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THERMAL STABILITY OF A FLUID LAYER IN A VARIABLE GRAVITATIONAL FIELD

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The instability of a heated layer of a viscous fluid confined between two horizontal planes and subjected to a variable gravitational field varying spatially with height is investigated. For a layer confined between two stress-free boundaries, irrespective of whether gravitational acceleration is increasing or decreasing with height, it is shown that (i) the principle of exchange of stabilities is valid when the layer is heated from below, and (ii) the layer is stable when it is heated from above. In the latter case, the complex growth rate of an arbitrary oscillatory mode lies outside of a circle whose radius depends on the wavelength of the mode and the Prandtl number of the fluid but not on the Rayleigh number. The underlying characteristic value problem is solved approximately for a linearly varying gravity field and stress-free boundaries. It is found that in the case of a layer heated from below, gravity increasing upward is a destabilizing influence.

1. INTRODUCTION

The importance of convection currents in our environment can scarcely be overestimated; we need only look to our immediate environment and note that the circulation of the Earth's atmosphere could not be explained without reference to convective motions induced by solar heating. However, the idealization of uniform gravity assumed in the theoretical investigations, although valid for laboratory purposes, can scarcely be justified for large-scale convection phenomena occurring in the atmosphere, the ocean or the mantle of the earth. It then becomes imperative to consider gravity as a variable quantity varying with distance from a reference point or surface.

Although Gresho and Sani[3] have considered a time-varying gravitational field in the convection problem of a horizontal fluid layer, the problem of a spatially-varying gravitational field in the plane layer case still remains open. Pradhan and Patra[4] considered the problem of onset of thermal instability in a cylindrical shell of self-gravitating fluid heated internally and from below i.e. from the inside boundary. In case of a shell heated from below, the narrow-gap approximation led to the plane layer problem with the gravity field varying linearly with height from the bottom surface. This prompted Pradhan and Samal[5] to consider the plane layer problem under a spatially-varying gravity field, neglecting viscosity. They found, among other things, that if gravity remained downward (upward) throughout the flow domain, neutral modes did not exist. A sufficient condition for stability of a layer heated from above was that gravity remain directed downward over a sufficiently large part of the flow domain. However, a sufficient condition for instability of a layer heated from below was that, besides remaining directed downward, the gravity profile must have concave curvature throughout the flow domain. There also existed a circle limiting the growth rates of an arbitrary oscillatory mode of the system. These results, which are indicative of the behaviour expected in the limit of small viscosity in the fluid, are here extended to the case of finite viscosity.

2. Perturbation Equations

Consider a layer of an incompressible, viscous fluid statically confined between two horizontal boundaries $z = 0$ and $z = d$ which are maintained at constant temperatures T_0 and T_1 respectively. The governing equations for flow and temperature, under the Boussinesq approximation, are

$$\rho_0\left(\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla\bar{u}\right) = -\nabla p - \rho_0\, g\,(z)\,\bar{k}\,(1 - \alpha\,(T - T_0)) + \mu\,\nabla^2\,\bar{u} \qquad \text{...(1)}$$

$$\nabla \cdot \bar{u} = 0 \qquad \text{...(2)}$$

and

$$\frac{\partial T}{\partial t} + \bar{u} \cdot \nabla\, T = \kappa\,\nabla^2\, T \qquad \text{...(3)}$$

where ρ is the density of the fluid, $\bar{u} = (u, v, w)$ the fluid velocity, $g\,(z)$ the gravitational acceleration in a coordinate system $Oxyz$ having Oz vertical and Ox, Oy in a horizontal plane, k is a unit vetor in the vertically upward direction; p the pressure in the fluid; μ the dynamic viscocity; T the obsolute temperature; κ the thermal diffusivity and ρ_0 the density at temperature T_0.

In the equilibrium state heat transport is by conduction alone and there are no velocities in the fluid. The solution of eqns. (1)–(3) is easily obtained in this case as the solution of (the basic state is denoted by an over bar).

$$\bar{u} = 0 \qquad \text{...(4)}$$

$$\nabla\,\bar{p} = g\,\rho_0\,\bar{k}\,\{1 - \alpha\,(\bar{T} - T_0)\} \qquad \text{...(5)}$$

$$\nabla^2\,\bar{T} = 0. \qquad \text{...(6)}$$

The basic temperature $\bar{T}$ is assumed to depend only on z, so we can integrate (6) to obtain

$$\bar{T} = T_0 + \beta\, z \qquad \text{...(7)}$$

where $\beta = (T_0 - T_1)/d$ is negative, since this is the solution which gives $T = T_0$, T_1 at $z = 0,\ d$.

