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ON *hv*-RECURRENT FINSLER CONNECTION

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The purpose of the present paper is to introduce a Finsler connection which is neither h-metrical nor v-metrical but it is recurrent with respect to both h and v-covariant derivatives. Such a Finsler connection will be called an hv-recurrent Finsler connection.

1. INTRODUCTION

Cartan[1] published his monograph "Les espaces de Finsler" and fixed his method to define a notion of connection in the geometry of Finsler space. His method was put in order by Matsumoto[6] and determined uniquely the Cartan connection by assuming four elegent axioms :

(1) The connection is metrical.

(2) The deflection tensor field vanishes.

(3) The torsion tensor field T vanishes.

(4) The torsion tensor field S^1 vanishes.

Hashiguchi[3] replaced the condition 2 by some weaker condition and determined a Finsler connection with the given deflection tensor field. He[4] also determined uniquely a Finsler connection, by replacing the condition 3. In almost all these works it has been assumed that the connection is metrical so that the covariant differentiations commute with the raising and lowering of indices.

Recently Prasad *et al.*[11] have introduced a Finsler connection with respect to which the metric tensor is h-recurrent. Such a Finsler connection has been called an h-recurrent Finsler connection. While introducing an h-recurrent Finsler connection, it has been assumed that the v-covariant derivative of the metric tensor vanishes and the torsion tensor fields T and S^1 also vanish. The notion of hv-recurrent Finsler conneetion has also been studied by Ghinea[2] from another standpoint.

A Finsler manifold (F^n, L) of dimension n is a manifold F^n associated with a fundamental function $L(x, y)$ where $x\ (= x^i)$ denote the positional variables of F^n and $y\ (= y^i)$ denote the components of a tangent vector with respect to x^i. The metric tensor of (F^n, L) is given by $g_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L^2$ where $\dot{\partial}_i = \partial/\partial y^i$.

A Finsler connection of (F^n, L) is tried $\left(F^i_{jk},\ N^i_k,\ C^i_{jk}\right)$ of a V-connection F^i_{jk}, a non-linear connection N^i_k and a vertical connection C^i_{jk} (Matsumoto[7]). If a Finsler connection is given, the h-and v-covariant derivatives of any tensor field V^i_j are defined as

$$V^i_{j|k} = d_k V^i_j + V^m_j F^i_{mk} - V^i_m F^m_{jk}$$

$$V^i_j\Big|_k = \dot{\partial}_k V^i_j + V^m_j C^i_{mk} - V^i_m C^m_{jk}$$

where

$$d_k = \partial_k - N^m_k \dot{\partial}_m, \partial_k = \partial/\partial x^k.$$

2. hv-Recurrent Finsler Connections

Let a_k be the components of a (0) p-homogeneous vector field and b_k be the components of a (-1) p-homogeneous vector field. Then a Finsler connection $F(a, b) = \{F^i_{jk}(a, b), N^i_k(a, b), C^i_{jk}(a, b)\}$ will be called hv-recurrent if the h-and v-covariant derivatives of the metric tensor g_{ij} with respect to $F(a, b)$ are recurrent, i. e , $g_{ij|k} = a_k g_{ij}$ and $g_{ij} \mid k = b_k g_{ij}$. In particular hv-recurrent Finsler connections $F(a, 0)$ and $F(0, b)$ will be called h and v-recurrent respectively. The quantities with respect to hv-, h-and v-recurrent Finsler connections will be denoted by putting (a, b), (a). and (b) respectively. The quantities without any parenthesis will correspond to the quantities with respect to the Cartan connection $C\Gamma$. To avoid confusions we use the h and v-covariant derivatives with respect to $C\Gamma$ by $\mid k$ and $\mid k$ while these covariant derivatives with respect to any hv-recurrent Finsler connection will be denoted by $\| k$ and $\| k$. To determine such an hv-recurrent Finsler connection we have the following.

