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THE EXTENDED MODULAR GROUP ACTING ON THE PROJECTIVE LINE OVER A GALOIS FIELD

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In his Oxford seminars, G. Higman considered actions of $\Delta(2, 3, 7) = < x, y : x^2 = y^3 = (xy)^7 = 1 >$ on the projective line over F_p (denoted by $PL(F_p)$) in the case when p is a prime (congruent to ± 1 mod 7). From the coset diagrams representing the natural action of $\Delta(2, 3, 7)$ on $PL(F_p)$ something of a pattern emerged: for each prime there were three diagrams, but just one of these three had an axis of symmetry passing through two vertices. It was easy to identify the permutation t, induced by the symmetry about this axis, as being odd or even according to the value of $p \equiv \pm 1$ (mod 4), and then correspondingly, the group $< x, y, t >$ was either $PGL(2, p)$ or just $PSL(2, p)$. In this paper we show that this pattern is not repeated for all values of p.

Let $\Delta(2, 3, 7)$ denote the abstract group with presentation $< x, y : x^2 = y^3 = (xy)^7 = 1 >$. It is well known that $PSL(2, Z)$ has the presentation $< x, y : x^2 = y^3 = 1 >$. Let p be a prime and F_p denote a finite field of order p. We use the notation $GL(2, p)$, $SL(2, p)$, $PSL(2, p)$ and $PGL(2, p)$ with its standard meaning.

Let $PL(F_p)$ denote the projective line over a finite field F_p. The points of $PL(F_p)$ are the elements of F_p together with the additional point ∞. It is well known that the extended modular group $PGL(2, Z)$ has the presentation

$$< x, y, t : x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1 >$$

and the modular group $PSL(2, Z)$ is of index 2 in $PGL(2, Z)$. The coset diagrams for the natural action of $PGL(2, Z)$ on $PL(F_p)$ are defined as follows.

The three cycles of y are denoted by triangles whose vertices are permuted anti-clockwise by y. Any two vertices which are interchanged by x are joined by an edge. The fixed points of y are denoted by heavy dots and the action of t is given by reflection in a vertical axis of symmetry.

For instance, the following diagram depicts a transitive action of $PGL(2, Z)$ on $PL(F_{13})$, in which

x acts as $(0, 11)(1, 12)(2, 7)(3)(4, 8)(5, 10)(6, \infty)(9)$

y acts as $(0)(\infty)(1, 9, 3)(2, 5, 6)(4, 10, 12)(7, 11, 8)$ and

t acts as $(0, \infty)(2, 7)(3, 9)(4, 10)(5, 8)(1)(12)$.

We have labelled each vertex to give a fuller illustration (Fig. 1).

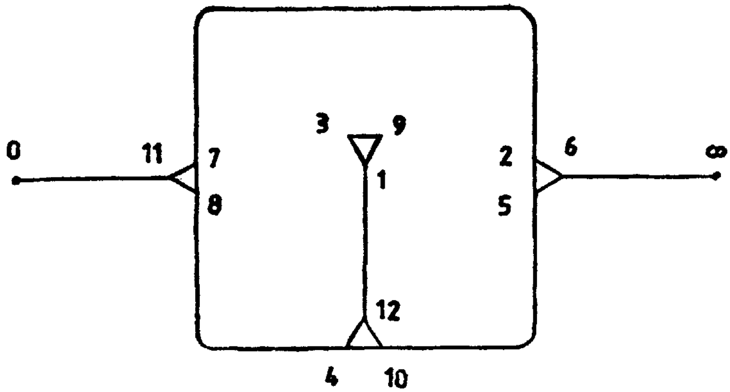

FIG. 1.

Notice that this diagram is symmetric : the permutation $t = (0, \infty)(2, 7)(3, 9)(4, 10)(5, 8)(1)(12)$ gives a reflection about a vertical axis. Also this permutation is the same as that induced by the transformation t which is an element of $PGL(2, 13)$ and it is easy to verify geometrically that the relations $t^2 = (xt)^2 = (yt)^2 = 1$ are satisfied.

Indeed in any case, if (x, y) is a $(2, 3, 7)$—generating pair for $PSL(2, p)$, then by the proof of Theorem 3 of Singerman[5], there has to be an automorphism σ of $PSL(2, p)$ such that $x\sigma = x^{-1}$ and $y\sigma = y^{-1}$. In terms of the associated coset diagram, this means there must be an axis of symmetry, with σ representable as a reflection about the axis. Hence all our diagrams for the groups $PSL(2, p)$ will be symmetric. It is important to note that, there are certain coset diagrams (e.g., the coset diagram for $PGL(2, 25)$) which admit more than one such symmetry. For details one can refer to Mushtaq[4]. Note that we can form the semi-direct product G of $PSL(2, p)$ by the cyclic group $<\sigma>$ and in this group the relations $\sigma^2 = (x\sigma)^2 = (y\sigma)^2 = 1$ are satisfied. It is worthy of mention that in Conder[1] has used these coset diagrams to show that all but a small number of alternating groups A_n are homomorphic images of the triangle group $\Delta(2, 3, 7)$.