Writing

$$p = \bar{p} + p',\ T = \bar{T} + T',\ \rho = \bar{\rho} + \rho',\ \bar{u} = \bar{u}' \qquad \text{...(8)}$$

where p', T', ρ' and $\bar{u}'$ are small perturbations with respect to which the equations are linearized and using the Boussinesq approximation, we obtain

$$\frac{\partial \bar{u}'}{\partial t} = \frac{1}{\rho_0} \nabla p' + \nu \nabla^2 \bar{u} + g \alpha T' \bar{k} \qquad \text{...(9)}$$

$$\nabla \cdot \bar{u}' = 0 \qquad \text{...(10)}$$

and

$$\frac{\partial T'}{\partial t} + \beta w = \kappa \nabla^2 T' \qquad \text{..(11)}$$

where ν is the kinematic viscosity.

Seeking solutions in terms of normal modes whose dependence on x, y and t is given by

$$\exp\ i\,(k_n x + k_y y + \sigma t) \qquad \text{..(12)}$$

and eliminating p', u' and v' from the resulting equations, we have

$$\nu (D^2 - k^2)\left(D^2 - k^2 - \frac{i\sigma}{\nu} \right) w = k^2 g \alpha T' \qquad \text{...(13)}$$

and

$$i \sigma T' - \kappa (D^2 - k^2) T' = -\beta w \qquad \text{...(14)}$$

where D denotes differentiation with respect to z and $k^2 = k_X^2 + k_Y^2$.

Assuming the boundaries to be perfectly heat conducting and either rigid or free we have

$$T' = 0 = w \text{ for } z = 0 \text{ and } d$$

and either $Dw = 0$ (in case of a rigid boundary)

or $D^2 w = 0$ (in case of a free boundary) .. (15)

Introducing the non-dimensional variables

$$z_* = z/d,\quad a = kd,\quad \sigma_* = \frac{i \sigma d^2}{\nu},$$

$$w_* = w\, d/k,\ D_* = dD,\ \theta_* = T/\beta d \qquad \text{..(16)}$$

and $g(z) = g_0 \gamma(z_*)$ where g_0 is the value of $g(z)$ at $z = 0$, we can write the eqns. (13) and (14), on suppressing the star subscripts, as

$$(D^2 - a^2 - \sigma)(D^2 - a^2)\, w = -Ra^2 \gamma(z)\, \theta \qquad ...(17)$$

and

$$(D^2 - a^2 - P_r)\, \theta = w \qquad ...(18)$$

where

$$R = \frac{-g_0 \alpha \beta d^4}{\nu \kappa} \qquad ...(19)$$

and

$$P_r = \nu/\kappa \qquad ...(20)$$

are the non-dimensional Rayleigh and Prandtl numbers, respectively.

The boundary conditions reduce to $w = 0 = \theta$ ($z = 0, 1$) and either

$$Dw = 0 \quad \text{(at a rigid boundary)}$$

or

$$D^2 w = 0 \text{ (at a free boundary).} \qquad ...(21)$$

Thus, for a given a, R and P_r, the equations (17) and (18) together with the boundary conditions (21) constitute an eigenvalue problem for σ and the system is unstable, neutral or stable according as the real part of σ, namely, σ_r is positive, zero or negative, respectively.

3. The Principle of Exchange of Stabilities

For fixed, stress-free boundaries at $z = 0, 1$ we have the kinematic boundary condition as

$$w = D^2 w = 0 \ (z = 0, 1). \qquad ...(22)$$

Since the thermal boundary condition is

$$\theta = 0 \ (z = 0, 1) \qquad ...(23)$$

the boundary condition (22) can be expressed in terms of θ by means of (18) and (23) as

$$D^2 \theta = D^4 \theta = 0 \ (z = 0, 1). \qquad ...(24)$$

So finally

$$\theta = D^2 \theta = D^4 \theta = 0 \ (z = 0, 1). \qquad ...(25)$$

Combining eqns. (17) and (18), we have

$$(D^2 - a^2 - P_r \sigma)(D^2 - a^2 - \sigma)(D^2 - a^2)\, \theta = -Ra^2 \gamma(z)\, \theta \qquad ...(26)$$

which must be considered together with the boundary conditions (25).

