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COSET DIAGRAMS FOR AN ACTION OF THE EXTENDED MODULAR GROUP ON THE PROJECTIVE LINE OVER A FINITE FIELD

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Higman has defined coset diagrams for the actions of $PGL(2, Z)$ on the projective line over a finite field F_q, denoted by $PL(F_q)$, where q is a prime power. A condition for the existence of a certain fragment of a coset diagram in a coset diagram for an action of $PGL(2, Z)$ on $PL(F_q)$ is known now: the condition is a polynomial $Z[z]$. There are special types of fragments of coset diagrams which occur quite frequently in certain coset diagrams. In this paper we have found conditions for their existence in coset diagrams representing these actions.

1. INTRODUCTION

If q is a power of a prime p then by $PL(F_q)$ we shall mean the projective line over the Galois field F_q. That is, $PL(F_q) = F_q \cup \{\infty\}$. The group $PGL(2, q)$ has its customary meaning, as the group of all transformations $z \to (az + b)/(cz + d)$ where a, b, c, d are in F_q and $ad - bc \neq 0$, while the group $PSL(2, q)$ is its subgroup consisting of all those where $ad - bc$ is a non-zero square in F_q. Mushtaq[4] it has shown that corresponding to each θ in F_q there exists a coset diagram which represents an action of the modular group $PSL(2, Z) = \langle x, y : x^2 = y^3 = 1 \rangle$ or the extended modular group $PGL(2, Z) = \langle x, y, t : x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1 \rangle$ on $PL(F_q)$. Consideration of these actions shows the importance of circuits contained in the coset diagrams representing these actions[5]. In this paper we have found values of θ and q, for the occurence of a certain circuit in the corresponding coset diagram.

2. COSET DIAGRAMS FOR $PGL(2, Z)$

The coset diagrams considered in this paper are defined for $PGL(2, Z)$ (see for details Conder[1, 3] and Mushtaq[4,5]). The three cycles of y are denoted by small triangles whose vertices are permuted counter-clockwise by y and any two vertices which are interchanged by x are joined by an edge. The action of t is given by a reflection in a vertical axis of symmetry. The fixed points of x and y are denoted by heavy dots. (Notice $(yt)^2 = 1$ is equivalent to $tyt = y^{-1}$, which means that t reverses the orientation of the triangles representing the three cycles of y (as reflection does); because of this, there is no need to make the diagram more complicated by introducing t-edges.)

The group $PGL(2, q)$ has a natural permutation representation on $PL(F_q)$ and therefore any non-degenerate homomorphism α of $PGL(2, Z)$ to $PGL(2, q)$ gives rise to an action of $PGL(2, Z)$ on $PL(F_q)$. Two such homomorphisms α and β are called conjugate if $\beta = \alpha\rho$ for some inner automorphism ρ of $PGL(2, q)$. We denote $x\alpha$, $y\alpha$ and $t\alpha$ respectively by $\bar{x}, \bar{y}, \bar{t}$. The actions corresponding to conjugate non-degenerate homomorphisms α and β (see Mushtaq[6] for definitions) will produce the same coset diagrams, except for the labelling of the vertices. (The vertices for one can always be re-labelled according to the action of ρ or ρ^{-1} in order to obtain the other).

It has been shown by Mushtaq[4], that the conjugacy classes of non-degenerate homomorphisms of $PGL(2, Z)$ into $PGL(2, q)$ correspond in a one-to-one fashion with the conjugacy classes of non-trivial elements of $PGL(2, q)$, under a correspondence which assigns to the homomorphism α the class containing the element $\bar{x}\,\bar{y}$.

Since it has been proved by Mushtaq[4] that there is a one-to-one correspondence between the conjugacy classes of elements of order greater than 2 in $PGL(2, q)$ and the non-zero elements of F_q, such that the class corresponding to an element θ in F_q consists of all elements represented by matrices M with $\theta = r^2/\Delta$ where $r = \text{trace}(M)$ and $\Delta = \det(M)$, it follows that we can actually parameterize the non-degenerate homomorphisms of $PGL(2, Z)$ into $PGL(2, q)$, exccpt for a few uninteresting ones, by the elements of F_q. If α is any such homomorphism, and X, Y and T are in $GL(2, q)$, which yield the elements $\bar{x}, \bar{y}, \bar{t}$ then letting $\theta = r^2/\Delta$ (where $r = \text{trace}(XY)$, $\Delta = \det(XY)$), we associate the parameter θ with the homomorphism α.