Earlier the author[3] has obtained a parametrization of all homomorphisms α from the extended modular group into $PGL(2, q)$ (where q is a prime-power), via non-zero elements of F_q. If neither of the generators x and y for the modular group lies in the kernel of α, so that their images $x\alpha$ and $y\alpha$ are of orders 2 and 3, respectively, then α is said to be a non-degenerate homomorphism. Two such homomorphisms α and β are called conjugute if $\beta = \alpha\rho$ for some inner automorphism ρ of $PGL(2, q)$. In this case the actions corresponding to α and β will produce the same coset diagrams, except for the labelling of the vertices. Thus, it has been shown[3] that corresponding to each

θ in F_q there exists a coset diagram $D(\theta, q)$ which represents the conjugacy class of non-degenerate homomorphisms α.

In the present case, since q is taken to be a prime $p \equiv \pm 1 \pmod 7$, according to Macbeath[2], there are three distinct traces $\lambda_1, \lambda_2, \lambda_3$ of elements of the group $SL(2, p)$ that yield elements of order 7 in $PSL(2, p)$ and thus, corresponding to $\theta_1 = \lambda_1^2$, $\theta_2 = \lambda_2^2$ and $\theta_3 = \lambda_3^2$, there are three conjugacy classes of non-degenerate homomorphisms from $\Delta(2, 3, 7)$ into $PGL(2, q)$. This means that there are three coset diagrams $D(\theta_1, p)$, $D(\theta_2, p)$ and $D(\theta_3, p)$ corresponding to the three conjugacy classes[3].

It is important to note that in this case, every element of $PSL(2, p)$ that comes from an element of $SL(2, p)$ with trace λ_1, λ_2 or λ_3 must have order 7. (Indeed except when the trace is ± 2, the trace of any element of $SL(2, p)$ determines its order).

G. Higman, in his Oxford seminars, considered conjugacy classes of non-degenerate homomophisms from $\Delta(2, 3, 7)$ into $PGL(2, p)$ where $p \equiv \pm 1 \pmod 7$. According to Macbeath[2], for each such prime p there are three conjugacy classes. Higman considered coset diagrams corresponding to each of these conjugacy classes. From the coset diagrams he produced in the cases with $p = 13, 29, 41, 43$ and 71, something of a pattern emerged : for each prime there were three diagrams, but just one of these

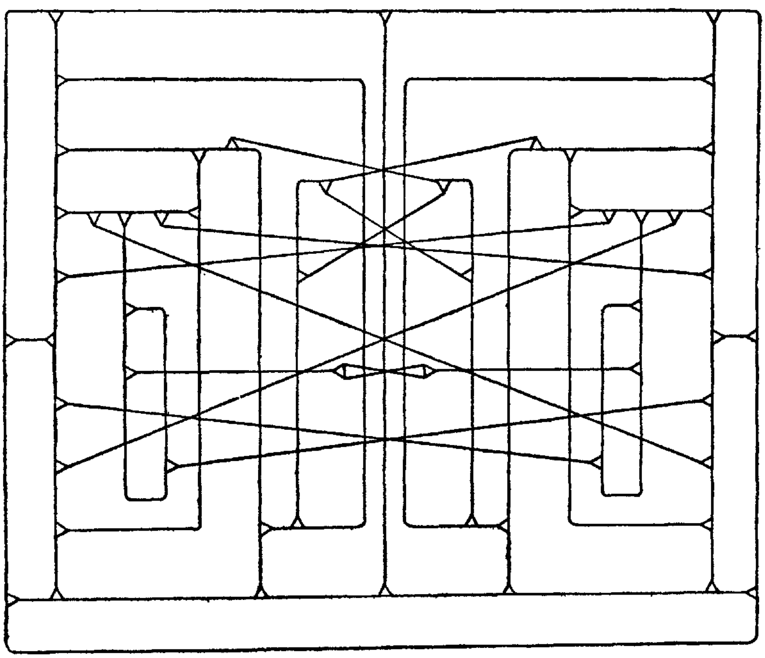

FIG. 2.

three had an axis of symmetry passing through two vertices. In other words, in one diagram the transformation t fixed two points of the projective line over F_p, while in the other two diagrams t fixed no points. It was easy to identify the permutation induced by t as being either odd or even, according to the value of p (mod 4), and then correspondingly, the group $<x, y, t>$ was either $PGL\ (2, p)$ or just $PSL\ (2, p)$. Unfortunately this pattern is not repeated for all longer values of p, as we shall see in the following example.

Following the method of Mushtaq[3], we have drawn three coset diagrams $D\ (\theta_1, p)$, $D\ (\theta_2, p)$, $D\ (\theta_3, p)$ for each value of $p = 7, 13, 29, 41, 43, 71, 83, 97, 113,$ $127, 139$ and 167. We have seen that $p = 167$ is the first case in which the three coset diagrams, namely, $D\ (21, 167)$, $D\ (27, 167)$ and $D\ (124, 167)$ are such that every vertex in them is fixed by $\{(xy)\ \alpha\}^7$ and the three diagrams contain an axis of symmetry passing through two vertices. Since no non-trivial linear-fractional transformation fixes more than two vertices of $PL\ (F_q)$, we have $\{(xy)\ \alpha\}^7 = 1$.