Multiplying eqn. (26) by θ^*, the complex conjugate of θ, and integrating the resulting equation over the range of z and making use of the boundary conditions (25), we have

$$P_r \sigma^2 \int_0^1 (\mid D\theta \mid^2 + a^2 \mid \theta \mid^2) dz + \sigma (1 + P_r) \int_0^1 \mid (D^2 - a^2) \theta \mid^2 dz$$

$$+ \int_0^1 (\mid D^3 \theta \mid^2 + 3a^2 \mid D^2 \theta \mid^2 + 3a^4 \mid D\theta \mid^2$$

$$+ a^6 \mid \theta \mid^2) dz - Ra^2 \int_0^1 \gamma (z) \mid \theta \mid^2 dz = 0. \quad \text{...(27)}$$

Now, for a neutral mode, we must have $\sigma = i\sigma_i$ with σ_i real, and then the real and imaginary parts of (27) give

$$P_r \sigma_i^2 \int_0^1 (\mid D\theta \mid^2 + a^2 \mid \theta \mid^2) dz - \int_0^1 (\mid D^3 \theta \mid^2 + 3a^2 \mid D^2 \theta \mid^2$$

$$+ 3a^4 \mid D\theta \mid^2 + a^6 \mid \theta \mid)^2 dz + Ra^2 \int_0^1 \gamma (z) \mid \theta \mid^2 dz = 0 \quad \text{...(28)}$$

and

$$(1 + P_r) \sigma_i \int_0^1 \mid (D^2 - a^2) \theta \mid^2 dz = 0. \quad \text{...(29)}$$

Equation (29) shows that $\sigma_i = 0$ and then (28) shows that the principle of exchange of stabilities is valid provided the neutral state exists, that is,

$$\int_0^1 (\mid D^3 \theta \mid^2 + 3a^2 \mid D^2 \theta^2 \mid^2 + 3a^4 \mid D\theta \mid^2 + a^6 \mid \theta \mid^2) dz$$

$$= Ra^2 \int_0^1 \gamma (z) \mid \theta \mid^2 dz. \quad \text{...(30)}$$

Equation (30) shows that, since $\gamma (z) > 0$, a necessary condition for the existence of neutral states is that

$$R > 0. \quad \text{...(31)}$$

If R is negative, there is no neutral state and a random disturbance will either be damped or amplified.

4. A Sufficient Condition for Stability

Since $\sigma \neq 0$ for R negative in the case of stress-free boundaries, multiplying eqn. (27) throughout by σ^*, the complex conjugate of σ and dividing by $\mid \sigma \mid^2$ we have

$$P_r \sigma \int_0^1 (|D\theta|^2 + a^2 |\theta|^2)\, dz + (1 + P_r) \int_0^1 |(D^2 - a^2)\theta|^2\, dz$$

$$+ \frac{\sigma^*}{|\sigma|^2} \int_0^1 (|D^3\theta|^2 + 3a^2 |D^2\theta|^2 + 3a^4 |D\theta|^2 + a^6 |\theta|^2)\, dz$$

$$- \frac{Ra^2 \sigma^*}{|\sigma|^2} \int_0^1 \gamma(z) |\theta|^2\, dz = 0. \qquad \text{...(32)}$$

Separating the real and imaginary parts of eqn. (32), we have

$$P_r \sigma_r \int_0^1 (|D\theta|^2 + a^2 |\theta|^2)\, dz + (1 + P_r) \int_0^1 |(D^2 - a^2)\theta|^2\, dz$$

$$+ \frac{\sigma_r}{|\sigma|^2} \int_0^1 (|D^3\theta|^2 + 3a^2 |D^2\theta|^2 + 3a^4 |D\theta|^2 + a^6 |\theta|^2)\, dz$$

$$- Ra^2 \frac{\sigma_r}{|\sigma|^2} \int_0^1 \gamma(z) |\theta|^2\, dz = 0 \qquad \text{...(33)}$$

and

$$P_r \sigma_i \int_0^1 (|D\theta|^2 + a^2 |\theta|^2)\, dz - \frac{\sigma_i}{|\sigma|^2} \int_0^1 (|D^3\theta|^2$$

$$+ 3a^2 |D^2\theta|^2 + 3a^4 |D\theta|^2 + a^6 |\theta|^2)\, dz$$

$$+ Ra^2 \frac{\sigma_i}{|\sigma|^2} \int_0^1 \gamma(z) |\theta|^2\, dz = 0. \qquad \text{...(34)}$$

Equation (34) shows that if $R < 0$, then $\sigma_r < 0$ provided $\gamma(z) > 0$ over most of the layer thus implying stability for the configuration.

