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ELLIPSOIDAL INCLUSIONS IN AN ELASTIC MEDIUM

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The problem of classical elasticity where a homogeneous linear ellipsoidal solid is embedded in an unbounded, homogeneous, isotropic, elastic medium is discussed. Exact solutions for the disturbance displacement field due to translational and rotational modes of the solid are obtained by distributing some types of concentrated singularities over the focal ellipse. These solutions extend earlier solutions for rigid spheroids.

1. INTRODUCTION

Many boundary value problems in different branches of mathematical physics are resolved by the distribution of singularities on, along or over some part of the body being investigated. This method, known as the singularity method, has proved to be effective in wide range of configurations in many fields such as low-Reynolds number hydrodynamics[1,2], potential theory[3], and scattering theory[4]. In elastostatics, Kanwal and Sharma[5] first use this method to investigate the displacement problems involving rigid spheroid translating or rotating about, one of its axis. The problem of translating thin oblate bodies has been solved by one of the authors[6].

Our aims in this paper is to extend this method to solve some displacement boundary value problems involving rigid ellipsoidal bodies. In section 2 we present the singularities needed in this analysis and in the rest of the paper we obtain the exact solutions for two modes of displacing an embedded rigid triaxial ellipsoid, first, when the ellipsoid is given a linear displacement and secondly when it rotates about any given axis. In these problems the displacement field is consturcted as superposition of suitable singularities distributed over the focal ellipse. Applying the boundary condition on the surface we obtain integral equations for the density of these distributions which are solved exactly.

Physical quantities such as the force and the torque are obtained. Other quantities such as the stress and strain tensors can be derived directly using their relations with the displacement field[7]. The results for spheroid and disk are obtained as degenerate ellipsoids.

2. SINGULAR SOLUTIONS

Consider a solid with surface S embedded in an unbounded, homogenuous, linearly elastic and isotropic medium, then the displacement field $\bar{u}$ satisfies the equilibrium equation

$$\mu\left[\frac{1}{1-2\nu}\nabla(\nabla.\bar{u}) + \nabla^2\bar{u}\right] + \bar{f} = 0 \qquad ...(2.1)$$

where μ is the shear modulus, ν the Poisson's ratio of the material and $\bar{f}$ the body force per unit volume.

The primary fundamental solution (singularity) of (2.1) corresponding to a concentrated force

$$\bar{f} = 16\,\pi\mu\,(1-\nu)\,\delta(x)\,\bar{\alpha} \qquad ...(2.2)$$

where $\delta(\bar{x})$ is the Dirac delta function, and $\bar{\alpha}$ represents the direction and the magnitude of the force, is called the Kelvin solution and it is given by[7]

$$\bar{U}^k(x;\alpha) = \frac{(3-4\nu)\,\alpha}{r} + \frac{(\alpha.\,\bar{x})\,\bar{x}}{r^3},\ r = |\,\bar{x}\,|\,. \qquad ...(2.3)$$

Another important singularity is called the doublet which is formed by two centres of dilatation[7] and it is given by

$$\bar{U}^k(\bar{x};\bar{\alpha}) = \nabla(\bar{\alpha}\ \nabla)\ \frac{1}{r}\,. \qquad ...(2.4)$$

Because of linearity of (2.1) any derivative of (2.3) and (2.4) is also a singular solution.

The first derivative of (2.3) in the direction of $\bar{\beta}$ is called the Kelvin doublet

$$\bar{U}^{kd}(\bar{x};\bar{\alpha},\bar{\beta}) = -(\bar{\beta}.\ \nabla)\ \bar{U}^k(\bar{x};\bar{\alpha}). \qquad ..(2.5)$$

The stresslet is the symmetric part of (2.5), that is,

$$\bar{U}^s(\bar{x};\bar{\alpha},\bar{\beta}) = \tfrac{1}{2}\,[\bar{U}^{kd}(\bar{x};\bar{\alpha},\bar{\beta}) + \bar{U}^{kd}(\bar{x};\bar{\beta},\bar{\alpha})]. \qquad ...(2.6)$$

