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A GENERALIZED CARLEMAN BOUNDARY VALUE PROBLEM FOR MULTIPLY CONNECTED DOMAINS

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In multiply connected domains D_1, D_2 in the z_1-plane and z_2-plane respectively with Ljapunov boundaries L^j $(j = 1,2)$, there is considered the following boundary value problem : To find two functions $\phi_1(z_1)$ and $\phi_2(z_2)$ analytic in D_1 and D_2 and H-continuous in $D_j + L^j$ according to the boundary condition

$$\phi_2[\alpha(t)] = a(t)\phi_1(t) + b(t)\overline{\phi_1(t)} + c(t) \qquad \ldots(A)$$

where the functions $a(t)$, $b(t)$ and $c(t)$ satisfy H-condition on L^1, and $\alpha(t)$ preserves the direction of the circuit on L^1.

The index of problem (A) is calculated, and conditions of its solvability are proved.

Let two complex planes $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be given, and $z_2 = \alpha(z_1)$ is the homeomorphism preserving the orientation of the z_1-plane onto the z_2-plane. We take a $(m + 1)$−connected domain D_1 in the z_1-plane bounded by the Ljapunov contour L^1 consisting of closed smooth non-intersecting contour L_1^1, L_2^1, ..., L_m^1, L_{m+1}^1 of which L_{m+1}^1 contains all the others. The domain D_2 bounded by smooth closed contours L_1^2, L_2^2, ..., L_m^2, L_{m+1}^2 not intersecting one another, the last of which encloses all the others, corresponds to the domain D_1 under the transformation $z_2 = \alpha(z_1)$.

Assume that a derivative $\alpha'(t)$ is different from zero and H-continuous.

We consider the following boundary value problem : To find two functions $\phi_1(z_1)$, $\phi_2(z_2)$ analytic in D_1 and D_2 and H-continuous in $D_j + L^j$ according to the boundary condition

$$\phi_2[\alpha(t)] = a(t)\phi_1(t) + b(t)\overline{\phi_1(t)} + c(t) \quad (t \in L^1) \qquad \ldots(1)$$

where the functions $a(t)$, $b(t)$, $c(t)$ satisfy the Hölder condition on L^1.

The following lemma holds.

Lemma—If $\alpha(t)$ preserves the direction of the circuit on L^1, then the functions $\phi_1(z_1)$, $\phi_2(z_2)$ analytic in D_1 and D_2 can be represented in the form

$$\left.\begin{aligned}\phi_1(z_1) &= -\frac{1}{\pi i}\int_{L^1}\frac{\overline{\phi(\tau)\,d\tau}}{\tau - z_1} - \int_{L^1_m}\overline{\phi(\tau)}\,[1 + |\alpha'(\tau)|]\,d\sigma \quad (z_1 \in D_1)\\ \phi_2(z_2) &= \frac{1}{\pi i}\int_{L^2}\frac{\phi[\beta(\tau)]\,d\tau}{\tau - z_2} + \int_{L^2_m}\phi(\tau)\,[1 + |\alpha'(\tau)|]\,d\sigma \quad (z_2 \in D_2)\end{aligned}\right\} \quad ...(2)$$

where σ is the arc coordinate of the point τ on the contour L^1; $\beta(\tau)$ is the inverse of the function $\alpha(t)$; the density $\phi(t)$ is determined to within a constant term of the form $\sum_{k=1}^{m-1}\lambda_k\beta_k(t)$, where λ_k are arbitrary complex constants.

PROOF : Consider the following integral representation

$$\phi_j(z_j) = \frac{1}{\pi i}\int_{L^j}\frac{\phi_j^*\,d\tau}{\tau - z_j} \quad (z_j \in D_j) \quad ...(3)$$

where the densities satisfy the condition

$$\phi_2^*[\alpha(t)] + \phi_1^*(t) = 0 \quad ...(4)$$

and are defined to within constant terms of the form $\sum_{k=1}^{m}\lambda_k\beta_k(t)$ and $-\sum_{k=1}^{m}\lambda_k\beta_k(t)$, respectively.

By means of the Sokhotski-Plemelj formulae[1] and condition (4) we obtain

$$\phi_2(t) - \phi_2^-(t) = 2\phi_2^*(t) \quad ...(5)$$

$$\phi_1(t) - \phi_1^-(t) = 2\phi_1^*(t)$$

whence

$$\phi_2^-[\alpha(t)] = -\phi_1^-(t) + \phi_2[\alpha(t)] + \phi_1(t) \quad (t \in L^1_k) \quad ...(6)$$

where the function $\phi_j^-(z_j)$, $(j = 1, 2)$ is analytic in the domain which is the complement of $D_j + L^j$ in the plane and is denoted by D^-_{j1}, D^-_{j2}, ..., $D^-_{j,m+1}$. As the functions $\phi_j(t)$ are given, then it is possible from equation (6) to find the functions $\phi^-_j(t)$ and from equations (5) the required densities $\phi^*_j(t)$. The equality (6) represents the boundary value problem for simply connected domains D^-_{j1}, D^-_{j2}, ..., $D^-_{j,m+1}$.

Let the functions $w = w^{(t)}_k(z_j)$ map conformally the domain D^-_{1k} $(k = 1, 2, \ldots, m + 1)$ in the z_1-plane and the domain D^-_{2k} in the z_2-plane into the exterior of a unit circle c in the w-plane. By $z_j = V^{(j)}_k(w)$ we denote the inverse of the functions $w = w^{(j)}_k(z_j)$. It is known from the theory of conformal transformation, under the assumed conditions, that not only the functions $w^{(j)}_k(z_j)$, $V^{(j)}_k(w)$ but also their derivatives are continuously continuable on L^j_k and c respectively and satisfy the H-condition.

