AN ELECTROSTATIC PROBLEM INVOLVING FOUR STRIPS PLACED INSIDE A GROUNDED CYLINDER

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We present here the solution of an electrostatic boundary value problem involving four non-intersecting conducting strips placed symmetrically inside a long grounded circular cylinder. Alternate strips are charged to the same unknown constant potential and are assumed to have the same known charge per unit length. By the usual Green's function approach, the solution of the problem is reduced to that of a pair of simultaneous Fredholm integral equations of the first kind which are subsequently solved by the regular perturbation technique, when the radius of the cylinder is large. Approximate expressions are obtained for the total charge densities per unit length of the strips. Finally, we deduce the solutions of the electrostatic problem of four non-intersecting strips in an unbounded medium as well as that of a cross lying in a grounded circular cylinder or in an unbounded medium.

1. INTRODUCTION

Some two-dimensional electrostatic boundary value problems of two charged coplanar parallel strips have been discussed by several authors. Tranter (1960) obtained the charge densities of the two strips, when these are charged to potentials ± 1 in an unbounded medium. Srivastava and Gupta (1971) studied the perturbation in the charge densities of the two strips, when these are placed symmetrically inside or outside a grounded cylinder by reducing each problem to a Fredholm integral equation of the second kind with the help of triple integral equations and the finite Hilbert transform technique given by Srivastava and Lowengrub (1970). Goel and Jain (1976) simplified the analysis of Srivastava and Gupta (1971) by applying the usual Green's function approach to reduce the solution of each problem to that of Fredholm integral equation of the first kind and then obtained approximate expressions for the charge densities of the two strips in each problem by solving the corresponding integral equation by the regular perturbation technique. Rooke and Sneddon (1969), Srivastava and Gupta (1973) considered the two-dimensional electrostatic problem of finding the stress distributions in an elastic medium containing a cross-shaped crack, when the crack is in a thin circular plate. Srivastava and Nath (1973) solved the corresponding elastostatic problem of four non-intersecting Griffith cracks, in an infinite elastic medium. Recently, Gupta and Gupta (1975) have discussed the two-dimensional electrostatic problems of two cross-shaped charged strips, when the strips are lying in an unbounded medium or inside a grounded cylinder by the dual integral equations method. The solution of each of these two problems is reduced to that of a Fredholm integral equation of the second kind, which is subsequently solved numerically to obtain expressions for the charge concentration factors. But the formulae for the charge concentration factors used in their analysis seem to be incorrect.

In this paper, we present the solution of the two-dimensional boundary value electrostatic problem of four non-intersecting conducting strips placed symmetrically inside a long grounded circular cylinder. Alternate strips are charged to the same unknown constant potential and are assumed to have the same total charge per unit length. By the usual Green's function approach, the solution of this problem is reduced to that of two Fredholm integral equations of the first kind, which are solved by the regular perturbation technique, when the radius of the cylinder is large. Approximate expressions are obtained for the charge densities per unit length of the strips as well as the unknown constant potentials of the strips in terms of the known total charges per unit length of the strips. Finally, by taking appropriate limits, we deduce solutions of the following three limiting cases:

(I) Four intersecting charged strips in the form of a cross when they are lying inside a grounded circular cylinder.

(II) Four non-intersecting charged strips lying in an unbounded medium when the alternate strips are at the same unknown constant potential or when all the four strips are at the same potential.

(III) Four intersecting charged strips in the form of a cross when they are lying in an unbounded medium.

To the authors' knowledge, even the results of the above limiting cases obtained from their analysis are new.

2. Formulation of Electrostatic Problem

We shall consider the problem of finding the electrostatic potential due to four charged strips in the form of a non-intersecting cross placed symmetrically inside a grounded circular cylinder of radius $c \gg 1$. We use here cylindrical polar co-ordinates (r, θ, z) and take the axis of the cylinder along the z-axis. The four strips occupy the regions

$$a < r < 1, \quad -\infty < z < \infty, \quad \theta = n\frac{\pi}{2}, \qquad n = 0, 1, 2, 3 \qquad \ldots(1)$$

The two strips along the lines $\theta = 0, \pi$, are charged to an unknown constant potential V_1, whereas the other two strips are charged to an unknown constant potential V_2. It is assumed that the total known charges on each of the two pairs of strips are Q_1 and Q_2 respectively. Thus, we have to solve the following two-dimensional electrostatic problem:

$$\nabla^2 \phi(r, \theta) = 0 \text{ in } D \qquad \ldots(2)$$

$$\phi(r, \theta) = 0, \quad 0 \leqslant \theta \leqslant 2\pi, \quad r = c \qquad \ldots(3)$$

$$\phi(r, \theta) = V_1, \quad \theta = 0, \pi, \quad a < r < 1 \qquad \ldots(4)$$

$$\phi(r, \theta) = V_2, \quad \theta = \pi/2, \ 3\pi/2, \quad a < r < 1 \qquad \ldots(5)$$

$$\left.\begin{array}{l} \phi, (\partial\phi/\partial\theta) \text{ are continuous across the line segments} \\ 0 \leqslant r < a, 1 < r < c, \quad \theta = 0, \quad \pi/2, \quad \pi, 3\pi/2 \end{array}\right\} \qquad \ldots(6)$$

where D is the whole region lying inside the grounded circular cylinder, except the four line segments $a < r < 1$, $\theta = 0$, $\pi/2$, π, $3\pi/2$.