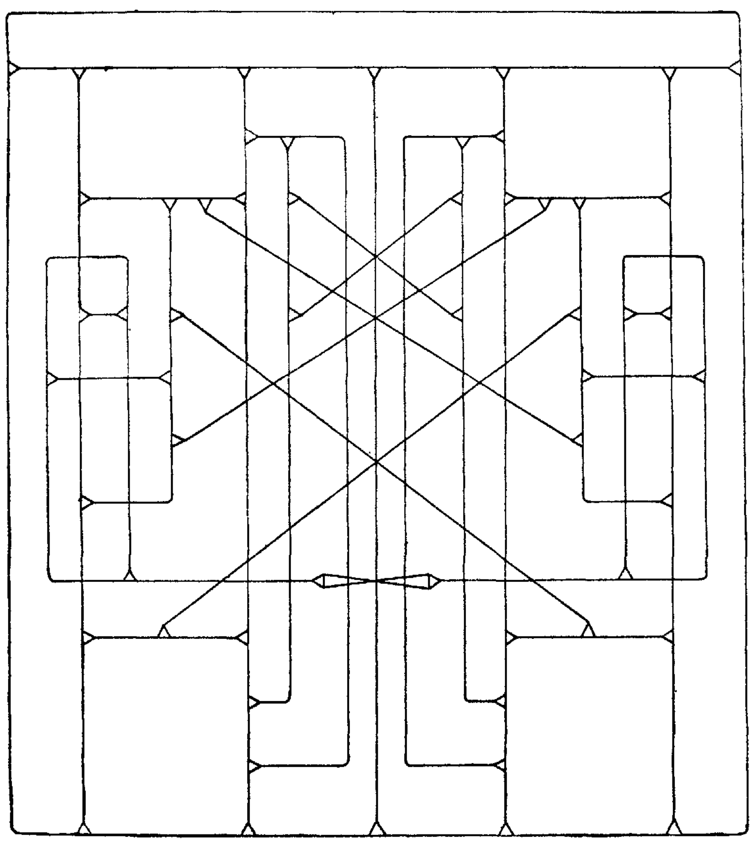

FIG. 3.

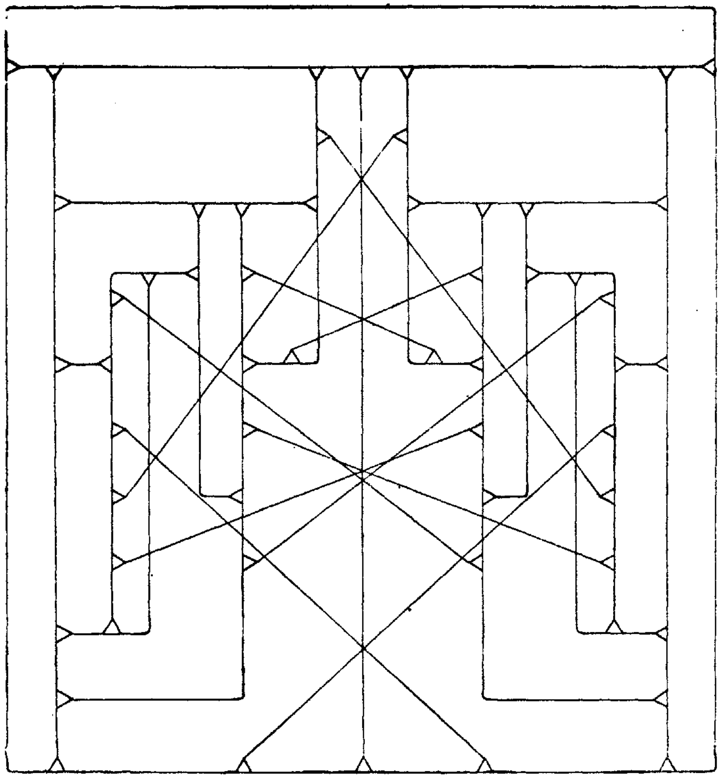

FIG. 4.

In other words, in the three diagrams, $\{(xy)\,\alpha\}^7 = 1$ and the permutation t fixes two points of $PL\,(F_{167})$. It is quite easy to see that in this case t is an odd permutation on $PL\,(F_{167})$ and as a consequence $t \in PGL\,(2,\,167) \setminus PSL\,(2,\,167)$.

We note that the three equations $\theta - 21 = 0$, $\theta - 27 = 0$ and $\theta - 124 = 0$, when multiplied together, yield the equation $f(\theta) = \theta^3 - 5\theta^2 + 6\theta - 1 = 0$. Notice that this is the same equation as discussed in Mushtaq[3].

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