Indian J. pure appl. Math., 20 (8). 804–811, August 1989

O-DISTRIBUTIVE POSETS

Y. S. PAWAR AND V. B. DHAMKE

Department of Mathematics, Shivaji University, Kolhapur 416004

(*Received 7 December* 1989)

In this paper O-distributivity in a partially ordered set (poset) is defined. Some equivalent formulations for O-distributivity, in poset, are obtained. It is shown that o-distributive poset is a generalization of pseudocomplemented poset. Mainly, we prove the following :

Theorem—For a O-distributive poset P, the set of all annihilator ideals, $A(P)$, is a Boolean algebra.

1. INTRODUCTION

Venkatanarasimhan[5] has defined pseudocomplemented partially ordered sets (posets). He proved that a poset P with O is pseudocomplemented if and only if $(a]^*$ is a principal ideal for every a in P. But it is observed that for $(a]^*$ to be an ideal (in the sense of Venkatanarasimhan[5], it is necessary and sufficient that, P is O-distributive. Also the purpose of this paper is to extend some of the results of Pawar[3] to partially ordered sets. Note that the definition of O-distributive semilattices given by Pawar[3] is different from that given by Varlet[4].

In section 1 we collect some known results and definitions which are used in subsequent sections. Section 2 deals with the definition, examples and several properties of O-distributive poset. In the concluding section annihilator ideals in a O-distributive poset are studied in detail.

2. PRELIMINARIES

P denotes a partially ordered set with the ordering relation $\leqslant$. For a finite set $A = \{a_1, a_2, \ldots a_n\}$ the least upper bound (l.u.b.) and the greatest lower bound (g.e.b.) of A are denoted by $a_1 \vee a_2 \vee \ldots \vee a_n$ and $a_1 \wedge a_2 \wedge \ldots \wedge a_n$ respectively. The least and the greatest elements of a poset, when they exist, are denoted by O and 1 respectively. A non-null subset A of P is called as a semi-ideal if $a \in A$, $b \leqslant a \Rightarrow b \in A$. A semi-ideal A of P is called as an ideal if the least upper bound of any finite number of elements of A, whenever it exists, belongs to A. This definition of an ideal in a poset given by Venkatanarasimhan[5] is different from that introduced by Frink[2]. Set inclusion, set intersection and set-union will be denoted by $\subseteq$ $\cap$ and U respectively.

An element a of a poset P with O is said to have the pseudocomplement a^* in P if there exists in P an element a^* such that (i) $(a] \subseteq (a^*] = (0]$ and (ii) if $(a] \cap (b] = (0]$ for b in P then $(b] \subseteq (a^*]$ A poset P is said to be pseudocomplemented if each of its element has a pseudocomplement. An element a of a poset P with 0 is said to be dense if $(a] \cap (b] = (0] \Rightarrow b = 0$ for $b \in P$. The set of all elements x of P such that $x \leqslant a$ for some fixed a in P forms an ideal of P. It is called the principal ideal generated by a and is denoted by $(a]$.

We need the following lemmas in sequel.

Lemma 1[5]—The set I of all ideals of a poset P with O is a complete lattice under set inclusion as ordering relation.

Lemma 2[5]—In a poset P a finite join $a_1 \vee a_2 \vee \ldots \vee a_n$ exists if and only if $(a] \vee (a_2] \vee \ldots \vee (a_n]$ is a principal ideal. Also whenever $a_1 \vee \ldots \vee a_n$ exists

$$(a_1 \vee \ldots \vee a_n] = (a_1] \vee (a_2] \vee \ldots \vee (a_n]$$

($\vee$ denotes the join in I_μ).

Lemma 3[5]—In a poset P with O the pseudocomplement a^* of an element a exists if and only if $(a]^* = \{x \in P/(x] \cap (a] = (0]\}$ is a principal ideal. Further whenever a^* exists, $(a]^* = (a^*]$.

Lemma 4[3]—Every distributive lattice with O (semilattice with O) is a O-distributive lattice (semilattice).

Lemma 5—Every pseudocomplemented semilattice is O-distributive. Throughout this paper the symbol P denotes a poset P with O.

2. O-DISTRIBUTIVE POSETS

We begin with

Definition 1—A poset P is called as a O-distributive poset if for $a, x_1, \ldots x_n \in P$ (n finite)

$$(a] \cap (x_i] = (\theta] \;\forall\; i,\ 1 \leqslant i \leqslant n \text{ imply } (a] \cap (x_1 \vee \ldots \vee x_n] = (0]$$

whenevvr $x_1 \vee x_2 \vee \ldots \vee x_n$ exists in P.

