEQUENCE SPACES

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The purpose of this paper is to characterise the matrices in the classes $(c(p), C_s)$, $(l_\infty(p), C_s)$ and $(l(p), C_s)$.

Let X and Y be any two nonempty subsets of the space of all sequences of complex numbers and let $A = (a_{nk})$, $(n, k = 1, 2, \ldots)$ be an infinite matrix of complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_k a_{nk} x_k$ converges for each n. (Throughout summation without limits runs from 1 to ∞). If $x = (x_k) \in X$ implies that $Ax = (A_n(x)) \in Y$, we say that A defines a matrix transformation from X into Y and we denote it by $A : X \to Y$. By (X, Y) we mean the class of matrices A such that $A : X \to Y$. If in X and Y there is some notion of limit or sum, then we write (X, Y, P) to denote the subset of (X, Y) which preserves the limit or sum.

If $p_k > 0$ and $\sup_k p_k < \infty$, we define (see Maddox[3])

$$c(p) = \{x : |x_k - l|^{p_k} \to 0 \text{ for some } l\}$$

$$l_\infty(p) = \{x : \sup_k |x_k|^{p_k} < \infty\}$$

$$l(p) = \{x : \sum_k |x_k|^{p_k} < \infty\}.$$

We define (see Stieglitz and Tietz[5])

$$C_S = \{x : \{\sum_{i=1}^{n} x_i\} \text{ is convergent}\}.$$

The purpose of this paper is to characterise the matrices in the classes $(c(p), C_S)$, $(l_\infty(p), C_S)$ and $(l(p), C_S)$.

The following notations are used throughout. For all integers $n \geq 1$ we write

$$t_n(Ax) = \sum_{i=1}^{n} A_i(x) = \sum b_{nk} x_k$$

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where

$$b_{nk} = \sum_{i=1}^{n} a_{ik}.$$

For all integers n, $t \geqslant 1$ and $1 \leqslant r \leqslant \infty$, we write

$$C(n, B, t, r) = \sum_{k=t}^{r} | b_{nk} |^{q_k} B^{-q_k}$$

where B is an integer and $p_k^{-1} + q_k^{-1} = 1$. We put

$$C(n, B, 1, \infty) = C(n, B) \text{ and } C(B) = \sup_n C(n, B).$$

We have

Theorem 1—$A \in (c(p), Cs)$ if and only if

(i) $D = \sup_n \sum_n | b_{nk} | B^{-1/p_k} < \infty$ for some integer $B > 1$

(ii) $\exists\ \alpha_k \in C$ such that

$$\lim_{n \to \infty} b_{nk} = \alpha_k \ (\forall\ k)$$

(iii) $\exists\ \alpha \in C$ such that

$$\lim_{n \to \infty} \sum b_{nk} = \alpha.$$

Proof : *Necessity*—Let $A \in (c(p), Cs)$. Put $t_n(Ax) = 6_n(x)$.

Since $(c(p), Cs) \subset (c_0(p), Cs)$, $\{6_n(x)\}$ is a sequence of continuous linear functionals on $c_0(p)$ (see Maddox[2]) such that $\lim 6_n(x)$ exists. Therefore by uniform boundedness principle for $0 < \delta < 1$, there exist $S_\delta[0] \subset c_0(p)$ and a constant K such that $\sigma_n(x) \leqslant K$ for each n and $x \in S_\delta[0]$. Define for each r :

$$y_n^r = \begin{cases} \delta^{M/p_k} \operatorname{sgn} b_{nk}, & 0 \leqslant k \leqslant r \\ 0, & r < k \end{cases}$$

where $M = \max(1, \sup_k p_k)$.

Now $y^r \in S_\delta[0]$ and

$$\sum_{k=1}^{r} | b_{nk} | B^{-1/p_k} \leqslant K$$

for each n and r where $B = \delta^{M}$. Therefore (i) holds. (ii) and (iii) trivially hold.

Sufficiency—Suppose (i)—(iii) hold and $x \in c(p)$. Then there exists l such that $|x_k - l|^{p_k} \to 0$. Hence for $0 < \epsilon < 1$ there exists $k_0 : \forall\, k > k_0$

$$|x_k - l|^{p_k/M} \leqslant \frac{\epsilon}{B(2D+1)} < 1$$

and therefore for $k > k_0$

$$B^{1/p_k} |x_k - l| < B^{M/p_k} |x_k - l| < \left(\frac{\epsilon}{2D+1}\right)^{M/p_k} < \frac{\epsilon}{2D+1}.$$

By (i) and (ii) we have

$$\sum_k |b_{nk} - \alpha_k| B^{-1/p_k} < 2D.$$

Hence

$$\sum_{k>k_0} |(b_{nk} - \alpha_k)(x_k - l)| < \epsilon.$$

Also

$$\sum_{k \leqslant k_0} |(b_{nk} - \alpha_k)(x_k - l)| \to 0 \text{ as } n \to \infty.$$

Therefore we have

$$\lim_{n\to\infty} \sum_k b_{nk}\, x_k = l\alpha + \sum_k \alpha_k (x_k - l).$$

This completes the proof.

Theorem 2—$A \in (l_\infty(p), Cs)$ if and only if

(i) for all integers $N > 1$

$$\sum_k |b_{nk}|\, N^{1/p_k} \text{ converges uniformly in } n$$

(ii) $\lim_{n\to\infty} b_{nk} = \alpha_k\ (\forall\, k)$.

Proof : Suppose that $A \in (l_\infty(p), Cs)$. Clearly (ii) holds. If (i) is false then the matrix $C = (C_{nk}) = (a_{nk} N^{1/p_k}) \notin (l_\infty, Cs)$ for some integer $N > 1$, see Stieglitz and Tietz[3]. So there exists $x \in l_\infty$ such that $Cx \notin Cs$. Now $y = (y_k) = (N^{1/p_k} x_k) \in l_\infty(p)$, but $Ay = Cx \notin Cs$ and this contradiction completes the proof.

For the sufficiency, suppose that the conditions hold. Take an integer $N >$ max $(1, \sup |x_k|^{p_k})$. We have

$$\left|\sum_k (b_{nk} - \alpha_k)\, x_k\right| < \sum_k |b_{nk} - \alpha_k|\, N^{1/p_k}.$$

By (i) and (ii) we have

$$\lim_{n\to\infty} \sum_k b_{nk} x_k = \sum_k \alpha_k x_k.$$

This completes the proofs.

Theorem 3—$A \in (l(p), C_S)$ if and only if

(i) there exists an integer $B > 1$ such that

$C(B) > \infty \; (1 < p_k < \infty)$.

$$\sup_{n'k} | \; b_{nk} \; |^{p_k} > \infty \; (0 < p_k \leqslant 1)$$

(ii) $\lim_{n\to\infty} b_{nk} = \alpha_k \; (A \;\; k)$.

This is an immediate consequence of the Corollary to Theorem 1 of Lascarides and Maddox[1], if one notices that $A \in (l(p), C_S)$ if and only if the matrix $B = (b_{nk}) \in (l(p), c)$.

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