Theorem 2.1—Given covariant vector fields a_k and b_k, there exists a unique Finsler connection $F(a, b)$ satisfying the axioms

$$g_{ij}\|_k = a_k g_{ij},\ g_{ij}\|_k = b_k g_{ij},\ N^i_k(a, b) = y^j F^i_{jk}(a, b)$$

$$F^i_{jk}(a, b) = F^i_{kj}(a, b),\ C^i_{jk}(a, b) = C^i_{kj}(a, b).$$

The coefficients are given by

$$F^i_{jk}(a, b) = \Gamma^{*i}_{jk} + Q^i_{jk} \quad \ldots(2.1)$$

$$N^i_k(a, b) = G^i_k + T^i_k \quad \ldots(2.2)$$

$$C^i_{jk}(a, b) = C^i_{jk} + \sigma^i_{jk} \quad \ldots(2.3)$$

where

$$C\Gamma = \left(\Gamma^{*i}_{jk},\ G^i_k,\ C^i_{jk}\right)$$

$$Q^i_{jk} = \tfrac{1}{2}\left\{a_0 C^i_{jk} + L^2\left(C^i_{jm} C^m_k + C^i_{km} C^m_j - C^m_{jk} C^i_m\right) - \left(C^i_j y_k + C^i_k y_j - C_{jk} y^i\right) - \left(a_k \delta^i_j + a_j \delta^i_k - a^i g_{jk}\right)\right\} \quad \ldots(2.4)$$

$$T^i_k = \tfrac{1}{2}\left(a^i y_k - a_k y^i - a_0 \delta^i_k L^2 C^i_k\right) \quad \ldots(2.5)$$

$$\sigma^i_{jk} = \tfrac{1}{2}\left(b^i g_{jk} - b_j \delta^i_k - b_k \delta^i_j\right) \quad \ldots(2.6)$$

$$C^i_k = a^m C^i_{km},\ b^i = g^{im} b_m$$

and 0 denotes the contraction with y^i.

In the following, we establish the relation between the torsion R', P' and cruvature tensors corresponding to $F(a, b)$ R^2, P^2, S^2 determined in Theorem (2.1) and $C\Gamma$. (k) denotes the h-covariant derivative with respect to the Berwald connection

$B\Gamma = (\dot{\partial}_k G^i_j,\ G^i_k,\ 0)$.

By direct calculations based on their definitions, we get

$$R^i_{jk}(a, b) = R^i_{jk} + T^i_{j(k)} - T^i_{k(j)} + T^m_j \dot{\partial}_m T^i_k - T^m_k \dot{\partial}_m T^i_j \quad \ldots(2.7)$$

$$P^i_{jk}(a, b) = P^i_{jk} + \dot{\partial}_k T^i_j - Q^i_{kj} \quad \ldots(2.8)$$

$$R^i_{hjk}(a, b) = R^i_{hjk} + Q^i_{hj|k} - Q^i_{hk|j} - T^m_k \dot{\partial}_m \Gamma^{*i}_{hj} - T^m_k \dot{\partial}_m Q^i_{hj} + T^m_j \dot{\partial}_m \Gamma^{*i}_{hk} + T^m_j \dot{\partial}_m Q^i_{hk} + Q^m_{hj} Q^i_{mk} - Q^m_{hk} Q^i_{mj}$$

(*equation continued on p. 793*)

$$+ \sigma^i_{hm} R^m_{jk} + \left(C^i_{hm} + \sigma^i_{hm}\right)\left(T^m_{j(k)} - T^m_{k(j)} + T^r_j \dot{\partial}_r T^m_k - T^r_k \dot{\partial}_r T^m_j \right) \quad ...(2.9)$$

$$P^i_{hjk}(a, b) = P^i_{hjk} + T^m_j \dot{\partial}_m \left(C^i_{hk} + \sigma^k_{hk}\right) + \left(C^i_{hm} + \sigma^i_{hm}\right) \dot{\partial}_k T^m_j + Q^i_{hj|k} - \sigma^i_{hk|j} + \sigma^i_{hm} P^m_{jk} + Q^i_{hm} C^m_{jk} - \sigma^m_{hk} Q^i_{mj} + \sigma^i_{mk} Q^m_{hj} \quad ...(2.10)$$