Remark: It is clear that our definition concides with the definition of Pawar[3] in a semilattice.

Examples of O-distributive posets are given in the following figures.

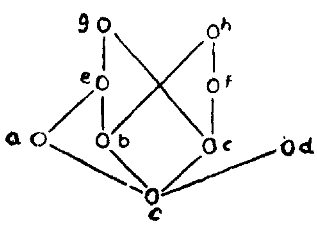

FIG. 1.

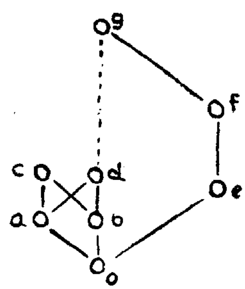

FIG. 2.

Examples—Note that every poset with O need not be O-distributive. The following is an example of a non-O-distributive poset.

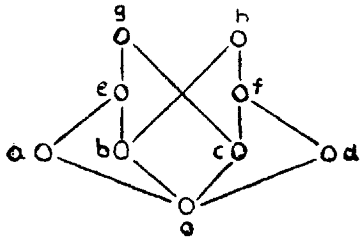

FIG. 3.

Example—In a poset P define

$$\{a\}^* = \{b \in P/(a] \cap (b] = (0]\}.$$

We characterize O-distributive poset as

Theorem 2—Poset P is O-distributive if and only if $\{a\}^*$ is an ideal for any a in P.

PROOF : Obviously in any poset $\{a\}^*$ is a semi-ideal. If for $x_1, \ldots, x_n$ (n finite) in $\{a\}^*, x_1 \vee \ldots \vee x_n$ exists then by O-distributivity $(a] \cap (x_1 \vee \ldots \vee x_n] = (0]$ proving that $x_1 \vee x_2 \vee \ldots \vee x_n \in \{a\}^*$ i.e. $\{a\}^*$ is an ideal. Conversely, if $\{a\}^*$ is an ideal then by the definition of an ideal, O-distributivity of P follows.

For any subset A of P If we denote $A^* = \{x \in P / (x] \cap (a] = (0]$ for all $a \in A\}$. Then obviously $A^* = \bigcap_{a \in A} \{a\}^*$. As arbitrary intersection of ideals is an ideal we get.

Corollary 3—P is O-distributive if and only if A^* is an ideal for any $A \subseteq P$.

By Lemma 3, it follows that P is pseudocomplemented if and only if $\{a\}^*$ is a principal ideal. Hence by Theorem 1 we get.

Corollary 4—Every pseudocomplemented poset is O-distributive.

The above corollary establishes that fact that O-distributive poset is a generalization of pseudocomplemented poset, as every O-distributive poset need not be pseudocomplemented. This is shown by a poset represented in the following figure.

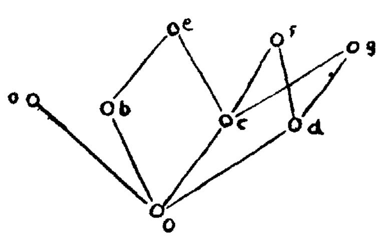

FIG. 4.

While generalizing the concept of disjunctivity to posets Venkatanarasimhan[6] defined disjunctive poset. Poset P is called disjunctive poset if $a \neq b$ in P implies the existence of c in P such that $(a] \cap (c] = (0]$ and $(b] \cap (c] \neq (0]$.

O-distributivity and disjunctivity are completely independent in a poset. This is cited by the following posets.

Example—Example of a poset which is O-distributive but not disjunctive.

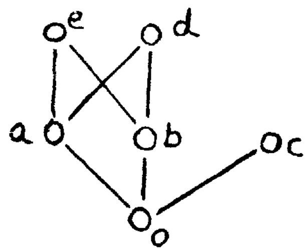

FIG. 5.

Example—Example of a poset which is disjunctive but not O-distributive.

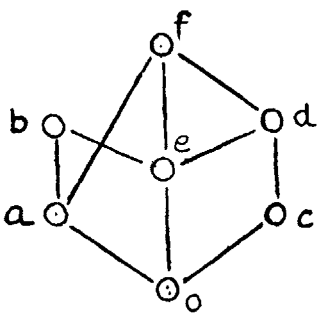

FIG. 6.

