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HYPERSURFACES WITH (f, g, u, v, λ)—STRUCTURE OF AN AFFINELY COSYMPLETIC MANIFOLD

DHRUWA NARAIN

Department of Mathematics, University of Gorakhpur, Gorakhpur 273001

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In this paper we shall show the existence of quartic structure of a hypersurface in an almost contact manifold. Applying cosymplectic conditions on almost contact manifold, we also study totally umbilical, geodesic and the existence of parallel vector fields in the noninvariant hypersurface with (f, g, u, v, λ)—structure and some properties of normal (f, g, u, v, λ)—structure.

1. QUARTIC STRUCTURE

Let M be $(2n+1)$ dimensional almost contact manifold with (1,1) type tensor ϕ, a fundamental vector field E and a contact form η. Let us consider a $2n$-dimensional manifold P embedded in M with embedding $i : P \to M$.

Let us choose on affine normal N on P in such a way that $\phi\, N$ is always tangent to hypersurface and satisfy following linear transformations :

$$\phi i_* X = i_* fX + u(X) N \qquad \ldots(1.1)$$

$$\phi N = -i_* U \qquad \ldots(1.2)$$

$$E = l_* V + \lambda N \qquad \ldots(1.3)$$

$$\eta(i_* X) = v(X) \qquad \ldots(1.4)$$

where f is a (1,1) type tensor; U, V are vector field, u, v are 1-forms and λ a C^∞-function. If $u \neq 0$ P is called a noninvariant hypersurface of M.

From (1.1), (1.2), (1.3), (1.4) and using properties of almost contact structure (ϕ, E, η), we have the following induced structure on P

$$\left.\begin{aligned} & f^2 X = -X + u(X) U + v(X) V \\ & u(fX) = \lambda v(X), v(fX) = -\eta(N) u(X), \\ & fu = -\eta(N) V, \quad f(V) = \lambda U, \\ & u(U) = 1 - \lambda\eta(N), u(V) = 0, \\ & v(U) = 0, v(V) = (1 - \lambda\eta(N)). \end{aligned}\right\} \qquad \ldots(1.5)$$

If the vector fields E and N are distinct affine normals on P, then P has a quartic structure

$$f^4 + (1 + \lambda\eta(N)) f^2 + \lambda\eta(N) I = 0. \quad \text{...(1.6)}$$

The above equation may be factorized as

$$(f^2 + \lambda\eta(N) I)(f^2 + I) = 0. \quad \text{...(1.7)}$$

If we put $\eta(N) = \lambda$ in (1.5), we have

$$f^2 = -I + u \otimes U + v \otimes V$$
$$fU = -\lambda V, fV = \lambda U$$
$$u \circ f = \lambda v, v \circ f = -\lambda u$$
$$u(U) = 1 - \lambda^2, u(V) = 0$$
$$v(U) = 0, v(V) = 1 - \lambda^2 \quad \text{...(1.8)}$$

which is an (f, U, V, u, v, λ) − structure[4]. Now if we introduce a metric g on the (f, U, V, u, v, λ) − structure, such that

$$\left.\begin{array}{l} g(U, X) = u(X), g(V, X) = v(X) \\ g(fX, fY) = g(X, Y) - u(X)u(Y) - v(X)v(Y). \end{array}\right\} \quad \text{...(1.9)}$$

Then above structure reduces to an (f, g, u, v, λ) − structure.

2. Hypersurfaces of an Affinely Cosymplectic Manifolds

If $M(\phi, E, \eta)$ is an affinely cosymplectic manifold, i.e. $\nabla\phi = 0$ and $\nabla\eta = 0$, where ∇ denotes the covariant differentiation on M. Since $\nabla\eta = 0$ implies that $\nabla E = 0$, i.e. the vector field E is parallel with respect to ∇. Let D be the induced connection on the hypersurface P of the affine connection ∇. Now using Gauss and Weingarten's equations

$$\nabla_{i_* X} i_* Y = i_* D_X Y + h(X, Y) N \quad \text{...(2.1)}$$

and

$$\nabla_{i_* X} N = i_* HX + \omega(X) N \quad \text{...(2.2)}$$

where h and H are the second fundamental tensors of type (0,2) and (1,1) and ω is a 1-form.

Now differentiating (1.1), (1.2), (1.3), (1.4) covariantly and using $\nabla\phi = 0$, $\nabla\eta = 0$, $\nabla E = 0$ and (2.1), (2.2) and reusing (1.1), (1.2), (1.3), (1.4), we get

(a) $(D_X f)(Y) = u(Y) HX - h(X, Y) U$

(b) $D_X V = \lambda HX$, $D_X U = fHX + \omega(X) U$

(c) $(D_X v)(Y) = \lambda h(X, Y)$, $(D_X u)(Y) = -h(X, fY) - \omega(X) u(Y)$

(d) $h(X, V) = -X\lambda - \lambda\omega(X)$. ...(2.3)

Theorem 2.1—If hypersurface P is endowed with (f, g, u, v, λ)—structure and if it is an affinely umbilical hypersurface of an affinely cosymplectic manifold, it is totally geodesic, iff

$$\omega = -d(\log \lambda).$$

PROOF : Since P is affinely umbilical, putting $H = \mu I$, (2.3)d yields

$$-X\lambda - \lambda\omega(X) = v(HX) = \mu v(X). \quad ...(2.4)$$

Thus if $\mu = 0$, P is totally geodesic from (2.4), we get

$$-X\lambda - \lambda\omega(X) = 0.$$

Hence

$$\omega(X) = -\frac{X\lambda}{\lambda} = -d(\log \lambda)(X)\,[df(X) = Xf]$$

i.e.

$$\omega = -d(\log \lambda).$$

Conversely, if $\omega = -d(\log \lambda)$, we get $-X\lambda - \lambda\omega(X) = 0$, which implies that $\mu=0$, since $v(X) \neq 0$.