$$S^i_{hjk}(a, b) = S^i_{hjk} + \tfrac{1}{2} b^m \left\{ C^i_{mk} g_{hj} + C_{mhk} \delta^i_j - C^i_{mj} g_{hk} - C_{mhj} \delta^i_k \right\} + \tfrac{1}{4} b^2 \left(g_{hk} \delta^i_j - g_{hj} \delta^i_k\right) + \tfrac{1}{4} b^i (g_{hj} b_k - g_{hk} b_j) + \tfrac{1}{4} b_h \left(b_j \delta^i_k - b_k \delta^i_j\right) + \dot{\partial}_k \sigma^i_{hj} - \dot{\partial}_j \sigma^i_{hk} \quad ...(2.11)$$

where

$$b^2 = b_m b^m.$$

3. h-Recurrent Finsler Connections

In this section we consider a particular one $F(a)$ of an h-recurrent Finsler connection in which a_k is the unit vector (Putting $a_k = l_k$, $b_k = 0$ in (2.4), (2.5) and (2.6) we get

$$Q^i_{jk} = -\tfrac{1}{2}\left(L C^i_{jk} - l_k \delta^i_j - l_j \delta^i_k + l^i g_{jk}\right) \quad ...(3.1)$$

$$T^i_k = -\tfrac{1}{2} L \delta^i_k, \; C^i_k = 0 \quad ...(3.2)$$

$$\sigma^i_{jh} = 0. \quad ...(3.3)$$

Thus from (2.1), (2.2) and (2.3) we get

$$F^i_{jk}(a) = \Gamma^{*i}_{jk} + \tfrac{1}{2}\left(L C^i_{jk} - l_k \delta^i_j - l_j \delta^i_k + l^i g_{jk}\right) \quad ...(3.4)$$

$$N^i_k(a) = G^i_k - \tfrac{1}{2} L \delta^i_k \quad ...(3.5)$$

$$C^i_{jk}\,(a) = C^i_{jk}\,. \qquad \ldots(3.6)$$

Since $L_{(k)} = 0$, in view of (2.7) and (3.2) we have

$$R^i_{jk}\;(a) = R^i_{jk} + \tfrac{1}{4}\left(y_j\,\delta^i_k - y_k\,\delta^i_j\right). \qquad \ldots(3.7)$$

By Matsumoto[9] (p. 168) a Finsler space of scalar curvature K is characterized by

$$R^i_{jk}\;y^j = K\left(L^2\;\delta^i_k - y_k\,y^i\right).$$

If K is constant then (F^n, L) is said to be of constant curvature[9] (p. 170). In view of the relation (3.7) we have the following

Theorem 3.1—If the (v) h-torsion tensor $R^i_{jk}\;(a)$ of an h-recurrent Finsler connection $F(a)$ with respect to $a_k = l_k$ vanishes then (F^n, L) is of constant curvature $(-\tfrac{1}{4})$.

Substituting (3.1) and (3.2) in (2.8) we get

$$P^i_{jk}\;(a) = P^i_{jk} - \tfrac{1}{2}\left(L\,C^i_{jk} - l_j\,\delta^i_k + l^i\,g_{jk}\right).$$

This relation gives

$$P^i_{0k}\;(a) = \tfrac{1}{2}\;L\;h^i_k\,,\quad P^0_{jk}\;(a) = -\tfrac{1}{2}\,L\,h_{jk}$$

$$P^i_{jk}\;(a) - P^i_{kj}\;(a) = \tfrac{1}{2}\left(l_j\;\delta^i_k - l_k\;\delta^i_j\right)$$

which give the following

Theorem 3.2—The (h) hv-torsion tensor $P^i_{jk}\;(a)$ of an h-recurrent Finsler connection $F(a)$ with respect to $a_k = l_k$ never vanishes.

