Indian J. pure appl. Math., **20** (8) : 834-839, August 1989

NONSTATIONARY LAW OF HEAT CONDUCTION IN CLASSICAL THERMOELASTIC SOLID

AMIYADEB MUKHERJEE*

N. D. College, Howrah, West Bengal

(*Received* 14 *September* 1987; *after revision* 9 *May* 1989)

The present paper seeks to envisage the classical nonstationary law of heat conduction at finite speed in thermoelastic solid continua. The covariant analysis is carried along purely phenomenological lines without kinetic-theory arguments and based on fundamental concepts of functional-theoretic continuum mechanics. As such, according to the nature of the problem, heat flux vector is considered as a linear functional of suitable well defined functions representable by Volterra-type Riemann convolution in normed Hilbert space which, under proper assumptions, yield the desired nonstationary law as revised version of Fourier's differential equation for anisotropic bodies in acceptable forms. The isotropic case is also treated. It appears, that the results allow transition to the stationary counterparts.

1. INTRODUCTION

Fourier's (linear) law of heat flux is known to render parabolic differential equation predicting infinite speed of thermal propagation in conductors[5]. But, for some reasons (vide section 2 of this text), the feature seems to be undesirable. As way out, notable attempts were made in classical physics, first by Cattaneo[2,3] and next by Vernotte[8], whence nonstationary thermodynamics came into being where hyperbolicity of the modified heat flux equation automatically leads to bounded speed of propagation. Later, the problem has been pursued by other authors with diverse motivations, accounts of which may be had in literatures on the topic (e.g. vide Bressan[1]).

The present studies is also an attempt in this direction, but with different aims and objectives. It seeks a solution to the classical nonstationary problem of modifying Fourier's constitutive law so as to account for heat conduction at finite speed in anisotropic thermoelastic solid continua, which has been hardly paid due attentions erstwhile. The analysis runs along purely phenomenological lines within the framework of axiomatic covariant continuum mechanics and functional theory. The rationale of the paper is as follows.

The heat flux vector is taken as a linear regular functional of well defined appropriate functions which, at some given point, is also a continuous function of time representable by Volterra-type Riemann convolution of tensor and vector-valued

* 69/A, Harish Mukherjee Road, Calcutta 700925.

bounded functions in normed Hilbert space. Finally, simple considerations of integro-differential equations, expansions under suitable assumptions and approximations reduce the integral into a covariant differential equation that appears to be the desired modification of classical Fourier's law for anisotropic solid continuum. Alternative versions of the equation are obtained and some important scalars, tensors and operators related to relaxation and nonstationary processes, are defined. The isotropic case is also discussed. The results appear to be agreeable and allow transition to the stationary counterparts.

2. Solution to Nonstationary Problem of Heat Conduction in Classical Thermoelastic Solid : Modified Fourier's Law without Paradox of Infinite Speed

First, let us consider an elastic solid continuum M in Euclidean space $E_{(3)}$ equipped with arbitrary space coordinates x_α and metric $g_{\alpha\beta}$ (functions of x_α), where (1, 2, 3) is the range of the Greek indices. Next, suppose that owing to some thermal process, M undergoes a deformation described by the strain tensor $e_{\alpha\beta}$ and absolute temperature T, relative to a reference state $R_{(3)}$ at initial temperature $\overset{\circ}{T}$ (constant) and time $t = 0$, where $e_{\alpha\beta}$, T are functions of x_α, t.

Then, Fourier's (linear) law of heat conduction stands as[4,6] :

$$q^\alpha = - K_T^{\alpha\beta} \overset{*}{T}_\beta \qquad \text{...(1.1)}$$

q^α being the heat flux vector, $k^{\alpha\beta}$ (positive definite) the conductivity tensor and $\overset{*}{T}_\beta = T_{|\beta}$ the temperature gradient which, since $g^\alpha_{\beta|\alpha} = 0$, may also be described by the equation

$$\overset{*}{T}_\beta = \overset{*}{T}_\alpha g^\alpha_\beta = \left(T g^\alpha_\beta \right)_{|\alpha} = \delta^\gamma_\alpha \left(T g^\alpha_\beta \right)_{|\lambda} \qquad \text{...(1.1')}$$

where δ^γ_α is the (mixed) Kronecker delta and the vertical stroke (|) denotes covariant derivative in $E_{(3)}$. Herein, we observe that $e_{\alpha\beta}$, $g_{\alpha\beta}$, $k^{\alpha\beta}$ and all other tensors to appear, are assumed to be symmetric in their covariant (or contravariant) indices and dummy suffix means summation.

