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ON THE EXISTENCE FOR A CLASS OF OPTIMAL CONTROL PROBLEMS

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A uniform theory for a class of optimal control problems governed by linear plants with generalized constraints on the control variable is developed. The results obtained by the method of functional analysis provide new insight. Some necessary and sufficient conditions are derived to obtain explicit expression for the optimal control function.

Krasovskii[9] applied methods of functional analysis to solve a class of control problems. Kranc and Sarachik[8] and Kreindler[10] solved a class of optimization problems with various types of constraints. Hermes and Lasalle[6] also considered time optimal control problem from functional analysis point of view for amplitude constraints. Chaudhuri and Mukherjee[2] also developed a uniform theory of time optimal control problems with generalized constraints. They[3] also discussed the global controllability. Porter[13] demonstrated as to how the function space approach could be utilised to obtain the optimum control for a wide class of minimum norm controls. Minamide and Nakamura[11] generalised the minimum cost problem of Porter. Burns[1] also considered the minimum effort problem and minimum cost problem in the Banach space setting.

The main purpose of this paper is to generalize the idea of Porter in locally convex linear topological space, so that the more generalized constraints on the control variable can be treated. The explicit expression under the generalized constraints on the control variable is obtained.

Now, let L be a locally convex linear topological space and L^* be the conjugate space of all continuous linear functionals defined on L.

Let M^π be the polar set of $M \subseteq L$ in L^* (Yosida[14], p. 136). Then the boundary of M^π will be denoted by δM^π and is defined as follows :

$$\delta M^\pi = \{ f \in L^* : \sup_{l \in M} \{ | < l, f > | \} = 1 \}.$$

Again let E_π be the polar set of $E \subseteq L^*$ in $L \subset L^{**}$ (Yosida[14], p. 136). Then the boundary of E_π will be denoted by δE_π and is defined as follows :

$$\delta E_\pi = \{l \in L;\ \sup_{f \in E} \{ \mid < l, f > \mid \} = 1\}.$$

Also $E_\pi = E^\pi_{\pi\pi}$ for a non-void set $E \subseteq L^*$.

Here, we shall define the optimal control problem as follows : Let L and L' be locally convex linear topological spaces and T, a continuous linear transformation from L into L'. For each ξ in the range of T, we find an element $l \in L$ that satisfies $\xi = Tl$

while minimizing

$$J(l) = \sup_{f \in E} \{ \mid < l, f > \mid \}$$

where E is any subset of L^*.

To solve the above problem we shall consider E_π to be the polar set in L of any arbitrary bounded subset E of L^*. Furthermore we shall assume that

(i) E_π is bounded,

(ii) for every neighbourhood, say, U of zero in L, $T(U)$ contains some set that is of the second category in L' and that satisfies the condition of Baire.

Definition 1 (Reachable set)—The set of all points $\xi \in L'$ such that $\xi = Tl$ for some $l \in E_\pi$ will be called the reachable set with respect to the linear transformation T, and will be denoted by $C = T(E_\pi)$. Also, the boundary of C will be denoted by δc.

We characterize the reachable set of C as follows :

Theorem 1—The reachable set C is bounded, convex body and circled with respect to the origin in L'.

Proof : Since E_π is convex and T is linear and since linear operators preserve convexity, then the reachable set C is convex. Also, since T is continuous linear transformation from L into L' and so bounded sets map into bounded sets, then by the hypothesis (i) the reachable set C is bounded. Again, if λ is any scalar with $\mid \lambda \mid \leqslant 1$, then for $\xi \in C$ we shall show that $\lambda \xi \in C$. Now $\xi = Tl$ for some $l \in E_\pi$. Then $\lambda\xi = \lambda Tl = T(\lambda l) \in C$ for some $\lambda l \in E_\pi$ because $\sup_{f \in ECL^*} \{ \mid < \lambda l, f > \mid \} \leqslant \mid \lambda \mid \sup_{f \in ECL^*} \{ \mid < l, f > \mid \} \leqslant 1$. (Yosida[14], p. 136).

Thus C is circled. Now we see that the polar set E_π of any arbitrary bounded subset E of L^* is a neighbourhood of zero in the original topology of L and for that neighbourhood of zero E_π in L, $T(E_\pi)$ contains some set that is of the second category in L' and that satisfies the condition of Baire [by the hypothesis (ii)]. Therefore, from the theorem on openness[7] (p 93), T is an open mapping onto L'. It follows that the open set is mapped into open set and hence C contains zero. Thus the reachable set C is a convex set with non-void interior and thus C is a convex body.