Substitting the values of Q^i_{jk} , T^i_j and σ^i_{jk} from (3.1), (3.2) and (3.3) in the relation (2.9) we get

$$R^i_{hjk}\;(a) = R^i_{hjk} + \tfrac{1}{2}\;L\left(P^i_{hjk} - P^i_{hkj}\right) + \frac{1}{4}\;L^2\,S^i_{hjk}$$
$$+\frac{1}{4}\left(\delta^i_k\,g_{hj} - \delta^i_j\,g_{hk}\right).$$

Since

$$P^i_{hjk} - P^i_{hkj} = -\,S^i_{hjk|0}$$

(Matsumoto[9], p. 115), we have the following

Theorem 3.3—If the h-curvature tensor $R^{i}_{hjk}(a)$ of an h-recurrent Finsler connection $F(a)$ with respect to $a_k = l_k$ vanishes and (F^n, L) is of constant curvature $(-\frac{1}{4})$ then

$$S^{i}_{hjk|0} = \frac{1}{2} L \; S^{i}_{hjk} \; .$$

To find the relation between the hv-curvature tensors of an h-recurrent Finsler connection $F(a)$ with respect to $a_k = l_k$ and $C\Gamma$, we differentiate (3.1) v-covariantly with respect to $C\Gamma$. Then we have

$$Q^{i}_{hj|k} = \frac{1}{2}\left(l_k\, C^{i}_{hj} + L\, C^{i}_{hi jk} - \frac{1}{L}\, h_{hk}\, \delta^{i}_{j} - \frac{1}{L}\, h_{jk}\, \delta^{i}_{h} + \frac{1}{L}\, h^{i}_{k}\, g_{hj}\right).$$

Substituting (3.1), (3.2) and (3.3) in (2.10) we get

$$P^{i}_{hjk}(a) = P^{i}_{hjk} + \frac{1}{2}\, L\, S^{i}_{hjk} + \frac{1}{2}\left(l_i\, C_{jkh} - l_h\, C^{i}_{jk}\right) + \frac{1}{2L}\left(h^{i}_{k}\, g_{hj} - h_{hk}\, \delta^{i}_{j} - h_{jk}\, \delta^{i}_{h}\right).$$

Since the vertical connection $C^{i}_{jk}(a)$ of an h-recurrent Finsler connection $F(a)$ is the same as the one C^{i}_{jk} of $C\Gamma$ cf. (3.6)), the v-curvatures of both the connections will be same.

4. v-Recurrent Finsler Connections

A v-Recurrent Finsler connection $F(b)$ is a particular hv-recurrent Finsler connection $F(a, b)$ obtained by putting $a_k = 0$. Then we have $T^{i}_{k} = 0$ and $Q^{i}_{jk} = 0$. Thus from (2.1), (2.2) and (2.3) we get

$$F^{i}_{jk}(b) = \Gamma^{*i}_{jk} \;, N^{i}_{k}(b) = G^{i}_{k} \;, C^{i}_{jk}(b) = C^{i}_{jk} + \sigma^{i}_{jk} \; .$$

It is to be noted that our v-recurrent Finsler connection is the generalized Cartan connection $C_\sigma\Gamma$ defined by Hōjō[5]. From (2.7), (2.8), (2.9) and (2.10), it follows that the torsion R^1, P^1 and curvature tensors R^2, P^2 of $F(b)$ are given by

$$R^{i}_{jk}(b) = R^{i}_{jk} \;, \quad P^{i}_{jk}(b) = P^{i}_{jk} \qquad \ldots(4.1)$$

$$R^i_{hjk}(b) = R^i_{hjk} + \tfrac{1}{2}\left(b^i R_{hjk} - b_m \delta^i_h R^m_{jk} - b_h R^i_{jk} \right) \quad \ldots(4.2)$$

$$P^i_{hjk}(b) = P^i_{hjk} - \tfrac{1}{2}\left(b^i_{|j} g_{hk} - b_{k|j} \delta^i_h - b_{h|j} \delta^i_k \right)$$
$$+ \tfrac{1}{2}\left(b^i P_{hjk} - b_m P^m_{jk} \delta^i_h - b_h P^i_{jk} \right). \quad \ldots(4.3)$$