Now, as it is well known, the relation (1.1) leads to parabolic differential equation which predicts infinite speed of heat conduction[4]. Incidentally, transmission of elastic waves (e.g. in incompressible fluid) and strains also offer similar examples. The results seem to be sufficient provided of course we treat systems, which are mechanically constrained, as first approximative ones. But, from standpoint of a more rigorous nature, propagation without bound is an undesirable feature in every such process which takes place in a material continuum. As such, we propose to revise equation (1.1) so that propagation of thermal disturbances in M is at finite speed.

To this end, it seems proper to compare (1.1) with the Hooke's law for stress-strain of similar type in elasticity theory and to note that the later also indicates instantaneous strain[7], which can be avoided through functional approach based on hereditary principles where relaxation phenomenon suppresses the infinite speed of propagation[5,9].

Likewise, as the above example suggests and also as it is justifiable from the axioms of determinism and memory in modern continuum mechanics[5], let us accept heat conduction as a hereditary—type event with fading memory so that the process is without the paradox of infinite speed[5].

With this point in view towards modifying a linear law, we propose to define q^α (response) as a Vector-valued first degree bounded functional of Q_α (action) which is a a vector assumed to be regular at the time-origin and some suitable function of x_λ, T, $\overset{*}{T}_\lambda$, $e_{\lambda s}$ such that heat flux q^α is also a continuous differentiable function of time over permissible domain. Now, collection of all histories of a physical process of fading memory is a Hilbert space where bounded linear functionals have defined norms and inner products. Hence, given x_μ, it is further assumed that q^α is a linear function in the Hilbert space of tensor- and vector-valued time-functions e.g. $\zeta(t)$, $\eta(t)$ on $[0, \infty)$ having finite norms $\|F\|^2 = \int_0^\infty |F(t)|^2 f(t)\,dt$, $(F = \zeta, \eta)$, and the inner product as required[4,10]:

$$<\zeta, \eta> = <\zeta(0), \eta(0)> + \int_0^\infty <\zeta(t), \eta(t)> f(t)\,dt \qquad ...(1.2)$$

with, obliviator (non-negative)

$$f(t) \to 0, \text{ as } t \to \infty. \qquad ...(1.2')$$

Then, as per representation of linear forms in Hilbert space consistent with the problem, we can, by proper choice of functions, write q^α as a Volterra-type Riemann convolution at time t and at point x_μ of M in $E_{(3)}$:

$$q^\alpha(t) = \int_0^t \overset{*}{d}{}^{\alpha\beta}(t, t')\, Q_\beta(t')\,dt'; \; t' \in [0, t], \; t \in [0, \infty) \qquad ...(1.3)$$

with,

$$d^{\alpha\beta}(t, t') = f^{\alpha\beta} e^{-(t-t')/k'}; \; \overset{*}{d}{}^{\alpha\beta}(t, t') = \frac{\partial d^{\alpha\beta}(t, t')}{\partial t'}$$

$$= \frac{f^{\alpha\beta}}{k'} e^{-(t-t')/k'} \qquad ...(1.3')$$

$$q^\alpha(t') \equiv Q_\alpha(t'), \; t' \leqslant 0; \text{ also } q^\alpha \equiv Q_\alpha \equiv 0, \text{ when } \overset{*}{T}_\beta = 0 \qquad ...(1.3'')$$

where, the tensors $d^{\alpha\beta}$ and $f^{\alpha\beta}$ (time-independent) are each assumed to be positive definite, and k' is a positive scalar (parameter) of the order of time such that for large t and fixed t', the expression exp $(-t/k')$ characterizes rapid decay of $d^{\alpha\beta}(t, t')$ which is also small when $t \gg k'$. Thereby, $d^{\alpha\beta}$ plays the role of relaxation tensor and k' the relaxation time which checks infinite speed of propagation.