Example—Example of a poset which is neither O-distributive nor disjunctive.

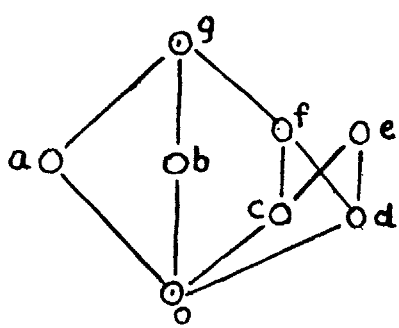

FIG. 7.

Every distributive lattice with O (semilattice with O) is O-distributive (see Lemma 4). Hence to keep up such a linking for posets, we define.

Definition 5—A poset Q is called as a distributive poset if $(a] \cap (b] \subseteq (c]$ $(a_1 b_1 c \in Q)$ implies the existence of x, y in Q $x \geqslant a, y \geqslant b$ such that $(x] \cap (y] = (c]$.

Theorem 6—Every distributive poset with 'o' is O-distributive.

PROOF : Let P be a distributive poset with O. Let $a, x_1, x_2, \ldots, x_n$ (n finite) be in P such that $(a] \cap (x_i] = (0]$ $\forall$ $1 \leqslant i \leqslant n$. Suppose that $x_1 \vee x_2 \ldots \vee x_n$ exists in P.

Now, $(x_1] \supseteq (x_2] \cap (a]$. Hence by distributivity there exist $y_2 \geqslant_1 x_2$ and $y_1 \geqslant a$ such that $(x_1] = (y_1] \cap (y_2]$. As $(y_1] \supseteq (y_1] \cap (y_2]$ we get

$$(y_1] \supseteq (x_1] \text{ i.e. } y_1 \geqslant x_1.$$

Further, $(x_1] \supseteq (x_r] \cap (a]$. Hence by distributivity there exist $y_r \geqslant x_r$ and $z_r > a$ such that

$$(x_1] = (y_r] \cap (z_r].$$

Thus we get, $y_1 \geqslant x_1, y_2 \geqslant x_2, \ldots\ldots, y_n \geqslant x_n$. Since the set of all ideals, I_μ, is a lattice (Lemma 1)

$$(y_1] \cap (y_2] \cap \ldots\ldots (y_n) \subseteq (x_1] \vee (x_2] \vee \ldots\ldots \vee (x_n]$$
$$= (x_1 \vee \ldots \vee x_n] \qquad \text{(see Lemma 2).}$$

Hence

$$(a] \cap \{(y_1] \cap (y_2] \cap \ldots (y_n]\} \supseteq (a] \cap \{(x_1] \vee (x_2] \vee \ldots \vee (x_n]\}$$

which in turn proves that

$$(a] \cap \{(x_1] \vee (x_2] \vee \ldots \vee (x_\mu]\} = (0].$$

Since $(a] \cap (y_2] = (0]$.

Hence $(a] \cap (x_1 \vee x_2 \vee \ldots \vee x_n] = (0]$ i.e. P is O-distributive.

Venkatanarasimhan[5] proved that for any poset I_μ is a complete lattice. (See Lemma 1). For I_μ to be pseudocomplemented we prove.

Theorem 7—P is O-distributive if and only if I_μ is pseudocomplemented.

PROOF : Let P be a O-distributive poset and $A \in I_\mu$. By Corollary 3 A^* is an ideal in P. We claim that A^* is the pseudocomplement of A in P. Clearly, $A \cap A^* = (0]$. If there exists B in I_μ such that $A \cap B = (0]$ then $B \subseteq A^*$. For $b \in B$ implies that $(a] \cap (b] = (0]$ for every a in A. This proves that I_μ is pseudocomplemented. Conversely, let I_μ be pseudocomplemented. For $a_1 x_1, x_2 \ldots x_n$ (n finite) in P suppose that $(a] \cap (x_i] = (0]$ for $i \leqslant i \leqslant n$. Assume that $x_1 \vee x_2 \ldots \vee x_n$ exists in P. By assumption, $(x_1] \subseteq (a]^*$ n, and hence $(x_1] \vee (x_2] \vee \ldots \vee (x_n] \subseteq (a]^*$ But, by Lemma 2, $(x_1] \vee (x_2] \vee \ldots \vee (x_n] = (x_1 \vee x_2 \vee \ldots x_n]$. Hence

$(x_1 \vee x_2 \vee \dots \vee x_n] \subseteq (a]^*$ proving that $(x_1 \vee \dots \vee x_n] \cap (a] = (0]$. Therefore P is O-distributive.