Theorem 2—If $\xi \in \delta C$, then the pre-image of ξ, say, l_ξ belongs to the boundary of E_π which must satisfy

$$\sup_{f \in ECL^*} \{ | < l_\xi, f > | \} = 1.$$

PROOF : Let $\xi \in \delta C$ and let each pre-image of ξ belong to the interior of E_π for which $\sup_{f \in ECL^*} \{ | < l, f > | \} < 1$ (Yosida[14], p. 136). But, by hypothesis (ii) we can say that $C = T(E_\pi)$ contains some set that is of the second category in L' and that satisfies the condition of Baire for the neighbourhood E_π of zero in L. Therefore from theorem on openness[7] (p. 93) that T is an open mapping from L onto L' and so T maps open sets into open sets. Thus, it follows that the pre-image of ξ can not belong to the interior of E_π. Therefore, the pre-image of ξ, say, l_ξ belongs to the boundary of E_π which must satisfy $\sup_{f \in ECL^*} \{ | < l_\xi, f > | \} = 1$.

Now, if $l \in L$ maps into $\xi \in \delta C$ then

$$\sup_{f \in ECL^*} \{ | < l, f > | \} \geqslant 1.$$

Now the set of all pre-images of the vector $\xi \in L'$ will be denoted by the set inverse notation $T^{-1}(\xi)$. We shall show that existence of a minimum element of $T^{-1}(\xi)$ in the next theorem.

Theorem 3—Let $\xi \in \delta c$. Then $T^{-1}(\xi)$ has a minimum element if and only if $\xi \in c$.

PROOF : Let $\xi \in c \cap \delta c$. Then $T^{-1}(\xi)$ contains an element of E_π. Since $\xi \in \delta c$, it follows from Theorem 2 that this element must satisfy $\sup_{f \in ECL^*} \{ | < l, f > | \} = 1$. Conversely, we suppose that l_ξ is a minimum element of $T^{-1}(\xi)$. Because T is homogeneous, it is clear that αl_ξ is a minimum element of $T^{-1}(\alpha\xi)$ for any scalar $\alpha > 0$. Now, if $\alpha < 1$ then $\alpha\xi \in C$ and so $\alpha\xi$ has a pre-image which satisfy

$$\sup_{f \in ECL^*} \{ | < \alpha l_\xi, f > | \} \leqslant 1 \text{ (Yosida}^{14}\text{, p. 136).}$$

Since α is arbitrary, it follows that $\sup \{ | < l_\xi, f > | \} \leqslant 1$ (Yosida[14], p. 136). Thus, $l_\xi \in \delta E_\pi$ and $\xi = Tl_\xi \in C$.

Corollary 3.1—$T^{-1}(\xi)$ has a minimum element for each $\xi \in L'$ if and only if C is closed.

PROOF : If $T^{-1}(\xi)$ contains a minimum element for each $\xi \in L'$, then from Theorem 3 any $\xi \in \delta C$ implies $\xi \in C$. Therefore, $\xi \in C \cap \delta C$ implies that C is closed. Conversely we suppose that C is closed and p is the Minkowski functional of C. If $\xi \neq 0$ belongs to L' then $\xi \in p(\xi) C$ i.e., $\xi = p(\xi) x$ where $x \in C$ and so $x = p(\xi)^{-1} \xi \in C$. Also, $x \in \delta C$ because C is closed. Therefore $p(\xi)^{-1} \xi \in C \cap \delta C$. So, by Theorem 3, $p(\xi)^{-1} \xi$ has a minimum pre-image which implies that ξ has a minimum pre-image.

Corollary 3.2—If L is semi-reflexive, then the optimal control problem which has been described before can be solved for every continuous linear transformation T from L into L' and for prescribed hypothesis (i) and (ii).

PROOF : From the criterion for semi-reflexiveness it is known that each bounded weakly closed set is weakly compact[7] (p. 190). So, we assume that L is semi-reflexive. Then the polar set E_π of any arbitrary bounded subset E of L^* in L is weakly compact. Because E_π is closed and by the hypothesis (i) E_π is bounded. Again since T remains continuous when both L and L' are equipped with their weak topologies, then the continuous image of weakly compact set is weakly compact. Consequently the reachable set C is weakly compact and it is weakly closed and therefore strongly closed. Then the set C is closed. Therefore we can prove that the optimal control problem is solvable from Corollary 3.1.