Now if $R^i_{hjk} = 0$ then $R^i_{jk} = R^i_{hjk}\, y^h = 0$. Hence from (4.2) we have $R^i_{hjk}(b) = 0$. Conversely, if $R^i_{hjk}(b) = 0$ then after contracting (4.2) with y^h we get

$$R^i_{jk} + \tfrac{1}{2}\left(b^i R_{0jk} - b_m y^i R^m_{jk} - b_0 R^i_{jk} \right) = 0. \quad \ldots(4.4)$$

Again contracting it with y_i we get

$$R_{0jk} = \tfrac{1}{2} L^2 b_m R^m_{jk}. \quad \ldots(4.5)$$

Transvecting (4.4) with b_i we get

$$(1 - b_0) R^m_{jk} b_m + \tfrac{1}{2} b^2 R_{0jk} = 0. \quad \ldots(4.6)$$

From (4.5) and (4.6) we get $R_{0jk} = 0 = R^m_{jk} b_m$ provided $4(1-b_0) + b^2 L^2 \neq 0$. Hence from (4.4) we get $R^i_{jk} = 0$ provided $b_0 \neq 2$. Therefore putting $R^i_{hjk}(b) = 0$, $R^i_{jk} = 0$ in (4.2) we get $R^i_{hjk} = 0$. Thus we get the following

Theorem 4.1—If a v-recurrent Finsler connection $F(b)$ satisfies $4(1 - b_0) + b^2 L^2 \neq 0$, $b_0 \neq 2$ then $R^i_{hjk}(b) = 0$ is equivalent to $R^i_{hjk} = 0$.

If $P^i_{hjk} = 0$ then $P^i_{jk} = P^i_{hjk}\, y^h = 0$. Hence from (4.3) we have $P^i_{hjk}(b) = 0$ provided $b_{i|k} = 0$. Conversely if $P^i_{hjk}(b) = 0 = b_{i|k}$ then we have

$$P^i_{hjk} + \tfrac{1}{2}\left(b^i P_{hjk} - b_m P^m_{jk} \delta^i_h - b_h P^i_{jk} \right) = 0. \quad \ldots(4.7)$$

Contracting (4.7) with y^h and using the fact that $P_{hjk}\, y^h = 0$, we get

$$(2 - b_0) P^i_{jk} = b_m P^m_{jk} y^i. \quad \ldots(4.8)$$

Again contracting (4.8) with b_i we get $b_m P^m_{jk} = 0$ provided $b_0 \neq 1$. Hence (4.8) yields $P^i_{jk} = 0$ if $b_0 \neq 2$. Thus (4.7) gives $P^i_{hjk} = 0$. Hence we have the following.

Theorem 4.2—If a ν-recurrent Finsler connection $F(b)$ satisfies $b_{i|k} = 0, b_0 \neq 1,2$ then $P^i_{hjk}(b) = 0$ is equivalent to $P^i_{hjk} = 0$.

Since the vertical connection $C^i_{jk}(a, b)$ of an $h\nu$-recurrent Finsler connection $F(a, b)$ is identical to the one $C^i_{jk}(b)$ of the corresponding ν-recurrent Finsler connection, $F(b)$ the ν-curvature tensor $S^i_{hjk}(b)$ of $F(b)$ is also given by (2.11).

If the recurrence vector b_k is such that $b_k = L^{-1} l_k$ then (2.11) is reduced to

$$S^i_{hjk}(b) = S^i_{hjk} - \frac{3}{4L^2}\left(h_{hk} h^i_j - h_{hj} h^i_k\right) \qquad \ldots(4.9)$$

A Finsler space of dimension $n \geqslant 4$ is called S_3-like if S^i_{hjk} is of the form[10]

$$L^2 S^i_{hjk} = S\left(h_{hj} h^i_k - h_{hk} h^i_j\right)$$

where S is a scalar. In this case the scalar S is a function of position alone[8]. Therefore (4.9) gives the following :

Theorem 4.3—If ν-curvature tensor $S^i_{hjk}(b)$ of a ν-recurrent Finsler connection $F(b)$ with respect to $b_k = L^{-1} l_k$ vanishes then (F^n, L) is S_3-like. In this case $S = -\frac{3}{4}$.

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