The antisymmetric part is called centre of rotation and it is given by

$$\bar{U}^r(\bar{x};\bar{\gamma}) = \nabla\times\ \frac{\bar{\gamma}}{r},\ (\bar{\gamma} = \bar{\alpha}\times\bar{\beta}). \qquad ..(2.7)$$

The first derivative of (2.4) in the direction of $\bar{\beta}$ is known as quadrupole.

$$\bar{U}^{dd}(\bar{x};\bar{\alpha},\bar{\beta}) = -\,(\bar{\beta}.\nabla)\ \bar{U}^d(\bar{x};\bar{\alpha}). \qquad ...(2.8)$$

For a control volume V enclose each of these singularities, the stresslet and quadrupole contribute neither a force nor a torque, while the centre of rotation exerted a torque

$$M = -\,8\,\pi\mu\,\bar{\gamma}. \qquad ...(2.9)$$

3. Statement of the Problem

Let S be the surface of the triaxial rigid ellipsoid

$$\sum_{i=1}^{3} \frac{x_i^2}{a_i^2} = 1, \; a_1 \geqslant a_2 \geqslant a_3 > 0 \qquad \ldots(3.1)$$

then the displacement field satisfies the homogenuous equations

$$\frac{1}{1-2\nu} \nabla (\nabla . \bar{u}) + \nabla^2 \bar{u} = 0 \qquad \ldots(3.2)$$

with the boundary conditions

$$\bar{u} = \bar{U} \text{ on } S \qquad \ldots(3.3)$$

and

$$\bar{u} \to 0, \text{ as } | \bar{x} | \to \infty \qquad \ldots(3.4)$$

where $\bar{U}$ is known and represents the mode of displacement.

We construct the solution of the above boundary value problem by distributing some of the singular solutions of section 2 over the interior of the focal ellipse $E(\bar{y})$ which is defined as the degenerate elliptical disk in the family of ellipsoids confocal to S. Its equation is

$$\frac{y_1^2}{h_1^2} + \frac{y_2^2}{h_2^2} = 1, \; y_3 = 0. \qquad \ldots(3.5)$$

where the major and minor semi axes of ϵ are defined by

$$h_i^2 = a_i^2 - a_3^2, \qquad i = 1, 2. \qquad \ldots(3.6)$$

The role of E in potential theory of ellipsoids has been investigated by Miloh[8].

The choice of the appropriate singularities depends on the form of $\bar{U}$, that is, on the mode of displacement involved. In low-Reynolds number flow problems some guidance rules are proposed[1,2] and those are found to be useful for many ptoblems in other fields including elastostatics[5,6].

We shall now consider two types of modes of displacement, translation and rotation of ellipsoid.

4. Translation of Ellipsoid

Let the ellipsoid S be given a linear displacement

$$\bar{U}(\bar{x}) = \bar{V}, \; (\bar{V} \text{ is constant}) \qquad \ldots(4.1)$$

then we construct the solution of (3.2) by distributions of Kelvins solutions and doublets over E directed along the cartesian axes, $\hat{e}_i$, that is,

$$\bar{u}(\bar{x}) = \sum_{i=1}^{3} \iint_E [A_i q^{-1} \bar{U}^k(\bar{x} - \bar{y}; \hat{e}_i) + B_i q \bar{U}^d(\bar{x} - \bar{y}; \hat{e}_i)] \, dA \quad \ldots(4.2)$$

where

$$\bar{x} = (x_1, x_2, x_3),\ y = (y_1, y_2, 0),\ dA = dy_1 \, dy_2 \quad \ldots(4.3)$$

and

$$q = \left[1 - \frac{y_1^2}{h_1^2} - \frac{y_2^2}{h_2^2}\right]^{1/2}. \quad \ldots(4.4)$$

The constants A_i and B_i represents the strength of the singularities and are to be determined. The function q plays a prominent role in the potential theory of ellipsoid particle[8].