5. A Circular Exclusion Theorem for Oscillatory Modes

Consider an arbitrary oscillatory mode of the system in the case of stress-free boundaries. For such modes, we have $\sigma_i \neq 0$. Hence eqn. (34) can be put as

$$\int_0^1 (|D^3\theta|^2 + 3a^2 |D^2\theta|^2 + 3a^4 |D\theta|^2 + a^6 |\theta|^2$$

$$- Ra^2 \gamma(z) |\theta|^2)\, dz - P_r \sigma^2 \int_0^1 (|D\theta|^2 + a^2 |\theta|^2)\, dz = 0.$$

$$\text{...(35)}$$

This can be rewritten as

$$\int_0^1 (|D^3\theta|^2 + 3a^2|D^2\theta|^2 + 2a^4|D\theta|^2)\,dz$$

$$+ (a^4 - P_r|\sigma|^2)\int_0^1 (|D\theta|^2 + a^2|\theta|^2)\,dz$$

$$= Ra^2 \int_0^1 \gamma(z)\,|\theta|^2\,dz. \qquad ...(36)$$

Hence if $R < 0$, then we must have

$$\sigma_r^2 + \sigma_i^2 > a^4/P_r. \qquad ...(37)$$

Hence the complex growth-rate of an arbitrary oscillatory mode (in the case of a layer heated from above in a variable gravitational field directed downward) must lie outside of a circle whose radius depends on the wavelength of the mode as well as the Prandtl number of the fluid (but not on the Rayleigh number of the configuration).

6. The Problem of a Layer with Free Boundaries

We solve the characteristic value problem consisting of (17), (18) and (21) for free upper and lower boundaries for the case of a linearly varying gravity field

$$\gamma(z) = 1 + Mz > 0. \qquad ...(38)$$

Since the principle of exchange of stabilities is true in this case and the marginal state is stationary, the equations to be solved are

$$(D^2 - a^2)^2 w = -Ra^2(1 + Mz)\theta \qquad ...(39)$$

and

$$(D^2 - a^2)\theta = w \qquad ...(40)$$

together with the boundary conditions

$$w = D^2 w = \theta = 0, \ (z = 0, 1). \qquad ...(41)$$

Eliminating w from eqns. (39) and (40) we get

$$(D^2 - a^2)^3\theta = -Ra^2(1 + Mz)\theta. \qquad ...(42)$$

The kinematic boundary condition can be expressed in terms of θ by eqn. (24). In view of (39) and (40) this gives

$$D^6\theta = D^8\theta = D^4\theta = \ldots = D^{2n}\theta = 0, \ (z = 0, 1). \qquad ...(43)$$

So θ must satisfy

$$\theta = D^2\theta = D^4\theta = \ldots = D^{2n}\theta = 0, \ (z = 0, 1). \qquad ...(44)$$

We have to solve eqn. (42) with the boundary condition (44). Rewriting eqn. (42) in the manner

$$(D^2 - a^2)^3 \theta = (1 + Mz) \psi \qquad ...(45)$$

and

$$\psi = - Ra^2 \theta \qquad ...(46)$$

we expand ψ and θ in the forms

$$\psi = \sum_{m=1}^{\infty} A_m \sin m \pi z \text{ and } \theta = \sum_{m=1}^{\infty} \frac{-A_m \sin m\pi z}{Ra^2} \qquad ...(47)$$

where θ is the solution of the equation

$$(D^2 - a^2)^3 \theta = (1 + Mz) \sum_{m=1}^{\infty} A_m \sin m \pi z \qquad ...(48)$$

which satisfies the boundary conditions (44).

Inserting the expansions (47) in eqn. (46) and multiplying the resulting equation by sin $n\pi z$ and integrating from 0 to 1 gives a set of homogeneous linear algebraic equations in the A_m. The requirement that the constants A_m in the resulting algebraic equations are not all zero leads to the secular equation. An approximation to the solution can be obtained by setting the determinant formed by the elements in the first n-rows and n-columns of the secular determinant equal to zero, successive approximations being obtained as n increases. The calculation can be done for various values of M, in each case the value of R which corresponds to a particular value of a being found by solving an algebraic equation. The minimum value of R with respect to a can then be found out for each M.

A first approximation to the solution of the secular equation is obtained by setting the (1, 1)-element of the matrix equal to zero and ignoring all the others. This corresponds to the choice of sin πz as a trial function for θ. As Chandrasekhar[1,2] has shown, very good approximation are obtained for $n = 1 = m$ and the solution converges rapidly for increasing values of m and n. Setting the (1, 1)-element of the secular determinant equal to zero, we find that

$$\tfrac{1}{2} (\pi^2 + a^2) \frac{1}{Ra^2} = \tfrac{1}{4} M + \tfrac{1}{2}. \qquad ...(49)$$