3. Occurrence of Circuits in $D(\theta, q)$

By a circuit (in a coset diagram for an action of $PGL(2, Z)$ on $PL(F_q)$) we shall means a closed path of triangles and edges. Coset diagrams arising from non-degenerate actions of $PGL(2, Z)$ on $PL(F_q)$ may be thought of as being composed of fragments, these fragements themselves being composed of single circuits, or a number of circuits[5]. Let us denote by $D(\theta, q)$ the diagram for a non-degenerate homomorphism of $PGL(2, Z)$ into $PGL(2, Z)$ with parameter θ. In order to know which class (es) a particular diagram comes from we need to consider this question: given a fragment, for which values of q and θ can that fragment be found in $D(\theta, q)$? The question has been answered by Mushtaq[5] for the fragments containing two or more intersecting circuits. In the following, we have considered the above question for single circuits.

Note that if v is a vertex of a triangle on a circuit in $D(\theta, q)$ then the circuit induces an element $\bar{g}$ of $PGL(2, q)$ fixing v. The element $\bar{g} \neq 1$ is conjugate either to $\bar{x}$ or to $\bar{y}^{\pm 1}$ or to an element of the form $\bar{x}\bar{y}^{\epsilon_1}\bar{x}\bar{y}^{\epsilon_2} \ldots \bar{x}\bar{y}^{\epsilon_k}$, $k > 1$ and $\epsilon_i = \pm 1$. The expression is unique up to cyclic permutations of $(\epsilon_1, \epsilon_2, \ldots \epsilon_k)$. If $\bar{g} = \bar{h}^{k'}$ for some $k' \geqslant 1$ then k' divides k and the cycle consists of $(\epsilon_1, \epsilon_2, \ldots, \epsilon_{k/k'})$ taken k'-times. It means that for any such $\bar{g}$ there is greatest integer k' such that $\bar{g}$ is a k'th power in $PGL(2, q)$. We say that $\bar{g}$ is a proper power if $k' > 1$.

Suppose $D(\theta, q)$ contains circuits which have more than two fixed vertices of the elements conjugate to $\bar{h}$ where $\bar{h}^k \neq 1$. Since the only element of $PGL(2, q)$ with more than two fixed vertices is the identity element, $\bar{g} = 1$. If $\bar{h} = \bar{x}\bar{y}$ the diagram will correspond to an action of $\Delta(2, 3, k) = < x, y: x^2 = y^3 = (x\,y)^k = 1 >$ on $PL(F_q)$. In the following theorems, we consider the coset diagrams which correspond to the actions of

$< x, y: x^2 = y^3 = xy^{\epsilon_1} xy^{\epsilon_2} \dots xy^{\epsilon_k} = 1, k > 1 >$ on $PL(F_q)$. Mushtaq[4] it has shown that in $D(\theta, q)$, there do not exist circuits which contain fixed points of $\bar{g} = (\bar{h})^k$ where $\bar{h} = (\bar{x}y^{-1})^{n_1} (\bar{x}\bar{y}^{-1})^{n_2} \dots (\bar{x}\bar{y}^{\,1})^{n_k}$, $k > 1$. We call such circuits periodic. A circuit which is not of this type is called a non-periodic circuit.

We are now going to consider the following question. Given a non-periodic circuit, for what values of q, θ can this circuit be found in the corresponding diagram, representing an action of $PGL(2, Z)$ on $PL(F_q)$? We have seen that $\bar{g}$ in $PGL(2, q)$ is either a proper power or it is not. The case where $\bar{g}$ is a proper power has already been considered by Mushtaq[6]. In the following Theorem we therefore deal with the case where $\bar{g}$ is not a proper power. For the meaning of homomorphic image of a circuit and how a non-periodic circuit can have a homomorphic image, we may refer to Mushtaq[5].

Theorem 3.1—Let C be a non-periodic circuit. Then there exists a polynomial f, with integer coefficients, such that if C occurs in $D(\theta, q)$ then $f(\theta)$ is a square in F_q, and if $f(\theta)$ is a square in F_q then either C, or a homomorphic image of it, occurs in $D(\theta, q)$.