The integrations over E can be carried out by making use of the following integral identities.

$$\iint_E \frac{y_i^{n+1}}{r} q^m \, dA = \frac{h_i^2}{m+2}\left[n \iint_E \frac{y_i^{n-1}}{r} q^{m+2} \, dA - \frac{\partial}{\partial y} \iint_E \frac{y_i^n}{r} q^{m+2} \, dA\right] m \neq -2 \quad \ldots(4.5)$$

$$\iint_E \frac{q^{2n-1}}{r} \, dA = \frac{(-1)^n (2n)! \, \pi \, h_1 h_2}{2^{2n} (n!)^2} \int_\lambda^\infty \left[\sum_{i=1}^{3} \frac{X_i^2}{a_i^2 + t} - 1\right]^n \frac{dt}{D(t)} \quad \ldots(4.6)$$

where

$$D(t) = \prod_{i=1}^{3} (a_i + t)^{1/2} \quad \ldots(4.7)$$

and λ is the positive root of the equation

$$\sum_{i=1}^{3} \frac{X_i^2}{a_i^2 + \lambda} = 1. \quad \ldots(4.8)$$

The proof of (4.5) is direct through integeration by parts and formula (4.6) has been established by Kim[2]. We note that on the surface of the ellipsoid $\lambda = 0$.

After some calculations the representation (4.2) can be written in the integerated form,

$$\bar{u}\,(\bar{x}) = \pi\, h_1\, h_2 \sum_{i=1}^{3} ([A_i \{(3 - 4\nu)\, I\,(\lambda) + (a_i^2 - a_3^2)\, I_i\,(\lambda)\}$$

$$- B_i\, I_i\,(\lambda)]\, \hat{e}_i - [A_i \{\nabla I\,(\lambda) - (a_i^2 - a_3^2)\, \nabla I_i\,(\lambda)\}$$

$$- B_i\, \nabla I_i\,(\lambda)]\, x_i) \qquad \ldots(4.9)$$

where

$$I\,(\lambda) = \int_{\lambda}^{\infty} \frac{dt}{D\,(t)},\; I_i\,(\lambda) = \int_{\lambda}^{\infty} \frac{dt}{(a_i^2 + t)\, D\,(t)} \qquad \ldots(4.10)$$

these integrals satisfy the relations

$$\sum_{i=1}^{3} I_i\,(\lambda) = \frac{2}{a_1\, a_2\, a_3} \qquad \ldots(4.11)$$

$$\sum_{i=1}^{3} a_i^2\, I_i\,(\lambda) = I\,(\lambda). \qquad \ldots(4.12)$$

The solution $\bar{u}\,(\bar{x})$ satisfies the condition at infinity (3.4) since each singularity does. Applying the surface condition (3.3) we obtain

$$\bar{V} = h_1\, h_2 \sum_{i=1}^{3} ([A_i \{(3 - 4\nu)\, I + (a_i^2 - a_3^2)\, I_i\} - B_i\, I_i]\, \hat{e}_i$$

$$- [(a_3^2\, A_i + B_i)\, x_i\, \nabla\, I_i]). \qquad \ldots(4.13)$$

where

$$I = I\,(0) \text{ and } I_i = I_i\,(0). \qquad \ldots(4.14)$$

Equation (4.13) is satisfied if

$$A_i = -\frac{1}{a_3^2} B_i = \frac{V_i}{\pi h_1 h_2} \{(3 - 4\nu)\, I + a_i^2\, I_i\}^{-1}. \qquad \ldots(4.15)$$

The net force experienced by the ellipsoid is obtained by adding the contributions of the forces (2.2) over the focal ellipse, hence,

$$F = 16\, \pi\mu\, (1 - \nu) \sum_{i=1}^{3} (\iint_E A_i\, q^{-1}\, dA)\, \hat{e}_i \qquad \ldots(4.16)$$

$$\bar{F} = 32\,\pi\mu\,(1-\nu)\sum_{i=1}^{3}\,[(3-4\nu)\,I + a_i^2\,I_i]^{-1}\,V_i\,\hat{e_i}. \qquad \ldots(4.17)$$

This complete the solution.

