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BOUNDS FOR THE ZEROS OF POLYNOMIALS

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In this paper a ring shaped region containing all the zeros of the polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ has been obtained. Our results improve upon a result proved by Deutsch [*Am. Math. Monthly* **88** (1981)], No. 3, and some well known classical results due to Cauchy.

1. INTRODUCTION AND STATEMENT OF RESULTS

Consider the polynomial

$$f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 \qquad \dots(1.1)$$

where $a_0, a_1, \dots, a_{n-1}$ are complex numbers.

For every zero z of $f(z)$ we have

$$|z| \leqslant \max \{ |a_0|, 1 + |a_1|, \dots, 1 + |a_{n-1}| \} \qquad \dots(1.2)$$

$$|z| \leqslant \max \{1, |a_0| + |a_1| + \dots + |a_{n-1}| \} \qquad \dots(1.3)$$

$$|z| \leqslant r \qquad \dots(1.4)$$

where r is the unique positive zero of

$$g(z) = z^n - |a_{n-1}| z^{n-1} - \dots - |a_1| z - |a_0| . \qquad \dots(1.5)$$

The above classical results are basically due to Cauchy (see Marden[3], pp. 96-97, Parodi[4], p. 126) Deutsch[1] obtained new upper bounds for the absolute values of the zeros of $f(z)$, which generalize those given by (1.2), (1.3) and (1.4) by proving the following :

Theorem A—Every zero of the complex polynomial

$$f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

satisfies

$$|z| \leqslant \max \{r_k, 1 + |a_{k+1}|, 1 + |a_{k+2}|, \dots, 1 + |a_{n-1}| \}$$

where $k \in \{0, 1, \dots, n-1\}$ and r_k is the unique positive zero of

$$g_k(z) = z^{k+1} - |a_k| z^k - |a_{k-1}| z^{k-1} - \dots - |a_1| z - |a_0| .$$

Theorem B—Every zero of the complex polynomial

$$f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

satisfies

$$|z| \leqslant \max \{1, |a_0| + |a_1| + \dots + |a_k|, 1 + |a_{k+1}|, \dots, 1 + |a_{n-1}|\}$$

where

$$k \in \{0, 1, \dots, n-1\}.$$

In the present paper we shall improve upon Theorems A and B by obtaining a ring shaped region containing all the zeros of the polynomial. In fact we prove

Theorem 1—Let $f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a polynomial complex coefficients then $f(z)$ has all its zeros in the ring-shaped region given by

$$R_2 \leqslant |z| \leqslant R_1.$$

Here

$$R_1 = \max \{r_k, 1 + |a_{k+1}|, 1 + |a_{k+2}| \dots, 1 + |a_{n-1}|\} \quad \text{..(1.6)}$$

where $k \in \{0, 1, \dots, n-1\}$ and r_k is the unique positive zero of

$$g_k(z) = z^{k+1} - |a_k| z^k - |a_{k-1}| z^{k-1} - \dots - |a_1| z - |a_0|$$

and

$$R_2 = \frac{1}{2M_1^2} [-R_1^2 |b| (M_1 - R_1 |a_0|) + \{R_1^4 |b|^2 (M_1 - R_1 |a_0|)^2 + 4 |a_0| R_1^3 M_1^3\}^{1/2}] \quad \text{..(1.7)}$$

where

$$M_1 = R_1^{n+1} \left(2 + \sum_{k=1}^{n-1} |a_k| + \frac{1}{R_1} \sum_{k=0}^{n-1} |a_k| \right)$$

$$b = |a_1| R_1 - |a_0| . \quad \text{...(1.8)}$$

Theorem 2—Let $f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a polynomial with complex coefficients, then $f(z)$ has all its zeros in the ring shaped region given by

$$R_4 \leqslant |z| \leqslant R_3.$$

Here

$$R_3 = \max\{1, |a_0| + |a_1| + \ldots + |a_k|,$$
$$1 + |a_{k+1}|, \ldots, 1 + |a_{n-1}|\} \qquad \ldots(1.9)$$

where $k \in \{0, 1, \ldots, n-1\}$ and

$$R_4 = \frac{1}{2M_2^2}\left[-R_3^2 |b| (M_2 - R_3 |a_0|) + \{R_3^4 |b|^2 (M_2 - R_3 |a_0|)^2 + 4 |a_0| R_3^3 M_2^3\}^{1/2}\right] \qquad \ldots(1.10)$$

and

$$M_2 = R_3^{n+1}\left(2 + \sum_{k=1}^{n-1} |a_k| + \frac{1}{R_3} \sum_{k=0}^{n-1} |a_k|\right) \qquad \ldots(1.11)$$

$$b = |a_1| R_3 - |a_0|.$$

Remark 1: It is easy to see that for $k = 0$ and $k = n - 1$, Theorem A fails to give an improvement of inequalities (1.2) and (1.4) while our Theorem 1 improves upon the inequalities (1.2) and (1.4) respectively, in this case also.

Remark 2 : Again for $k = 0$ and $k = n - 1$, Theorem B yields the inequalities (1.2) and (1.3) respectively while Theorem 2 obviously improves the inequalities (1.2) and (1.3) respectively.