3. Reducton of the Problem to Integral Equations

The integral representation formula for $\phi(r, \theta)$ follows from the usual Green's function approach. Indeed, the potential $\phi(r, \theta)$ satisfying the relations (2), (3) and (6) is

$$\phi(r, \theta) = \sum_{n=0}^{3} \int_a^1 \sigma\left(r_0, \frac{n\pi}{2}\right) g\left(r, \theta \,\middle|\, r_0, \frac{n\pi}{2}\right) dr_0 \qquad \ldots(7)$$

where $\sigma(r_0, n\pi/2)$, $n = 0, 1, 2, 3$ are the unknown charge densities of the four strips and Green's function $g(r, \theta \,|\, r_0, \theta_0)$, as given by Stakgold (1968) is

$$g(r, \theta \,|\, r_0, \theta_0) = -\frac{1}{4\pi} \log\left[r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)\right]$$

$$+ \frac{1}{4\pi} \log\left[c^2 + \frac{r^2 r_0^2}{c^2} - 2rr_0 \cos(\theta - \theta_0)\right] \qquad \ldots(8)$$

If we use the boundary conditions (4) and (5) in (7) and appeal to the symmetrical relations,

$$\sigma(r_0, r) = \sigma(r_0, \pi); \qquad \sigma\left(r_0, \frac{\pi}{2}\right) = \sigma\left(r_0, \frac{3\pi}{2}\right), \qquad \ldots(9)$$

we obtain the following pair of governing simultaneous Fredholm integral equations of the first kind:

$$\int_a^1 \left\{\sigma(r_0, 0)\left[g(r, 0 \,|\, r_0, 0) + g(r, 0 \,|\, r_0, \pi)\right]\right.$$

$$\left. + 2\sigma\left(r_0, \frac{\pi}{2}\right) g\left(r, 0 \,\middle|\, r_0, \frac{\pi}{2}\right)\right\} dr_0 = V_1, \qquad a < r < 1 \qquad \ldots(10)$$

$$\int_a^1 \left\{2\sigma(r_0, 0)\, g\left(r, \frac{\pi}{2}\middle| r_0, 0\right) + \sigma\left(r_0, \frac{\pi}{2}\right)\left[g\left(r, \frac{\pi}{2}\middle| r_0, \frac{\pi}{2}\right)\right.\right.$$

$$\left.\left. + g\left(r, \frac{\pi}{2}\middle| r_0, \frac{3\pi}{2}\right)\right]\right\} dr_0 = V_2, \qquad a < r < 1. \qquad \text{...(11)}$$

We now substitute the special values of the Green's functions from (8) into (10) and (11), add and subtract the resulting relations and obtain

$$\int_a^1 \left[\sigma(r_0, 0) + \sigma\left(r_0, \frac{\pi}{2}\right)\right] M_1(r, r_0)\, dr_0 = 2\pi V, \qquad a < r < 1 \qquad \text{...(12)}$$

$$\int_a^1 \left[\sigma(r_0, 0) - \sigma\left(r_0, \frac{\pi}{2}\right)\right] M_2(r, r_0)\, dr_0 = 2\pi V_0, \qquad a < r < 1 \qquad \text{...(13)}$$

where

$$M_1(r, r_0) = -\log\left|1 - \frac{r_0^4}{r^4}\right| + \log\left(1 - \frac{r^4 r_0^4}{c^8}\right) - 4\log\left(\frac{r}{c}\right) \qquad \text{...(14)}$$

$$M_2(r, r_0) = \log\left|\frac{r^2 + r_0^2}{r^2 - r_0^2}\right| - \log\left[\frac{1 + (r^2 r_0^2/c^4)}{1 - (r^2 r_0^2/c^4)}\right] \qquad \text{...(15)}$$

$$V = V_1 + V_2, \qquad V_0 = V_1 - V_2. \qquad \text{...(16)}$$

We now present approximate solutions of the two integral equations (12) and (13) by a regular perturbation technique, when the perturbation parameter c is very large compared to unity.