5. Rotation of Elliposid

Suppose that the elipsoid S rotates with angular velocity $\bar{\Omega}$ then the boundary condition on the surface is

$$\bar{u}\,(\bar{x}) = \bar{U} = \bar{\Omega} \times \bar{x}, \bar{x} \in S. \qquad \ldots(5.1)$$

In this case we write the displacement field as

$$u\,(x) = \sum_{i=1}^{3} \iint_E C_{ta}\,\bar{U}^r\,(\bar{x} - \bar{y}, \hat{e_i})\,dA$$

$$+ \sum_{\substack{i,j=1\\ i\neq j}} \iint_E [q\,S_{ij}\,\bar{U}^s\,(\bar{x} - \bar{y};\,\hat{e_i},\,\hat{e_j})$$

$$+ q^3\,Q_{ij}\,\bar{U}^{dd}\,(\bar{x} - \bar{y};\,\hat{e_i},\,\hat{e_j})]\,dA \qquad \ldots(5.2)$$

where C_i, S_{ij} and Q_{ij} are constants to be determined. Due to symmetry we have

$$S_{ij} = S_{ji},\; Q_{ij} = Q_{ji}. \qquad \ldots(5.3)$$

Performing the integrations in (5.2) using the identities (4.5) and (4.6) we obtain

$$\iint_E q\,\bar{U}^r\,(\bar{x} - \bar{y};\,\hat{e_i})\,dA = -\pi\,h_1\,h_2\,\hat{e_i} \times \sum^{3} x_k\,I_k\,(\lambda)\,\hat{e_k} \qquad \ldots(5.4)$$

$$\iint q\,\bar{U}^s\,(\bar{x} - \bar{y};\,\hat{e_i},\,\hat{e_i})\,dA = -\pi\,\frac{h_1\,h_2}{2}\,[\,\{(3-4\nu)\,I_j\,(\lambda) - I_i\,(\lambda)\}\,x_j\hat{e_i}$$

$$+ \{(3-4\nu)\,I_i\,(\lambda) - I_j\,(\lambda)\}\,x_i\,\hat{e_j} - x_i\,x_j\,\{\nabla I_i\,(\lambda) + \nabla I_j\,(\lambda)\}$$

$$+ \sum_{k=1}^{2} h_k^2\,[\{x_j\,I_{kj}\,(\lambda)\,\delta_{ik} + x_i\,I_{ki}\,\delta_{jk}\}\,\hat{e_k}$$

$$+ \{I_{kj}\,(\lambda)\,\delta_{ik}\,\hat{e_j} + I_{ki}\,\delta_{jk}\,\hat{e_i} + x_j\,\delta_{ik}\,\nabla I_{kj}\,(\lambda) + x_i\,\delta_{jk}\,\nabla I_{ki}\,(\lambda)\}$$

$$\times\, x_k] \qquad \ldots(5.5)$$

and

$$\iint_E q^3 \bar{U}^{dd} (\bar{x} - \bar{y}; e_i, \hat{e}_j) \, dA$$

$$= - 3\pi h_1 h_2 [(x_i \hat{e}_j + x_j \hat{e}_i) I_{ij} (\lambda) + x_i x_j \nabla I_j (\lambda)] \quad ...(5.6)$$

where

$$I_{ij} (\lambda) = \int_\lambda^\infty \frac{dt}{(a_i^2 + t) \; (a_j^2 + t) \, D(t)} \quad ...(5.7)$$

which satisfy the identity

$$(a_i^2 - a_j^2) \, I_{ij} = I_j - I_i. \quad ...(5.8)$$

Applying the boundary condition (5.1) and evaluating (5.2) on the surface, we obtain after a straight forward calculations

$$C_1 = \Omega_1 J_{23} K_{23}, \; C_2 = \Omega_2 J_{13} \; K_{13}, \; C_3 = \Omega_3 J_{12} K_{12} \quad ...(5.9)$$

