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FUNCTIONAL LIMITS

G. DAS AND S. NANDA*

Department of Mathematics, Utkal University, Bhubaneswar 751004

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The concepts of Banach limits and almost convergence of functions have been discussed respectively by Raimi[8] and Tien-Kung Ho[4]. The purpose of this paper is to introduce the continuous analogue of (Z, p) summable sequences. Several properties of such functions including their relationship with almost convergent functions are discussed.

INTRODUCTION

Let E be a Banach space of measurable essentially bounded real-valued functions defined on the real line R with the norm $\|f\| = \text{ess. sup}\,\{\,|f(x)| : x \in R\}$ and let L be the class of functions Lebesgue integrable in every finite interval. That E and L are actually made up of equivalent classes of such functions will be ignored in the sequel. For $c \in R$ and $f \in E$ we define f_c by

$$f_c(x) = f(x + c).$$

We assume that E contains all bounded uniformly continuous functions and the functions f_c for each $f \in E$ and $c \in R$. Let e denote the constant function defined by $e(x) = 1$ for all $x \in R$.

Let E^* denote the conjugate space of E. An element $\Phi \in E^*$ is said to be a Banach limit (see Raimi[8]) if

$$\|\Phi\| = 1$$

$$\Phi(e) = 1$$

and

$$\Phi(f_c) = \Phi(f) \text{ for all } f \in E \text{ and } c \in R.$$

This is the continuous analogue of Banach limits for bounded sequences (see Banach[1] and Lorentz[7]).

It has been shown by Raimi[8] that if f is a bounded uniformly continuous function, then all its Banach limits coincide and equal to $\lim\limits_{x\to\infty} f(x)$†.

†In the sequal $\alpha, x \to \infty$ in the limits.

*On leave of absence from Indian Institute of Technology, Khargpur 721302.

For each $\alpha \geqslant 0$ and $x \in R$ we define the operator T_α by

$$T_\alpha f(x) = \begin{cases} f(x) & (\alpha = 0) \\ \dfrac{1}{\alpha} \displaystyle\int_x^{x+\alpha} f(t)\, dt & (\alpha > 0). \end{cases}$$

Let F denote the set of all almost convergent functions, that is

$$F = \{ f \in E : \lim_{\alpha \to \infty} T_\alpha f(x) \text{ exists uniformly in } x \}.\dagger$$

This has been discussed by Raimi[8] and Hö[4]. Note that F is the continuous analogue of almost convergent sequences [see Lorentz[3]]. It has been demonstrated by Raimi[8] that if $f \in F$, then all its Banach limits coincide with $\lim_\alpha T_\alpha f(x)$.

We now introduce the set

$$S_\alpha = \{ f \in L : \lim_{x \to a} T_\alpha f(x) \text{ exists for fixed } \alpha \geqslant 0 \}.$$

It is obvious that

$$S_0 = \{ f \in L : \lim_x f(x) \text{ exists} \}.$$

It should be noted here that $T_\alpha f(x)$ is a special case of the functional Nörlund transform of $f(t + \alpha)$ generated by the function

$$p(t) = \begin{cases} 1 & (0 \leqslant t \leqslant \alpha) \\ 0 & (t > \alpha). \end{cases} \qquad \text{...(1)}$$

Although the general functional Nörlund methods have been discussed by Knopp and Vanderburg[5] and Knopp[6], but still there is scope for considering the special and interesting method defined by (1) which has so far not been studied. The series analogue of this special method has been discussed by Silverman and Szasz[9], Hill and Sledd[3] and Das[2]. If $f \in S_\alpha$ is such that $T_\alpha f(x) \to s$ as $x \to \infty$ for fixed α, then we sometimes write this as $f(x) \to s\ (S_\alpha)$. It should be remarked here that f could be in S_α even if it is not essentially bounded and is merely integrable in any finite range, whereas it should be essentially bounded in order to be in F.