PROOF : Let $\bar{g} = \bar{x}\bar{y}^{\epsilon_1}\ \bar{x}\bar{y}^{\epsilon_2} \dots \bar{x}\bar{y}^{\epsilon_k}$, where $\epsilon_i = \pm 1$ and $i = 1, 2, \dots, k$, be an element of $PGL(2, q)$ fixing a particular vertex v on the circuit C. The matrices X, Y in $GL(2, q)$, yielding the elements $\bar{x}, \bar{y}$, induce the matrix $M = XY^{\epsilon_1}\ XY^{\epsilon_2} \dots XY^{\epsilon_k}$ where $\epsilon_i = \pm 1$ and $i = 1, 2, \dots, k$.

Since $\bar{x}^2 = \bar{y}^3 = 1$, the matrices X and Y can be taken as the matrices with $\det(X) = \Delta$, trace $(X) = 0$ and $\det(Y) = 1$, trace $(Y) = -1$. Thus, as in a paper by Mushtaq[5] characteristic equations of X, XY and Y will be

$$X^2 + \Delta I = 0 \qquad \dots(1)$$

$$(XY)^2 - r(XY) + \Delta I = 0 \qquad \dots(2)$$

$$Y^2 + Y + I = 0 \qquad \dots(3)$$

where the trace of XY is r. Since $\det(XY) = \Delta$, the determinant of M will be Δ^k. Using equation (1), (2) and (3) the matrix M can be expressed as $\lambda_0 1 + \lambda_1 X + \lambda_2 Y + \lambda_3 XY$, where λ_i is a polynomial in r and Δ. So the trace of will be the trace of $M\ \lambda_0 1 + \lambda_1 X + \lambda_2 Y + \lambda_3 XY$. That is, trace $(M) = 2\lambda_0 - \lambda_2 + \lambda_3 r$. So the characteristic equation of M will be

$$M^2 - (2\lambda_0 - \lambda_2 + \lambda_3 r)\,M + \Delta^k I = 0. \qquad \dots(4)$$

The discriminant $(2\lambda_0 + \lambda_2 + \lambda_3 r)^2 - 4\,\Delta^k$ of the characteristic equation of M is a polynomial in r and Δ. It can be seen (by induction on k), that if we regard r as of degree 1, and Δ as of degree 2, then this polynomial is homogeneous of degree $2k$. It follows that for a suitable $h(\theta)$, the discriminant is $h(\theta)\,\Delta^k$.

Now $\bar{g}$ has a fixed vertex in $D(\theta, q)$ if the characteristic equation of M has roots in F_q. This means that $\bar{g}$ has a fixed vertex in $PL(F_q)$, if the discriminant $h(\theta)\,\Delta^k$ is a square in F_q. Since Δ is a square if and only if θ is, we let $f(\theta) = h(\theta)$ if k is even and $f(\theta) = h(\theta)\,\theta$ if k is odd and obtain the result in the above theorem.

The preceding theorem has the following interesting corollary.

Corollary 3.2—Let C be circuit such that it contains a vertex which is fixed by $\bar{x}\bar{y}^{-1}\bar{x}\bar{y}$. Then C, or a homomorphic image of it occurs in $D(\theta, q)$ if and only if $\theta^2 - 2\theta - 3$ is a square in F_q.

PROOF : Consider the vertex v (on the circuit) fixed by $\bar{x}\bar{y}^{-1}\bar{x}\bar{y}$. Let $\bar{g} = \bar{x}\bar{y}^{-1}\bar{x}\bar{y}$, then by the direct application of Theorem 3.1, the characteristic equation of the matrix M, corresponding to $\bar{g}$, will be

$$M^2 - (-r^2 + \Delta)\,M + \Delta I = 0. \qquad \ldots(5)$$

Substituting $r^2 = \theta\Delta$ in the discriminant of (5) we get $f(\theta) = \theta^2 - 2\theta - 3$. Thus the circuit C, or its homomorphic image, exists in $D(\theta, q)$ if and only if $f(\theta)$ is a square in F_q.

Note that the circuit C, which contains a vertex v fixed by $\bar{x}\bar{y}^{-1}\bar{x}\bar{y}$, will be as follows

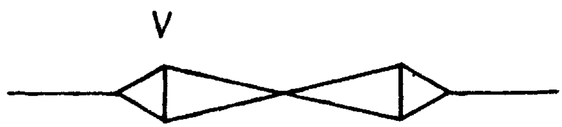

FIG. 1.

Remarks 3.3 : (i) The degree of the polynomial $f(\theta)$ will be k or $k + 1$, where k is the number of triangles on the circuit C.