$$S_{12} = (-3/a_3^2) \; Q_{12} = \Omega_3 H_{12} K_{12} \quad ...(5.10)$$

$$S_{13} = (-3/a_2^2) \, Q_{13} = \Omega_2 H_{13} K_{13} \quad ...(5.11)$$

$$S_{23} = (- \; 3/a_1^2) \, Q_{23} = \Omega_{23} = \Omega_1 H_{23} K_{23} \quad ...(5.12)$$

where

$$J_{ij} = 2 \, \{(1 - \nu) \, a_j^2 + \nu \, a_i^2 \} \, I_i - 2 \, \{(1 - \nu) \, a_i^2 + \nu a_j^2 \} \, I_j \quad ...(5.13)$$

$$K_{ij} = \frac{1}{\pi h_1 h_2} \{a_i^2 I_i - a_j^2 I_j^2 - (3 - 4\nu) \, (a_i^2 - a_j^2) \, I_i I_j\}^{-1} \quad ...(5.14)$$

and

$$H_{ij} = 2 \, (a_j^2 - a_j^2) \, (I_i - I_j). \quad ...(5.15)$$

The torque exerted on the ellipsoid is obtained by adding the contributions of the centers of rotation and using formula (2.9), thus

$$\bar{M} = - 8 \pi \mu \sum_{i=1}^{3} (\iint_E C_i \, q \, dA) \, \hat{e}_i \quad ...(5.16)$$

$$\bar{M} = - \frac{16}{3} \pi^2 \mu \; h_1 h_2 \sum^{3} C_i \hat{e}_i. \quad ...(5.17)$$

In the limit $\nu \to 1/2$ and $2\mu\nu/(1-2\nu)\ \nabla.\ \bar{u} \to -P$, the elastostatic equation (3.2) reduces to Stokes equations of fluid mechanics, where $\bar{u}$ denotes the velocity field and P denotes the pressure. In this case formulas (4.17) and (5.17) reduces to the drag and torque formulas given in Jeffery[9].

6. Degenerate Ellipsoids

(a) *Spheroid*

When $a_2 = a_3$ the ellipsoid (3.1) degenerates into the prolate spheroid

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2 + x_3^2}{a_2^2} = 1. \qquad \text{...(6.1)}$$

In this case all the ellipsoid integrals in the previous analysis are elementary and are given by

$$I = L/(ea_1) \qquad \text{...(6.2)}$$

$$I_1 = \frac{1}{e^3 a_1^3} \ (-2e + L) \qquad \text{...(6.3)}$$

$$I_2 = I_3 = \frac{1}{2e^3 a_1^3} \ (-2e/(1-e^2) + L) \qquad \text{...(6.4)}$$

where $L = \ln\left(\frac{1+e}{1-e}\right)$, and e is the eccentricity of the spheroid defined by

$$e^2 = 1 - a_2^2/a_1^2. \qquad \text{...(6.5)}$$

Substituting these into (4.17) and (5.17) we obtain the force and the torque exerted by the prolate spheroid. Their components are

$$F_1 = e^3 a_1 V_1 [2e + (3e^2 - 4\nu e^2 - 1) L]^{-1} \qquad \text{...(6.6)}$$

$$\frac{F_2}{V_2} = \frac{F_3}{V_3} = 2e^3 a_1 [-2e^2 - (1 - 7e^2 + 8\nu e^2) L]^{-1} \qquad \text{...(6.7)}$$

$$M = -(32/3)\ \pi\ \mu\ \Omega_1 e^3 a_1 [(2e/1-e) - L]^{-1} \qquad \text{...(6.8)}$$

and

$$\frac{M_2}{\Omega_2} = \frac{M_3}{\Omega_3} = -\frac{64}{4}\pi\mu\ e^3 a^3 [2e\{3 - (4-\nu)e + 2\ (1-\nu)e^4$$

$$- (1-e^2)(3 - 2e^2 + \nu e^2) L] \Delta^{-1} \qquad \text{...(6.9)}$$

where $\Delta = 4e^2\{3 + 2e^2 - 8\nu e^2\} - 4e\{3 + (3-8\nu)e^2 - (3-4\nu)e^4\} L$

$$+ (1-e^2)\{3 + (7-8\nu)e^2\} L^2.$$

The corresponding results for oblate spheroid can be obtained by replacing e by $ie(1-e^2)^{1/2}$. Formula (6.8) agrees with the one obtained by Kanwal and Sharma[5] for the torque exerted by a prolate spheroid rotating about its major axis and (6.9) coincide with that for rotation about the minor axis.