We define the set

$$| S_\alpha | = \{ f \in L : T_\alpha f(x) \in BV_x \text{ for fixed } \alpha \}.$$

Then

$$| S_0 | = \{ f \in L : f(x) \in BV \}.$$

It is obvious that $| S_\alpha | \subset S_\alpha$ for every $\alpha \geqslant 0$.

If we write

†In the sequal $\alpha, x \to \infty$ in the limits.

$$C_1 = \left\{ f \in L : \lim_{\alpha} T_\alpha f(0) = \lim_{\alpha} \frac{1}{\alpha}\int_0^\alpha f(t)\, dt \text{ exists} \right\}$$

then it is obvious that $F \subset C_1 \cap E$.

The purpose of the present paper is to study the properties of the sets S_α and $| S_\alpha |$ only.

The Results

Theorem 1—The translation of the argument of f has no influence on its generalised limits, that is, for a constant $c > 0$,

$$f(x) \to s\,(S_\alpha) \Leftrightarrow f(x + c) \to s\,(S_\alpha),\ \alpha \geqslant 0.$$

Proof : Let

$$\bar{T}_\alpha f(x) = \frac{1}{\alpha}\int_x^{x+\alpha} f(t + c)\, dt.$$

Hence

$$\bar{T}_\alpha f(x + c) = T_\alpha f(x)$$

and this proves the theorem.

Theorem 2—$S_0 \subset S_\alpha$ for every $\alpha > 0$.

$$| S_0 | \subset | S_\alpha | \text{ for every } \alpha > 0.$$

Proof : Since, for given $\epsilon > 0$ and for $x \geqslant x_0$, $| f(x) - s | < \epsilon$

implies

$$\left| \frac{1}{\alpha}\int_0^\alpha f(x + t)\, dt - s \right| \leqslant \frac{1}{\alpha}\int_0^\alpha \left| f(x + t) - s \right| dt < \epsilon$$

it follows that $S_0 \subset S_\alpha$. In fact

$$S_0 = \bigcap_{\alpha \geqslant 0} S_\alpha.$$

It is easy to verify that

$$T_\alpha f(x) = T_\alpha f(0) + \frac{1}{\alpha}\int_0^x \{f(t + \alpha) - f(t)\}\, dt. \qquad \ldots(2)$$

Thus it follows from (2) that $f(x) \to s\ (S_\alpha)$ if and only if

$$\int_0^x \{f(t+\alpha) - f(t)\}\ dt \to \alpha\,(s - T_\alpha f(0)) \text{ as } x \to \infty. \qquad \ldots(3)$$

It follows from (3) that the definition of $| S_\alpha |$ is equivalent to

$$\int_0^\infty | f(t+\alpha) - f(t) |\ dt < \infty \ (\alpha > 0).$$

If we write for $g \in L$, $f(t) = \int_0^t g(u)\ du$, then

$$\begin{aligned} \int_0^\infty | f(t+\alpha) - f(t) |\ dt &= \int_0^\infty dt\ \Big| \int_0^\alpha g(u+t)\ du \Big| \\ &\leqslant \int_0^\alpha du \int_0^\infty | g(u+t) |\ dt \\ &\leqslant \int_0^\alpha du \int_0^\infty | g(t) |\ dt \\ &= \int_0^\alpha du \int_0^\infty | df(t) | \\ &\leqslant \alpha V(f) < \infty. \end{aligned}$$

This shows that $| S_0 | \subset | S_\alpha |$ for every $\alpha > 0$. In fact

$$| S_0 | = \bigcap_{\alpha > 0} | S_\alpha | .$$

Theorem 3—If $T_\alpha f(x) \to s$ and $T_\beta f(x) \to s'$ as $x \to \infty$, then $s = s'$.

PROUF : This follows from Theorem 2 and from the fact that

$$T_\alpha (T_\beta f(x)) = T_\beta (T_\alpha f(x)).$$

Because of Theorem 3 the statment $S_\beta \subset S_\alpha$ means that if $f(x) \to s\ (S_\beta)$, then $f(x) \to s\ (S_\alpha)$.