The equation (1.3) with (1.3′) - (1.3″) then defines, in integral from, a modified version of the classical Fourier's law (1.1) for heat conduction at finite speed in the continuum M.

Next, we observe that though the kernel $\overset{*}{d}{}^{\alpha\beta}(t, t')$ is not a polynomial in t, it is degenerate and such that, by (1.3′), $\overset{*}{d}{}^{\alpha\beta}(t, t) \equiv f^{\alpha\beta}/k$ (a nonzero time-independent quantity) has a 0-fold root for $t = 0$. Hence, in order that a solution of (1.3) may exist and be finite at the origin, we should necessarily have an 1-fold root for $q^{\alpha}(t)$ at $t = 0$. Then, Q_{β} must satisfy the following first order differential equation obtained by differentiating (1.3) once with respect to t and eliminating the integral :

$$q^{\alpha} + k' \dot{q}^{\alpha} = f^{\alpha\beta} Q_{\beta} \qquad ...(1.4)$$

with

$$\dot{q}^{\alpha} = Dq^{\alpha}(t);\ D \equiv \frac{d}{dt} = v_{\alpha} \frac{\partial}{\partial x_{\alpha}} + \frac{\partial}{\partial t} \left(\simeq \frac{\partial}{\partial t}, \text{ for small motions} \right),$$

$$v_{\alpha} = \frac{dx^{\alpha}}{dt} \qquad ...(1.4\,)$$

where $\dot{q}^{\alpha}$ may be taken as a vector in $E_{(3)}$; since Q_{β} is regular at origin and $q^{\alpha}(0) \equiv Q_{\alpha}(0) \equiv 0$, the differential equation (1.4) yields a solution which is also an unique solution of the integral equation (1.1)[7,9].

As for the arbitrary function Q_{β}, we note that linear expansion yields,

$$Q_{\beta} = \overset{\circ}{Q}_{\beta} - h^{\lambda}_{\beta} \overset{*}{T}_{\lambda} + \omega^{\lambda s}_{\beta} e_{\lambda s} \qquad ...(1.5)$$

where $\overset{\circ}{Q}_{\beta}$, $h^{\beta\lambda}$ (positive definite), and $\omega^{\lambda s}_{\beta}$ depends on T, x_{μ} only. But, since $q^{\beta} \equiv Q_{\beta} \equiv 0$ when $\overset{*}{T}_{\lambda} = 0$, we must have $\overset{\circ}{Q}_{\beta} \equiv 0$ and $\overset{\circ}{\omega}{}^{\lambda s}_{\beta} \equiv 0$ irrespective of $e_{s\lambda}$; for, otherwise the tensorial equation (1.5) would become inconsistent having a scalar zero (a tensor of order zero) and a nonzero vector (a tensor of order one) on either sides of it. Hence,

$$Q_{\beta} = - h^{\lambda}_{\beta} \overset{*}{T}_{\lambda} \qquad ...(1.5')$$

so that

$$f^{\alpha\beta} Q_\beta = - \kappa^{\alpha\lambda} \overset{*}{T}_\lambda \qquad \ldots(1.5'')$$

with

$$k^{\alpha\lambda} = - f^{\alpha\beta} h^\lambda_\beta \qquad \ldots(1.5''')$$

where the form shown by (1.5″) must be quadratic due to the minus sign before $k^{\alpha\beta}$ (positive definite) and on account of $- q^\alpha \overset{*}{T}_\alpha \geqslant 0$ in keeping with the fact that heat flows from higher to lower temperature zones in M^6.

Taking (1.4) and (1.5″), we finally arrive at the result

$$q^\alpha + k' \dot{q}^\alpha = - k^{\alpha\beta} \overset{*}{T}_\beta \qquad \ldots(1.6)$$

where $k^{\alpha\beta}$ is identified as the thermal conductivity tensor, for (1.6) reduces to (1.1) when $k' = 0$.