Corollary 3.3—Let S be a continuous linear transformation from a locally convex linear topological space Y into another locally convex linear topological space X. Then the optimal control problem has a solution provided $S(Y)$ is closed when X is Hausdorff and the adjoint S^* of S is onto between X^* and Y^* i.e. for every $\xi \in L' = Y^*$ there is a pre-image $l_\xi \in L = X^*$ of ξ under $T = S^*$ with $\sup_{f \in ECL^*} \{ | < l_\xi, f > | \} = 1$.

PROOF : Since S is a continuous linear transformation of Y into X, then the adjoint S^* of S is also a continuous linear transformation from X^* into Y^*. Again since $S(Y)$ is closed when X is Hausdorff, it follows that S^* is an open mapping from X^* into Y^* (Kelley *et al.*[7], p. 204, Theorem 21.6). Also from hypothesis S^* is onto mapping. Furthermore, S^* is continuous when both X^* and Y^* are equipped with their weak* topology. Now it is known that the polar set M^π of $M \subset X$ where M is taken to be a convex balanced neighbourhood of zero in X, is weak* compact [application of Tychonov's theorem, (Yosida[14], p. 137)]. Therefore, we can conclude from the continuity of S^* that the reachable set $S^*(M^\pi)$ is closed and hence from Corollary 3.1 the optimal control problem is solvable for each $\xi \in L' = Y^*$ which has a minimum pre-image $l_\xi \in L = X^*$ with $\sup_{f \in ECL^*} \{ | < l_\xi, f > | \} = 1$ under the transformation $T = S^*$.

Now, applying application of Tychonov's theorem (Yosida[14], p. 137) and invoking Krein-Milman theorem on locally convex linear topological space (Yosida[14],

p. 362) we can infer that the polar set E_π^π in L^* of $E_\pi \subset L$ has at least one extreme point. Therefore, for every $0 \neq l \in L$ there exist at least one $f \in E_\pi^\pi \subset L^*$ such that $\sup_{l \in E_\pi CL} \{ | < l, f > | \} = 1$ and $f(l) = | < f, l > | = \sup_{f \in E_\pi^\pi CL^*} \{ | < f, l > | \}$.

Thus, such a vector $f \in E_\pi^\pi \subset L^*$ is called an extremal of $l \in L$.

Again, for every $0 \neq f \in L^*$ if there exists at least one $l \in E_\pi \subset L$ such that $f(l) = | < l, f > | = \sup_{l \in E_\pi} \{ | < l, f > | \}$ and $\sup_{f \in ECL^*} \{ | < f, l > | \} = 1$. Then such a vector $l \in E_\pi$ is called an extremal of $f \in L^*$.

Now we are to obtain the form of the optical control to the given optimal control problem.

Theorem 4—Let $\phi \in L'^*$ and let L^*_{w*} be the locally convex linear topological space with respect to the weak* topology of L^*. Corresponding to an element $T^* \phi \in L^*_{w*}$ if there exists at least one non-zero extremal l_ϕ then $T(l_\phi) \in C \cap \delta C$ holds. Conversely, each $\xi \in C \cap \delta C$ can be written as $T(l_\phi)$ where l_ϕ (i.e. it exists) is a non-zero extremal of $T^* \phi$ for some $\phi \in L'^*$.

PROOF : We suppose that $\phi \in L'^*$ and suppose that $T^* \phi \in L^*_{w*}$ has a non-zero extremal $l_\phi \in E_\pi \subset L$.

We put $\xi_\phi = Tl_\phi$. Clearly, $\xi_\phi \in C$. Now, we are to show that $\xi_\phi \in \delta C$. Let any $\eta \in C$. Then $\eta = Tl$ for some $l \in E_\pi$ and consequently

$$| < \eta, \phi > | = | < Tl, \phi > | = | < l, T^* \phi > |$$

$$\leqslant \sup_{l \in E_\pi} \{ | < l, T^* \phi > | \} = | < l_\phi, T^* \phi > | = | < Tl_\phi, \phi > |$$

$$= | < \xi_\phi, \phi > | \quad [\because \ \xi_\phi = Tl_\phi].$$

It follows that the functional ϕ assumes its maximum value on C at the vector ξ_ϕ.

Again, since ϕ is continuous linear functional, open sets map into open sets. Thus, we show that ξ_ϕ can not belong to the interior of C. Therefore, $Tl_\phi \in C \cap \delta C$ for some $\phi \in L'^*$ and for $T^* \phi \in L^*_{w*}$.