At this stage we make a conjecture :

Conjecture 4—$S_\alpha \cap E \subset F$.

We now prove :

Theorem 5—$S_\alpha \subset C_1$ $(\alpha \geqslant 0)$.

PROOF : If $\alpha = 0$, the theorem is obvious. Suppose now that $0 < \alpha \leqslant 1$. We write $N = [x]$. Then

$$\frac{1}{x}\int_0^x f(t)\,dt = \frac{1}{x}\left(\int_0^{N\alpha} + \int_{N\alpha}^x\right) f(t)\,dt \qquad \ldots(4)$$

Now

$$\int_0^{N\alpha} f(t)\,dt = \sum_{r=0}^{n-1}\int_0^{\alpha} f(t + r\alpha)\,dt.$$

Without loss of generality we take $s = 0$. Hence by hypothesis

$$\theta_r = \frac{1}{\alpha}\int_0^{\alpha} f(t + r\alpha)\,dt \to 0 \text{ as } r \to \infty$$

and so

$$\frac{1}{N}\sum_{r=0}^{N-1} \theta_r \to 0 \text{ as } N \to \infty.$$

Since $\frac{N}{x} \to 1$ as $x \to \infty$ we get

$$\frac{1}{x}\int_0^{N\alpha} f(t)\,dt \to 0 \text{ as } x \to \infty. \qquad \ldots(5)$$

Now we have to show that

$$\frac{1}{x}\int_{N\alpha}^{x} f(t)\,dt \to 0 \text{ as } x \to \infty. \qquad \ldots(6)$$

If $\alpha = 1$, then (6) obviously holds. If $0 < \alpha < 1$, we have

$$\frac{1}{x}\int_{N\alpha}^{x} f(t)\,dt = \frac{1}{x}\sum_{r=1}^{m}\int_0^{\alpha} f(t + [x]\alpha + (r-1)\alpha)\,dt$$

$$+ \frac{1}{x}\int_{[x]\alpha + m\alpha}^{x} f(t)\,dt \qquad \ldots(7)$$

where m is an integer such that

$$[x]\,\alpha + m\,\alpha \leqslant x \leqslant [x]\,\alpha + (m+1)\,\alpha.$$

Clearly

$$\frac{1}{x} \int_{[x]\alpha+m\alpha}^{x} f(t)\,dt = o\left(\frac{1}{x}\right) = o(1), \text{ as } x \to \infty. \qquad \ldots(8)$$

Since $T_\alpha\ f(x) \to 0$ as $x \to \infty$, choose $\epsilon > 0$ and m_0 such that, for $r > m_0$

$$\int_0^\alpha f(t + [x]\,\alpha + (r-1)\,\alpha)\,dt < \epsilon.$$

Hence

$$\frac{1}{x} \sum_{r=1}^{m} \int_0^\alpha f(t + [x]\,\alpha + (r-1)\,\alpha)\,dt$$

$$= \frac{1}{x}\left(\sum_{r=1}^{m_0} + \sum_{r=m_0+1}^{m}\right) \int_0^\alpha f(t + [x]\,\alpha + (r-1)\,\alpha)\,dt$$

$$= o(1/x) + o(m/x) = o(1).$$

Now the theorem follows from (4) – (8).

When $\alpha > 1$ the theorem can be proved similarly.

On the contrary, we have

Theorem 6—For $\alpha > 0$,

$$f \in C_1 \Rightarrow T_\alpha f(x) = o(x).$$

PROOF : It follows from (2) that

$$\alpha\ \frac{d}{dx} T_\alpha f(x) = f(x+\alpha) - f(x)$$

almost everywhere. Hence

$$\int_0^x f(t)\,dt = \int_0^x \left\{ f(t+\alpha) - \alpha\ \frac{d}{dt}\ T_\alpha f(t) \right\} dt$$

(equation continued on p. 818)

$$= \int_0^x f(t+\alpha)\,dt - \alpha \int_0^x \frac{d}{dt} T_\alpha f(t)\,dt$$

$$= \int_0^x f(t+\alpha)\,dt - \alpha\,\{T_\alpha f(x) - T_\alpha f(0)\}. \qquad \ldots(9)$$

It follows from (9) that when $f \in C_1$, then

$$T_\alpha f(x) - T_\alpha f(0) = o(x)$$

and this completes the proof of the theorem.