(ii) According to Theorem 3.1, the circuit (Fig. 2) occurs in $D(\theta, q)$ if and only if -3 is a square in F_q.

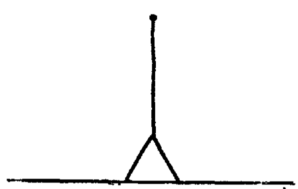

FIG. 2.

(iii) The circuit (Fig. 3) occurs in $D(\theta, q)$ if and only if $-\theta$ is a square in F_q. This is an immediate consequence of Theorem 3.1 and the fact that discriminant in (1) is equal to -4Δ, $r^2 = \Delta\theta$ and

FIG. 3.

(iv) The elements $\bar{x}$, $\bar{t}$ generate a 4-group. In characteristic 2, the only irreducible linear representation of the 4-group is the trivial representation and so in any projective representation there is a fixed vertex. This means $\bar{x}$ and $\bar{t}$ must have a common vertex. Thus a circuit (Fig. 4) falls on the vertical axis of symmetry of $D(\theta, q)$ if and only if q is a power of 2.

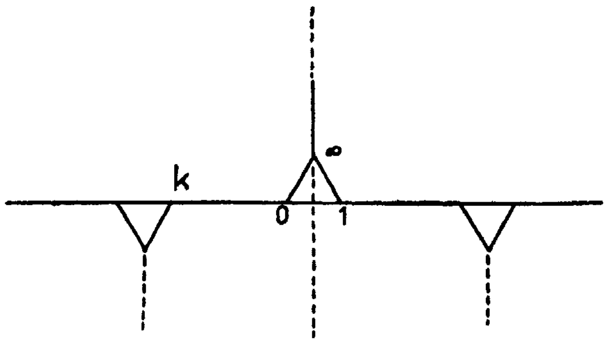

FIG. 4.

(v) If q is a power of 3, then $\bar{y}$ has a unique fixed vertex because the number of vertices in $PL(F_q)$ is $3^k + 1$; $\bar{t}$ normalizes $<\bar{y}>$ and so $\bar{t}$ must also fix this vertex. Thus the circuit (Fig. 5) falls on the vertical axis of symmetry of $D(\theta, q)$.

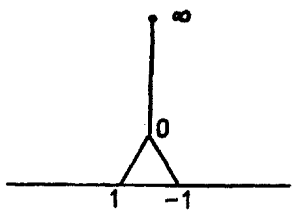

FIG. 5.

Corollary 3.4—A circuit, containing a vertex fixed by $\bar{x}\,\bar{y}$, will exist in $D(\theta, q)$ if and only if $\theta(\theta - 4)$ is a square in F_q.

PROOF : The proof is an easy consequence of Theorem 3.1.

Note that the circuit which contains a fixed point of $\bar{x}\,\bar{y}$ will be (see Fig. 6)

FIG. 6.

4. CIRCUITS WHEN THE DISCRIMINANT IS ZERO

An obvious question which arises here is, what happens when the discriminant of eqn. (4) is equal to zero? Exactly what this means depends upon the circuit.

Let C be a circuit such that a vertex in it is fixed by an element $\bar{g}$ of $PGL\ (2, q)$. The characteristic equation of matrix corresponding to $\bar{g}$ will, of course, have equal eigen values if the discriminant of the characteristic equation is equal to zero. This means that $\bar{g}$ will have just one fixed vertex in $D\,(\theta, q)$, but the exact type will depend upon the circuit concerned.

For instance, assume that the homomorphic image of the circuit (Fig. 7) occurs

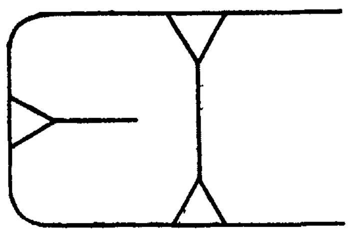

FIG. 7.

in the coset diagram $D\ (\theta, q)$. Since $D\ (\theta, q)$ admits the axis of symmetry, the image of the circuit under the permutation $\bar{t}$, will also occur. The vertices v and $v\ \bar{t}\ \bar{y}^{-1}$ on