(b) *Disks*

When $a_3 \to 0$, the ellipsoid degenerates into an elliptic disk whose axis lies along the x_3-axis. The elliptic integrals in this limit become

$$I = \frac{2}{a_1} K \qquad ...(6.10)$$

$$I_1 = \frac{2}{a_1^3 k^2} (K - E) \qquad ...(6.11)$$

$$I = \frac{2}{a_1^3 k^2 k'^2} (E - k'^2 K) \qquad ...(6.12)$$

where K and E are the complete elliptic integrals of the first and second kind, respectively, with argument

$$k^2 = 1 - a_2^2/a_1^2 \qquad ...(6.13)$$

and k'^2 is defined by the relation

$$k^2 + k'^2 = 1. \qquad ...(6.14)$$

The components of the force experienced by the translating elliptic disk are

$$F_1 = -16\pi a_1 k^2 (1 - \nu) V_1 [\{1 + (3 - 4\nu) k^2\} K - E]^{-1} \qquad ...(6.15)$$

$$F_2 = 16 \pi a_1 k^2 (1 - \nu) V_2 [\{(1 - 4 (1 - \nu) k^2\} K - E]^{-1} \qquad ...(6.16)$$

and

$$F_3 = 16 \pi a_1 \mu (1 - \nu) V_3 \{(3 - 4\nu) K\}^{-1} \qquad ...(6.17)$$

and the components of the torque exerted by a rotating elliptic disk are

$$M_1 = - (32/3)\pi\mu (1 - \nu) \Omega_1 \{(3 - 4\nu) I_2\}^{-1} \qquad ...(6.18)$$

$$M_2 = - (32/3) \pi\mu (1 - \nu) \Omega_2 \{(3 - 4\nu) I_1\}^{-1} \qquad ...(6.19)$$

and

$$M_3 = - \frac{32}{3} \pi \mu \Omega_3 [\{(1 - \nu) a_2^2 + \nu a_1^2\} I_1 - \{(1 - \nu) a_1^2 + \nu a_2^2\} I_2]$$

$$+ [a_1^2 I_1^2 - a_2^2 - (3 - 4\nu) (a_1^2 - a_2^2) I_1 I_2]^{-1}. \qquad ...(6.20)$$

The limiting case of circular disk ($a_1 \to a_2$) yields

$$\frac{F_1}{V_1} = \frac{F_2}{V_2} = - \frac{64 a_1 \mu (1 - \nu)}{7 - 8\nu}, \; F_3 = \frac{32 a_1 \mu (1 - \nu)}{3 - 4\nu} \qquad ...(6.21)$$

$$\frac{M_1}{\Omega_1} = \frac{M_2}{\Omega_2} = \frac{64a_1^3 \mu (1 - \nu)}{3 (3 - 4\nu)}, \; M_3 = - \frac{32a_1^3 \mu}{3}. \qquad \ldots(6.22)$$

Formulas (6.21) and (6.22) agree with results in Lur'e[10].

7. Conclusion

In this paper we extend the method of singularities to solve two problems of clastostatics involving translational and rotational ellipsoidal bodies and their degenerate cases. Compared with the usual separation of variables method we note that the singularity solutions are far more simple in from since it does not depend on the choice o f the appropriate coordinate system and therefore instead of using ellipsoidal harmonics expansions to represent the solution, it is now represented in terms of simple elliptic integrals. The only difficulty in using the singularity method is the choice of the proper singularities for each mode of displacement. It is hoped that by this method more boundary value problems in elasticity are to be solved, specifically the stress type problems. Extension of the singularity method for elastodynamic problems of ellipsoid are under investigation.

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