The following theorems give inclusion relations for S_α.

Theorem 7—If α is an integral multiple of β, then $S_\beta \subset S_\alpha$.

PROOF : Suppose that $\alpha = m\beta$ where m is an integer. We exclude the cases when $\beta = 0$ or $\alpha = \beta$ which are trivial. Now suppose that $f \in S_\beta$ such that $f(x) \to s(S_\beta)$. We have

$$T_\alpha f(x) = \frac{1}{\alpha} \int_x^{x+m\beta} f(t)\,dt$$

$$= \frac{1}{\alpha} \sum_{r=1}^{m} \int_{x+(r-1)\beta}^{x+r\beta} f(t)\,dt$$

$$= \frac{1}{\alpha} \sum_{r=1}^{m} \int_x^{x+\beta} f(t+(r-1)\beta)\,dt$$

$$\to \frac{\beta}{\alpha} \sum_{r=1}^{m} s = s$$

(by Theorem 1 and the hypothesis). Hence $f \in S_\alpha$.

Theorem 8—(i) $S_\alpha \cap S_\beta \subseteq S_{\alpha+\beta}$

(ii) $S_\alpha \cap S_\beta \subset S_{\alpha-\beta}\ (\alpha > \beta)$.

PROOF : We have from Theorems 1 and 3 that

$$(\alpha+\beta)\ T_{\alpha+\beta} f(x) = \left(\int_x^{x+\alpha+\beta} + \int_x^{x+\alpha} + \int_{x+\alpha}^{x+\alpha+\beta} \right) f(t)\,dt$$

$$\to \alpha s + \beta s$$

and this proves (i). It is easily verified that for $\alpha > \beta$

$$\frac{\alpha T_\alpha f(x) - \beta T_\beta f(x)}{\alpha - \beta} = \frac{1}{\alpha - \beta} \int_x^{x+\alpha+\beta} f(t+\beta)\, dt. \qquad \ldots(10)$$

Now suppose that $f(x) \to s\ (S_\beta)$. Hence by Theorem 3 the left hand side of the expression in (10) converges to

$$\frac{\alpha s - \beta s}{\alpha - \beta} = s.$$

Now by Theorem 1 it follows that $f(x) \to s\ (S_{\alpha-\beta})$ and this completes the proof.

Theorem 9—Let $\alpha, \beta > 0$ be such that $\alpha = p\delta$, $\beta = q\delta$, and p, q are positive integers with $(p, q) = 1$. Then $S_\alpha \cap S_\beta = S_\delta$.

PROOF : Suppose that $f \in S_\delta$. Then by Theorem 7, $f \in S_{p\delta} = S_\alpha$ and $f \in S_{q\delta} = S_\beta$. Hence $f \in S_\alpha \cap S_\beta$. We have now to show that $S_\alpha \cap S_\beta \subset S_\delta$. Choose positive integers h and k such that $ph - qk = 1$. Hence $\delta ph - \delta qk = \alpha h - \beta k = \delta$. Now suppose that $f \in S_\alpha \cap S_\beta$. Hence by Theorem 7.

$$f \in S_\alpha \subset S_{\alpha h} \text{ and } f \in S_\beta \subset S_{\beta k}$$

and by Theorem 8,

$$f \in S_{\alpha h - \beta k} = S_\delta.$$

This completes the proof.

Remark : It may be observed that the results of Theorems 7, 8 and 9 remain true if the set S_α is replaced by $| S_\alpha |$.

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