This gives

$$R = \frac{2}{2 + M} \frac{(\pi^2 + a^2)^3}{a^2}. \qquad ...(50)$$

We can get a better approximate solution by taking the first two terms in the expansion for θ and w. In this case, the secular determinant reduces to the form

$$\begin{vmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{vmatrix} = 0 \qquad ...(51)$$

where

$$S_{11} = -\frac{(\pi^2 + a^2)^3}{2Ra^2} + \tfrac{1}{4} M + \tfrac{1}{2}$$

$$S_{12} = -\frac{8M}{\pi^2 + a^2}\left\{\frac{a^2 + 13\pi^2}{9\pi^2} + \frac{3\pi^2 (2a^4 + 8a^2 \pi^2 + 9\pi^4)}{(a^2 + 4\pi^2)(a^2 + \pi^2)^2}\right\}$$

$$S_{21} = \frac{8M (8\pi^2 - a^2)}{9\pi^2 (\pi^2 + a^2)} + \frac{M\pi^2 (\pi^2 + a^2)}{a^2 (4\pi^2 + a^2)^2}\left[\frac{(4\pi^2 - 3a^2)}{4\pi^2 + a^2}\right.$$

$$\left.\{1 - \cosh a + \sinh a\,(1 + \cosh a)\} + \frac{4a\,(1 + \cosh a)}{\sinh a}\right]$$

$$S_{22} = -\frac{(4\pi^2 + a^2)^3}{2Ra^2} + \tfrac{1}{4} M + \tfrac{1}{2}. \qquad \text{...(52)}$$

Equation (51) is quadratic in R. The minimum positive value of R with respect to a for each M can be calculated from this equation.

7. Discussion

Numerical results summarized in Figs. 1—3 were obtained by evaluating (51). For comparison purposes, separate curves (dotted ones) have been drawn using (50). It is seen that a variable gravitational field with gravity increasing linearly upward has a destabiling effect (Fig. 1), the neutral stability curves shifting progressively downward

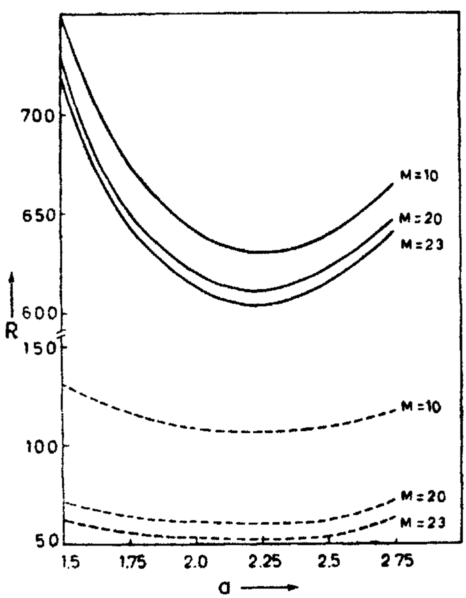

Fig. 1. Neural stability curves for different values of the gravity parameter.

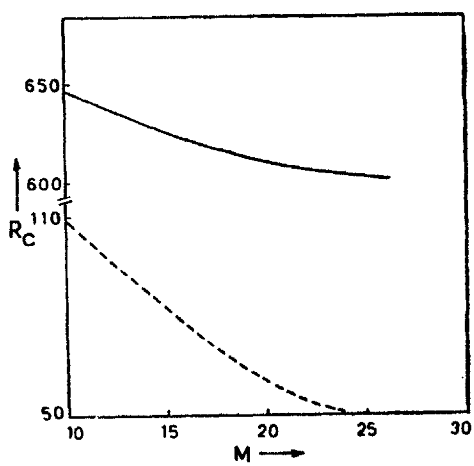

Fig. 2 Behaviour of the critical Rayleigh number with increase in gravity parameter.

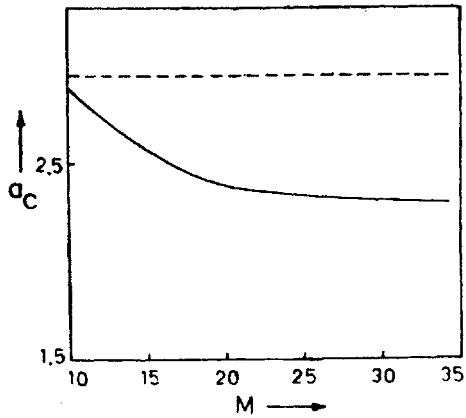

FIG. 3 Behaviour of critical wave number a_c with increase in gravity parameters.

as the gravity parameter increases. The monotonic increase in the destabilizing effect with increase in the gravity parameter is further brought out in Fig. 2. Figure 3 shows that the critical wave-number decreases monotonically with increase in the gravity parameter.

We expect similar behaviour for other boundary conditions.

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