Introduce the new functions $\psi^-_j(w) = \phi^-_j[V^{(j)}_k(w)]$. It is easily seen that the boundary conditions (6) assume the form

$$\psi^-_2[\alpha_1(\zeta)] = -\psi^-_1(\zeta) + \psi_2[\alpha_1(\zeta)] + \psi_1(\zeta) \quad (\zeta \in c) \qquad \ldots(7)$$

where

$$\alpha_1(\zeta) = w^{(2)}_k\left\{\alpha\left[V^{(1)}_k(\zeta)\right]\right\}, \quad \psi_j(\zeta) = \phi_j\left[V^{(j)}_k(\zeta)\right].$$

The function $\alpha_1(\zeta)$ has on c a derivative $\alpha'_1(\zeta)$ different from zero, satisfying the H-condition, and transforms c one-to-one onto itself while preserving the direction of the circuit when $\alpha(t)$ possesses this property on L^1.

Thus, the problems (6) by means of the conformal mapping have been reduced to the problems of Carleman's type for a simply connected domain which are solvable, and the solution of each of them depends linearly on an arbitrary complex constant[2]. The equality (6) in the domain $D^-_{j,m+1}$ assigns the boundary condition of an exterior problem of Carleman's type and is solvable if to find a solution bounded at infinity; here, the constant $\lambda_{m+1} = \lambda$ is determined uniquely by functions $\phi_j(z_j)$. Having defined $\phi_j(z_j)$ in that way and substituted them in (5), we find the densities $\phi^*_j(t)$.

$$\phi^*_1[\alpha(t)] + \phi^*_2(t) = 0$$

and also the densities $\phi_j(t)$ shall be defined to within constant terms of the form

$$\sum_{k=1}^{m} \lambda_k \beta_k(t), \; - \sum_{k-1}^{m} \bar{\lambda}_k \beta_k(t).$$

Putting $\phi_2(t) = \phi[\beta(t)]$, then it is obvious that $\phi_1^*(t) = -\overline{\phi(t)}$, and the function $\phi(t)$ is defined to within a constant term of the form

$$\sum_{k=1}^{m} \lambda_k \beta_k(t).$$

Thus, we have shown that for the functions $\phi_j(z_j)$ there exist the integral representations

$$\left.\begin{aligned} \phi_1(z_1) &= -\frac{1}{\pi i}\int_{L^1} \frac{\phi(\tau)\,d\tau}{\tau - z_1} - \bar{\lambda} \; (z_1 \in D_1) \\ \phi_2(z_2) &= \frac{1}{\pi i}\int_{L^2} \frac{\phi[\beta(\tau)]\,d\tau}{\tau - z_2} + \lambda \; (z_2 \in D_2). \end{aligned}\right\} \quad ..(8)$$

We shall show how to obtain formulae (2) from formulae (8). With that end in view, we take any interior contour, for instance L_m^1, and choose a constant δ so as the following equality holds

$$\delta = \int_{L_m^1} \phi(\tau)[1 + |\alpha'(\tau)|]\,d\sigma.$$

Then changing the density $\phi(t) + \delta$ to $\phi(t)$ on the contour L_m^1 and leaving $\phi(t)$ unchangeable on the others, we obtain the representation (2) where the density is defined completely on the contour L_m^1 and on the others to within a constant term of the form

$$\sum_{k=1}^{m-1} \lambda_k \beta_k(t)$$

where λ_k is an arbitrary complex constant.

The lemma has been proved.

With the aid of the integral representation (2) we reduce the boundary value problem (1) to the singular integral equation

$$[1 + b(t)]\,\phi(t) + a(t)\,\overline{\phi(t)} + \frac{1}{\pi i}\int_{L^1} \left[\frac{\alpha'(\tau)}{\alpha(\tau) - \alpha(t)}\right.$$

(equation continued on p. 833)

$$- \frac{b(t)\overline{\tau'^2(\sigma)}}{\bar{\tau} - \bar{t}}\Big]\phi(\tau)\, d\tau + \frac{a(t)}{\pi i}\int_{L^1} \frac{\overline{\phi(\tau)}\, d\tau}{\tau - t}$$

$$+ [1 + b(t)]\int_{L_m^1} \phi(\tau)[1 + |\alpha'(\tau)|]\, d\sigma + a(t)\int_{L_m^1} \overline{\phi(\tau)}$$

$$[1 + |\alpha'(\tau)|]\, d\sigma + a(t) = c(t) \qquad (t \in L^1). \qquad \ldots(9)$$

The index of this equation over the field of real numbers is equal to 2 ind $b(t)$ Litvinchuk and Hasabov[3]. The boundary condition of the problem adjoint to (1) is written in the form

$$\psi_1(t) = a(t)\,\alpha'(t)\,\psi_2[\alpha(t)] + \overline{b(t)\, t^2(s)\, \alpha(t)\, \psi_2[\alpha(t)]}\ (t \in L) \qquad \ldots(10)$$

Let l and l' be the numbers of linearly independent solutions of the homogeneous problem (1) and the adjoint problem (10), then it can be shown that

$$l - l' = 2 \text{ ind } b(t) - 2m + 2. \qquad \ldots(11)$$

For solvability of equation (9) it is neccessary and sufficient that there hold the condition[2]

$$\text{Re} \int_{L^1} c(t)\, \psi_2^{(k)}[\alpha(t)]\, \alpha'(t)\, dt = 0 \ (k = 1, 2, \ldots) \qquad \ldots(12)$$

where $\left\{\psi_j^{(k)}(t)\right\}$ is the complex system of linearly independent solutions of the adjoint problem (10).

References

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