Conversely, let $\xi \in C \cap \delta C$. Then from Theorem 3 we see that the minimum pre-images of ξ exist and we shall show that each pre-image of ξ is an extremal l_ϕ of $T^* \phi$ for some $\phi \in L'^*$. Now since $\xi \in \delta C$ and C is a convex body, we may consider that $\phi \in L'^*$ such that

$$\sup_{l \in E_\pi} \{ | < l, T^* \phi > | \}$$

$$\leqslant | < l, T^* \phi > | \text{ [for some } l \in \delta E_\pi]$$

$$= | < Tl, \phi > |$$

$$= | < \xi, \phi > | \; [Tl = \xi].$$

Again since $\xi \in C$, $\xi = TL$ for some $l \in E_\pi$ and so

$$\sup_{l \in E_\pi} \{ | < l, T^* \phi > | \} \geqslant | < l, T^* \phi > | \text{ for any } l \in E_\pi$$

$$= | < Tl, \phi > | = | < \xi, \phi > | .$$

These two inequalities show that

$$| < l, T^* \phi > | = | < \xi, \phi > | = \sup_{l \in E_\pi} \{ | < l, T^* \phi > | \}.$$

Again, since $l \in E_\pi$ and obviously $\sup_{T^* \phi \in ECL^*} \{ | < l, T^* \phi > | \} = 1$, l is an extremal of $T^* \phi$ and so $\xi = Tl = Tl_\phi$ for some $\phi \in L'^*$ and for $T^* \phi \in L^*_{w*}$.

Corollary 4.1—If $\xi \in C \cap \delta C$, then $\xi = T(l_\phi)$ for some $\phi \in L'^*$ and for some $T^* \phi \in L^*_{w*}$ if and only if ϕ defines a supporting hyperplane to C at ξ.

PROOF : If $\xi \in C \cap \delta C$ and $\xi = T(l_\phi)$ for some $\phi \in L'^*$, and $T^* \phi \in L^*_{w*}$, then for each $l \in E_\pi$

$$| < Tl, \phi > | = | < l, T^* \phi > | \}$$

$$\leqslant \sup_{l \in E_\pi} \{ | < l, T^* \phi > | \}$$

$$= | < l_\phi, T^* \phi > |$$

$$= | < Tl_\phi, \phi > |$$

$$= | < \xi, \phi > | .$$

Consequently, we have the supporting hyperplane

$$\{\eta : | < \eta, \phi > | = | < Tl, \phi > | = | < \xi, \phi > | \} \text{ to } C \text{ at } \xi.$$

Again, if $\phi \in L'^*$ defines a supporting hyperplane at ξ, then proof is obvious from the converse part of Theorem 4.

Corollary 4.2—If $\xi \in C \cap \delta C$ and ϕ defines a supporting hyperplane to C at ξ, then $T^* \phi$ attains its supremum at $l_\phi \in E_\pi$. Conversely, if $T^* \phi$ attains its supremum at $l_\phi \in E_\pi$ for some $\phi \in L'^*$, then $T^{-1}(\xi_\phi)$ will contain a minimum element for which $\xi_\phi = Tl_\phi$.

PROOF: We suppose that ϕ supports C at ξ. Then from Corollary 4.1 and again from Theorem 4, $T^* \phi$ has at least one non-zero extremal l_ϕ for $\phi \in L'^*$. This shows that $T^* \phi$ attains its supremum at $l_\phi \in E_\pi$. Conversely, if $T^* \phi$ attains its supremum at $l_\phi \in E_\pi$ for some $\phi \in L'^*$, then by Theorem 4, $T(l_\phi) \in C \cap \delta C$. Hence by Theorem 3, $T^{-1}(\xi_\phi)$ has a minimum element l_ϕ for which $\xi_\phi = Tl_\phi$.

Thus, we can conclude that if L is semi-reflexive or the conditions of Corollary 3.3, then the Corollary 3.1 holds and combining Corollary 3.1, Corollary 4.2, and Theorem 4, we see that $T^{-1}(\xi)$ has minimum elements for every $\xi \in L'$ and each minimum element must have the form $l_\xi = p(\xi)\, l_\phi$ for some outward normal ϕ to C at $p(\xi)^{-1}\, \xi$ and some extremal l_ϕ of $T^* \phi \in L^*_{w*}$.

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