Theorem 2.2—Let P be a noninvariant hypersurface of an affinely cosymplectic manifold with (f, g, u, v, λ)—structure. Then if the linear transformation field f is a parallel field, we have

$$(1 - \lambda^2)^2 h(X, Y) = \nu u(X) u(Y) \quad ...(2.5)$$

$$\omega = -d(\log \lambda) \quad ..(2.6)$$

where $\nu = h(U, U)$, i.e. P is cylindrical hypersurface.

PROOF : Since the linear transformation field f is a parallel field, from (2.3) a, we have

$$u(Y) u(HX) = (1 - \lambda^2) h(X, Y). \quad ...(2.7)$$

Since h is symmetric, $u(Y) u(HX) = u(X) u(HY)$ and putting $Y = U$, we get

$$\nu u(X) = (1 - \lambda^2) u(HX). \quad ...(2.8)$$

Now from (2.7) and (2.8), we get (2.5), which shows that P is cylindrical. Further putting $Y = V$ in (2.3) a, we get $h(X, V) = 0$, which from (2.3) d is $X\lambda = -\lambda\omega(X)$, implies $\omega(X) = -\dfrac{X\lambda}{\lambda} = -d(\log \lambda)\, X$.

Theorem 2.3—Let P be a noninvariant hypersurface of an affinely cosymplectic manifold with (f, g, u, v, λ) - structure. Then if P is totally geodesic, f is parallel with respect to induced connection.

PROOF : Since P is totally geodesic, $h = 0$, implies $H = 0$, which yields $Df = 0$, i.e. f is a parallel vector field.

Theorem 2.4—The noninvariant totally geodesic hypersurface P with (f, g, u, v, λ) - structure of an almost cosymplectic manifold, the vector fields U, V are parallel vector fields, if λ is constant.

PROOF : Since P is totally geodesic, we have $\omega = -d(\log \lambda)$. If λ is constant $\omega(X) = 0$. Now from (2.3 b), we get $DU = 0$ and $DV = 0$.

3. NORMAL (f, g, u, v, λ)—STRUCTURE

An (f, g, u, v, λ) - structure is said to be normal if the torsion tensor S of f satisfies

$$S(X, Y) = N(X, Y) + du(X, Y)U + dv(X, Y)V = 0 \quad \ldots(3.1)$$

where N is the Nijenhuis tensor, and

$$du(X, Y) = (D_X u)(Y) - (D_Y u)(X)$$

$$dv(X, Y) = (D_X v)(Y) - (D_Y v)(X).$$

Theorem 3.1—The noninvariant hypersurface P with (f, g, u, v, λ)—structure of affinely cosymplectic manifold is normal, if f commutues with H and $\omega = \alpha' u$.

PROOF : The Nijenhuis tensor N of f is given by

$$N(X, Y) = (D_{fX} f)(Y) - (D_{fY} f)(X) - f(D_X f)(Y)$$
$$+ f(D_Y f)(X). \quad \ldots(3.2)$$

Using (2.3 a), we get

$$N(X, Y) = u(Y)(HfX - fHX) + u(X)(fHY - HfY)$$
$$+ (g(HfY, X) - g(HfX, Y))U.$$

Now from (2.3 c), $dv = 0$

and

$$du(X, Y) = -h(X, fY) - \omega(X)u(Y) + h(Y, fX) + \omega(Y)u(X).$$

Now putting $fH = Hf$ and $\omega = \alpha' u$ in (3.1), yields $S = 0$.

Theorem 3.2—If the hypersurface P with (f, g, u, v, λ)—structure is normal, we have

$$\eta^a(N(X, Y) + (1 - \lambda^2)d\eta^a(X, Y) = 0 \quad \ldots(3.3)$$

$$\overline{\overline{N(X, Y)}} + \lambda^2 N(X, Y) = 0 \quad \text{...(3.4)}$$

$$(1 - \lambda^2)(d\eta^a(X, \bar{Y}) + d\eta^a(\bar{X}, Y)) + \eta^a(N(X, \bar{Y}) + N(\bar{X}, Y)) = 0 \quad \text{...(3.5)}$$

where

$$\bar{X} = fX \text{ and } a = 1,2;\ \eta^1 = u,\ \eta^2 = v.$$

PROOF : Operating (3.1) by η^a, we get (3.3). Barring (3.1) twice and using (1.8), we get

$$\overline{\overline{N(X, Y)}} - \lambda^2 \{du(X, Y)\, U + dv(X, Y)\, V\} = 0. \quad \text{...(3.6)}$$

Now multiplying (3.1) by λ^2 and adding with (3.6) we get (3.4). Barring X, Y in (3.1) respectively and adding them, we get

$$N(\bar{X}, Y) + N(X, \bar{Y}) + (d\,\eta^a(\bar{X}, Y) + d\,\eta^a(X, \bar{Y}))\, E_a = 0.$$

Operating the above by η^a, we get (3.5).

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