Example—Consider the polynomial

$$f(z) = z^3 + 0.3\, z^2 + 0.7\, z + 0.7.$$

Then by inequalities (1.2) and (1.3), $f(z)$ has all its zeros in $|z| \leqslant 1.7$, by Theorem B, with $k = 1$, $f(z)$ has all its zeros in $|z| \leqslant 1.4$ whereas by Theorem 2, with $k = 1$, $f(z)$ has all its zeros in

$$0.67 \leqslant |z| \leqslant 1.4$$

2. Lemmas

Lemma 1—If $f(z)$ is analytic in $|z| \leqslant 1$, $f(0) = a$, where $|a| < 1$, $f'(0) = b$, $|f(z)| \leqslant 1$ on $|z| = 1$, then for $|z| \leqslant 1$,

$$|f(z)| \leqslant \frac{(1-|a|)|z|^2 + |b|\,|z| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |b|\,|z| + (1-|a|)}. \qquad \ldots(2.1)$$

The example $f(z) = \left(a + \frac{b}{1+a} z - z^2\right) \Big/ \left(1 - \frac{b}{1+a} z - az^2\right)$ shows that the estimate is sharp.

The above lemma is due to Govil, *et al.*[2]. One gets easily from Lemma 1, the following

Lemma 2—If $f(z)$ is analytic in $|z| \leqslant R, f(0) = 0, f'(0) = b$. and $|f(z)| \leqslant M$ for $|z| = R$, and then for $|z| \leqslant R$

$$|f(z)| \leqslant \frac{M|z|}{R^2} \frac{M|z| + R^2|b|}{M + |z||b|}. \quad ...(2.2)$$

3. Proofs of Theorems

Proof of Theorem 1. In view of Theorem A it is sufficient to prove that $f(z) \neq 0$ if

$$|z| < R_2$$

where R_2 is defined in (1.7).

For arbitrary $\gamma \in$ R we consider

$$F(z) = (R_1 - z)\, f(e^{i\gamma} z)$$

$$= R_1 a_0 + \sum_{k=1}^{n} (R_1 a_k e^{i\gamma} - a_{k-1})\, e^{(k-1)i\gamma} z^k$$

$$- e^{ni\gamma} z^{n+1},\ a_n = 1$$

$$= R_1 a_0 + P(z), \text{ say.} \quad ...(3.1)$$

Clearly

$$|P(z)| \leqslant |z|^{n+1} + \sum_{k=1}^{n} |R_1 e^{i\gamma} a_k - a_{k-1}|\, |z|^k,\ a_n = 1$$

and hence

$$M(R_1) := \max_{|z|=R_1} |P(z)|$$

$$\leqslant R_1^{n+1} + R_1^n \sum_{k=1}^{n} |R_1 e^{i\gamma} a_k - a_{k-1}|,\ a_n = 1$$

$$\leqslant 2R_1^{n+1} + R_1^{n+1} \sum_{k=1}^{n-1} |a_k| + R_1^n \sum_{k=0}^{n-1} |a_k|$$

$$= R_1^{n+1} \left(2 + \sum_{k=1}^{n-1} |a_k| + \frac{1}{R_1} \sum_{k=0}^{n-1} |a_k| \right)$$

$$= M_1, \text{ say.} \quad ...(3.2)$$

Now we note that $P(0) = 0$, $|P'(0)| = |R_1 a_1 e^{i\gamma} - a_0| = |b|$ if γ is appropriately chosen. Choosing γ subject to this requirement and then applying Lemma 2 we obtain

$$|P(z)| \leqslant \frac{M_1 |z|}{R_1^2} \frac{M_1 |z| + R_1^2 |b|}{M_1 + |b| |z|} \qquad \ldots(3.3)$$

for $|z| \leqslant R_1$.

Combining (3.1) and (3.3), we get, for $|z| \leqslant R_1$

$$|F(z)| \geqslant R_1 |a_0| - \frac{M_1 |z|}{R_1^2} \frac{M_1 |z| + R_1^2 |b|}{M_1 + |b| |z|}$$

$$= \frac{-1}{R_1^2 (M_1 + |b| |z|)} \{ |z|^2 M_1^2 + R_1^2 |b| |z|$$

$$\times (M_1 - R_1 |a_0|) - |a_0| R_1^3 M_1\}$$

$$> 0$$

if

$$|z| < \frac{-R_1^2 |b| (M_1 - R_1 |a_0|) + \{R_1^4 |b|^2 (M_1 - R_1 |a_0|)^2 + 4 |a_0| R_1^3 M_1^3\}^{1/2}}{2M_1^2}$$

$$= R_2.$$

Since the zeros of $f(ze^{i\gamma})$ lie exactly in the same disc (centred at the origin) as the zeros of $f(z)$ the polynomial $f(z)$ has no zeros in

$$|z| < R_2.$$

This completes the proof of Theorem 1.

We omit the proof of Theorem 2 as it is analogous to that of Theorem 1 except that we have to use Theorem B instead of Theorem A.

References

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