Corollary 8—When P is a pseudocomplemented poset I_μ is pseudocomplemented.

As we know that every pseudocomplemented lattice is O-distributive (Lemma 5) one more generalization of O-distributivity is obtained. This is given in the following.

Theorem 9—P is O-distributive if and only if I_μ is O-distributive.

PROOF: Let I_μ be O-distributive. Let $x_1, x_2, .., x_n$ be in P such that $(x_i] \cap (a] = (0]$ for every i, $1 \leqslant i \leqslant n$. Assume that $x_1 \vee x_2 \vee \dots \vee x_n$ exists in P. As $(x_1] \cap (a] = (0], \dots (x_n] \cap (a] = (0]$ in I_μ and I_μ is O-distributive we get

$(x_1] \vee (x_2] \vee \dots \vee (x_n] \cap (a] = (0]$. But by Lemma 2

$(x_1] \vee (x_2] \vee \dots \vee (x_n] = (x_1 \vee x_2 \vee \dots \vee x_n]$. Hence

$(x_1 \vee x_2 \vee \dots \vee x_n] \cap (a] = (0]$

which in turn proves O-distributivity of P. Conversely, let P be O-distributive. By Theorem 7, I_μ is pseudocomplemented. But every pseudocomplemented lattice being O-distributive (see Lemma 5) I_μ is O-distributive.

A poset Q is said to satisfy the ascending chain condition if any increasing chain terminates in Q i.e. if $x_i \in P$, $i = 0, 1, 2, \dots$ and $x_0 \leqslant x_1 \leqslant \dots \leqslant x_n \dots$ then for some on we have $x_m = x_{m+1} = \dots\dots\dots$

Clearly, in a poset satisfying ascending chain condition every ideal is principal. Using this we prove.

Theorem 10—Every O-distributive poset satisfying ascending chain condition is pseudocomplemented.

PROOF: Let P be O-distributive poset satisfying ascending chain condition. For any a in P, $(a]^*$ is an ideal in P, by Theorem 1. As P satisfies ascending chain condition, $(a]^*$ is a principal ideal. Hence P is pseudocomplemented. (See Lemma 3).

A sufficient condition for $(a]^* = (b]^*$ in a O-distributive poset for $a \neq b$ is stated in the following.

Theorem 11—If a and b are the elements of a O-distributive poset such that $(a] \cap (d] = (b] \cap (d]$ for some dense element $dE \in P$ then $(a]^* = (b]^*$.

PROOF: $(a]^{**} = (a]^{**} \cap P = (a]^{**} \cap (d]^{**}$ (d any dense element in P) $= \{(a] \cap (d]\}^{**} = \{(b] \cap (d]\}^{**} = (b]^{**} \cap (d]^{**} = (b]^{**} \cap P = (b]^{**}$. Hence $(a]^* = (b]^*$.

A property of the set of dense elements in a O-distributive poset is investigated in the following.

Theorem 12—In a O-distributive poset P if $\{0\} \neq A$ is the intersection of all nonzero ideals of P then $A^* = P - D$ where D is set of all dense elements of P.

PROOF : $A \neq \{0\}$ implies that $\{x\}^* \neq \{0\}$ for any x in P. i.e. $x \in A^* \Rightarrow x \in P - D$. Hence $A^* \subseteq P - D$. On the other hand, P being O-distributive, $\{d\}^*$ is a non-zero ideal of P for every $d \notin D$. But $A \subseteq \{d\}^*$ implies $A^* \supseteq d^{**}$. But $d \in \{d\}^{**}$ implies $d \in A^*$. Thus $P - D \subseteq A^*$. This proves that $A^* = P - D$.

3. ANNIHILATOR IDEALS

In this section we deal with annihilator ideals in a O-distributive poset.

Cornish[1] has defined annihilator ideal in a distributive lattice. On the same lines we define annihilator ideals in a O-distributive poset, as follows.

Definition—An ideal J of a O-distributive poset P is called an annihilator ideal if $J = J^{**}$ i.e. $J = S^*$ for some subset S of P.

The collection of all annihilator ideals in a O-distributive poset P is denoted by $A(P)$.