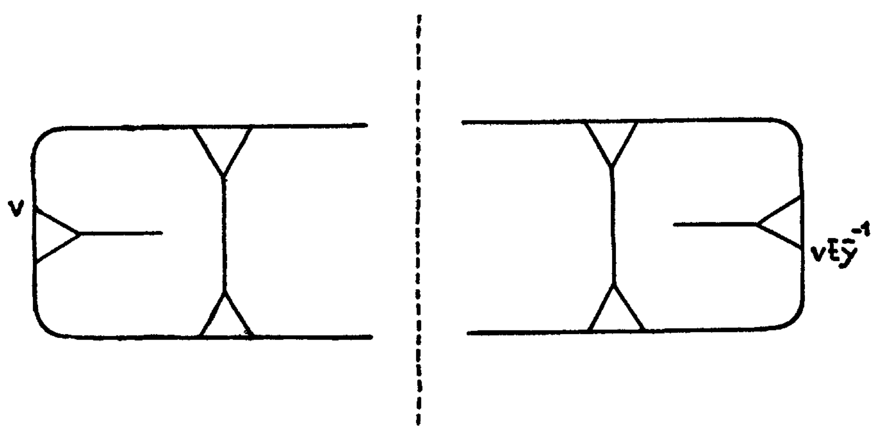

FIG. 8.

the circuits (Fig. 8) are both fixed by $\bar{g}$, where $\bar{g} = \bar{x}\,y\,\bar{x}\,y\,\bar{x}\,y^{-1}$. So if the discriminant of the characteristic equation of the matrix corresponding to $\bar{g}$ is equal to zero then $v = v\,\bar{t}\,y^{-1}$. This means that the circuit, which has a symmatry, lies on the vertical axis of symmetry Diagrammatically it means, (Fig. 9).

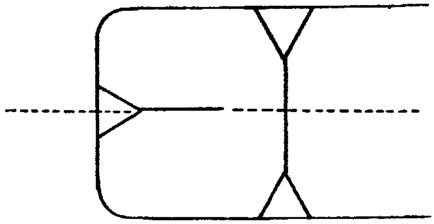

FIG. 9.

Consider another example in which the vertices v_1 and v_2 on the circuit (Fig. 10) are fixed points of $\bar{g}$, where $\bar{g} = \bar{x}\,y\,\bar{x}\,y\,\bar{x}\,y^{-1}\,\bar{x}\,y^{-1}$. In this case when the discriminant of the characteristic equation of the matrix corresponding to $\bar{g}$ is zero we must,

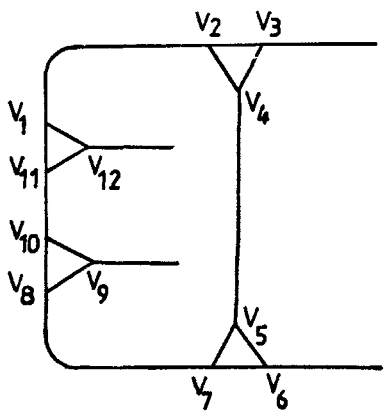

FIG. 10.

get a homomorphic image of the circuit in which v_1 and v_2 are identified, say with $v\,\bar{x} = v$, and so a homomorphic image of the circuit will be (Fig. 11) and it will be symmetrical about the vertical line of axis because $(\bar{x}\,y)^2 = 1$, that is because $v_2 = v_0\,\bar{t}$.

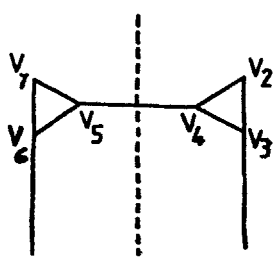

FIG. 11.

If the discriminant $(\theta - 4)\,\Delta$ of the characteristic equation in Corollary 3.4 is equal to zero then $\theta = 4$ because Δ, being the determinant of the non-singular matrix XY, cannot be zero. This implies that the characteristic equation will have equal roots and so there will be only one vertex, namely ∞, in $D(\theta, q)$ fixed by $\bar{x}$. Since the action of t represent reflection about the vertical line of symmetry, the circuit (Fig. 12) will, in this case, lie on the vertical axis of symmetry.

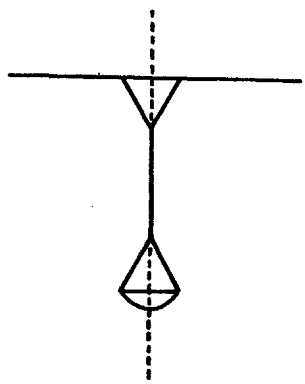

FIG. 12.

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