The equation (1.6) is then a covariant version of (1.3), which constitutes a possible modification of the equation (1.1) for heat propagation in M without the paradox of infinite speed. Note that the other equation in q_α would be

$$q_\alpha + k' \dot{q}_\alpha = - k^\beta_\alpha \overset{*}{T}_\beta. \qquad \ldots(1.6a)$$

If inner product with the metric tensors be performed, we can also write the respective differential equations (1.6) and (1.6a) in operational notations as follows :

$$D^\alpha_\beta \, q^\beta \equiv D^{\alpha\beta} q_\beta = - k^{\alpha\beta} \overset{*}{T}_\beta \qquad \ldots(1.7)$$

and

$$D_{\alpha\beta} \, q^\beta \equiv D^\beta_\alpha \, q_\beta = - k^\beta_\alpha \overset{*}{T}_\beta \qquad \ldots(1.7a)$$

with

$$D_{\alpha\beta} = g_{\alpha\beta} + g_{\alpha\beta} k' D; \; D^{\alpha\beta} = g^{\alpha\beta} + g^{\alpha\beta} k' D; \; D^\alpha_\beta = g^\alpha_\beta + g^\alpha_\beta k' D \qquad \ldots(1.7')$$

where $D_{\alpha\beta}$ may be called the classical covariant relaxation operator.

In particular, for an isotropic model, we may set $k^{\alpha\beta} = k \, \delta^{\alpha\beta}$ where the positive scalar k is the thermal conductivity. Then (1.7)—(1.7a) respectively reduce to

$$D^\alpha_\beta \, q^\beta \equiv D^{\alpha\beta} q_\beta = - k \, \delta^{\alpha\beta} \overset{*}{T}_\beta \qquad \ldots(1.8)$$

and

$$D_{\alpha}^{\beta} q_{\beta} \equiv D_{\alpha\beta} q^{\beta} = -k \delta_{\alpha}^{\beta} \overset{*}{T}_{\beta} \qquad \ldots(1.8a)$$

which agree in essence with the result obtained by Cattaneo[2] and Vernotte[8] from different approaches, assumed that $D \simeq \partial/\partial t$ for small motions.

3. Conclusion

Looking back, we find that it is possible to obtain the classical nonstationary law of heat conduction in thermoelastic solid continua on phenomenological lines, by proper choice of assumptions, mathematical techniques, and right type of independent arguments for dependent variables consistent with the basic principles of functional theory and tensorial continuum mechanics. Accordingly, heat flux vector is considered as a bounded linear functional of space coordinates, absolute temperature, temperature gradient and strain tensor, being itself a differentiable function of time representable by Volterra-type Riemann convolution integral in Hilbert space, which finally yields modified classical covariant version of Fourier's differential equation for thermal propagation without infinite speed. The isotropic case is also treated where the results are found to agree in essence with those obtained by others (e.g. vide[2,5,8]). In particular, the stationary case also appears to be recoverable.

Acknowledgment

The author is thankful to Professor D. K. Sinha, formely of Jadavpur University, for his kind supervision and to the U. G. C., Jadavpur University and N. D. College (West Bengal, India) for necessary supports in the work carried out under F. I. P. of the U. G. C.

References

1. A. Bressan, *Atti Accad. Naz. Lincei* **17**, (1983), 59–100.
2. C. Cattaneo, *Atti Semin. Mat. Fis. Univ. Modena* **3** (1948), 3-21.
3. C. Cattaneo, *C. R. Acad. Sci. (Paris)* **A247** (1958), 431–33.
4. A. C. Eringen, Continuum Physics (Ed.), Pts. I & II, *Vol.* 12, Academic Press, N. Y., 1975.
5. G. Maugin, *J. Phys.* **A7** (1974), 465-84.
6. W. Nowacki, *Dynamic Problems of Thermoelasticity*, Noordhoff Int. Pub., Netherlands, 1975.
7. Yu. N. Rabotnov, *Elements of Hereditary Solid Mechauics*, Mir Pub., Moscow (1980).
8. F. Vernotte, *C. R Acad. Sci. (Paris)* **A246** (1958), 3154-55.
9. V. Volterra, *The Theory of Functionals and of Integral And Integro-Differential Equations*, Dover Pub. Inc., N. Y. (1959).
10. K. Yosida, *Functional Analysis* (Fourth Edition; Springer International Student Edition) Narosa Pub. House, New Delhi, India (1974).