Theorem 13—For a O-distributive poset P, the set of all annihilator ideals $A(P)$ forms a Boolean algebra.

PROOF : For I and J in $A(P)$ define

$$I \wedge J = I \cap J \text{ and } I \vee J = (I^* \cap J^*)^*.$$

(i) As $I = I^{**}$ and $J = J^{**}$ we get $I \cap J$ is the g.l.b. of I and J. Further $(I \cap J)^{**} \supseteq (I \cap J)$ and $I \subseteq I^{**}$, $J \subseteq J^{**}$ implies $(I \cap J) \subseteq I^{**} \cap J^{**}$ proving that $I \cap J = (I \cap J)^{**}$ i.e. $I \cap J$ is in $A(P)$. Hence $I \cap J \in A(P)$ for I, J in $A(P)$.

(ii) Again $I, J \in A(P) \Rightarrow I \subseteq (I^* \cap J^*)^*$ and $J \subseteq (I^* \cap J^*)^*$. If $I \subseteq K$ and $J \subseteq K$ for some $K \in A(P)$ then $I^* \supseteq K^*$, $J^* \supseteq K^*$ will imply $I^* \cap J^* \supseteq K^*$ i.e. $(I^* \cap J^*)^* \subseteq K^{**} = K$. But then this shows that $(I^* \cap J^*)^*$ is the l.u b. of I and J in $A(P)$. Hence $I \vee J \in A(P)$.

From (i) and (ii) we get $\langle A(P); \wedge, \vee \rangle$ is a lattice.

Since $(0] = P^*$ and $P = (0]^*$, $(0]$ and P are the elements of $A(P)$. Further $(0]$ and P are the least and the greatest elements of $A(P)$.

Thus $A(P)$ is a bounded lattice.

Next we show that $A(P)$ is complemented. Let $I \in A(P)$. Then obviously $I^* \in A(P)$. Further $I \vee I^* = (I^* \cap I^{**})^* = (I^* \cap I)^* = (0]^* = P$ and $I \cap I^* = (0]$ show that I^* is the complement of I in $A(P)$.

It only remains to show $A(P)$ is distributive that, for $I, J, K \in A(P)$ we have to show that

$$I \vee (J \wedge K) = (I \vee J) \wedge (I \vee K)$$

But $I \vee (J \wedge K) \leqslant (I \vee J) \wedge (I \vee K)$ is true always.

Hence we have to only prove that

$$(I^* \cap J^*)^* \cap (I^* \cap K^*)^* \subseteq [I^* \cap (J \cap K)^*]^*.$$

To prove this we need to prove the following set inclusion

$$(I^* \cap J^*)^* \cap K \subseteq [I^* \cap (J \cap K)^*]^*.$$

Let $I, J, K \in A(P)$. Now $I \cap K \subseteq I \subseteq [I^* \cap (I \cap K)^*]^*$.

Similarly $J \cap K \subseteq [I^* \cap (J \cap K)^*]^*$.

Now $I \cap K \subseteq [I^* \cap (J \cap K)^*]^* \Rightarrow I \cap K \cap [I^* \cap (J \cap K)^*]^{**} = (0]$

that is $I \cap K \cap [I^* \cap (J \cap K)^*] = (0]$.

Silimarly $J \cap K \cap [I^* \cap (J \cap K)^*] = (0]$

that is $J \cap [K \cap I^* \cap (J \cap K)^*] = (0]$.

But this imply

$$[K \cap I^* \cap (J \cap K)^*] \subseteq J^*. \quad \text{Similarly}$$

$$[K \cap I^* \cap (J \cap K)^* \subseteq I^*$$

$\Rightarrow [K \cap I^* \cap (J \cap K)^*] \subseteq I^* \cap J^*$

$\Rightarrow [K \cap I^* \cap (J \cap K)^*] \cap (I^* \cap J^*)^* = (0]$

that is $I^* \cap (J \cap K)^* \cap [K \cap (I^* \cap J^*)^*] = (0]$

$\Rightarrow K \cap (I^* \cap J^*)^* \subseteq [I^* \cap (J \cap K)^*]^*$

i.e. $(I^* \cap J^*)^* \cap K \subseteq [I^* \cap (J \cap K)^*]^*$

i.e. $(I \vee J) \wedge K \leqslant I \vee (J \wedge K)$ providing that $A(P)$ is distributive.

Thus $A(P)$ is a complemented, distributive, lattice and hence a Boolean algebra.

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