4. Solutions of the Problem by Perturbation Technique

We substitute the series expansions of (14) and (15) into (12) and (13) and obtain

$$\int_a^1 \sigma_1(r_0) K_1(r, r_0)\, dr_0 = -2\pi V, \qquad a < r < 1 \qquad \text{...(17)}$$

$$\int_a^1 \sigma_2(r_0) K_2(r, r_0)\, dr_0 = 2\pi V_0, \qquad a < r < 1 \qquad \text{...(18)}$$

where

$$K_1(r, r_0) = \log|R| + 4\log\left(\frac{r}{c}\right) + \frac{r^4 r_0^4}{c^8} + \frac{r^8 r_0^8}{2c^{16}} + o(c^{-24}) \qquad \text{...(19)}$$

$$K_2(r, r_0) = \log|S| - 2\left(\frac{r^2 r_0^2}{c^4} + \frac{r^6 r_0^6}{3c^{12}}\right) + o(c^{-20}) \qquad \text{...(20)}$$

$$\sigma_1(r_0) = \sigma(r_0, 0) + \sigma\left(r_0, \frac{\pi}{2}\right); \qquad \sigma_2(r_0) = \sigma(r_0, 0) - \sigma\left(r_0, \frac{\pi}{2}\right) \qquad \text{...(21)}$$

and we have further used the functions R and S, where

$$R(r, r_0) = 1 - \frac{r_0^4}{r^4}; \qquad S(r, r_0) = \frac{r^2 + r_0^2}{r^2 - r_0^2}. \qquad \ldots(22)$$

The expansions of $K_1(r, r_0)$ and $K_2(r, r_0)$, as given by relations (19) and (20) suggest that the integral equations (17) and (18) can be solved by setting

$$\begin{aligned}\sigma_1(r_0) &= r_0^{-1} h(r_0^4)\\ &= r_0^{-1}\{h_0(r_0^4) + c^{-8} h_8(r_0^4) + c^{-16} h_{16}(r_0^4) + o(c^{-24})\} \qquad \ldots(23)\\ \sigma_2(r_0) &= 2r_0 H(r_0^4)\\ &= 2r_0\{H_0(r_0^4) + c^{-4} H_4(r_0^4) + c^{-8} H_8(r_0^4) + c^{-12} H_{12}(r_0^4) + o(c^{-16})\}. \qquad \ldots(24)\end{aligned}$$

Since V and V_0 are unknown constants, we represent them by perturbations

$$-2\pi V = A_0 + c^{-8} A_8 + c^{-16} A_{16} + o(c^{-24}) \qquad \ldots(25)$$

$$2\pi V_0 = B_0 + c^{-4} B_4 + c^{-8} B_8 + c^{-12} B_{12} + o(c^{-16}) \qquad \ldots(26)$$

Firstly, we solve eqn. (17). For this purpose, we now substitute from (19), (23) and (25), the expansions of the unknown charge densities $\sigma_1(r_0)$, the kernel $K_1(r, r_0)$ and the unknown constant potential V into (17), equate the coefficients of various powers of c and obtain the following set of Fredholm integral equations of the first kind:

$$\int_a^1 r_0^{-1} h_0(r_0^4) \log|R|\, dr_0 = A_0 - 4Q\log\left(\frac{r}{c}\right), \qquad a < r < 1 \qquad \ldots(27)$$

$$\int_a^1 r_0^{-1} h_8(r_0^4) \log|R|\, dr_0 = A_8 - r^4 \int_a^1 h_0(r_0^4)\, r_0^3\, dr_0, \qquad a < r < 1 \qquad \ldots(28)$$

$$\int_a^1 r_0^{-1} h_{16}(r_0^4) \log|R|\, dr_0 = A_{16} - r^4 \int_a^1 h_8(r_0^4)\, r_0^3\, dr_0 - \tfrac{1}{2} r^8 \int_a^1 h_0(r_0^4)\, r_0^7\, dr_0, \qquad a < r < 1 \qquad \ldots(29)$$

and so on, where the constant Q is given by

$$Q = Q_1 + Q_2. \qquad \ldots(30)$$

To effect inversions of (27) to (29), we transform to the new variables x, x_0 defined by

$$r_0^2 = x_0, \qquad r^2 = x \qquad \ldots(31)$$

and also use the contractions b and Y defined by

$$a^2 = b, \qquad Y(x, x_0) = 1 - \frac{x_0^2}{x^2}. \qquad \ldots(32)$$

Eqns. (27)–(29) then reduce to

$$\int_b^1 x_0^{-1} h_0(x_0^2) \log|Y|\, dx_0 = 2A_0 - 4Q\log\left(\frac{x}{c^2}\right), \quad b < x < 1 \quad \ldots(33)$$

$$\int_b^1 x_0^{-1} h_8(x_0^2) \log|Y|\, dx_0 = 2A_8 - x^2 \int_b^1 h_0(x_0^2)\, x_0\, dx_0, \; b < x < 1 \quad \ldots(34)$$

$$\int_b^1 x_0^{-1} h_{16}(x_0^2) \log|Y|\, dx_0 = 2A_{16} - x^2 \int_b^1 h_8(x_0^2)\, x_0\, dx_0 - \tfrac{1}{2} x^4 \int_b^1 h_0(x_0^2)\, x_0^3\, dx_0, \quad b < x < 1. \quad \ldots(35)$$

We invert these integral equations successively by using the inversion formula (A2) given in Appendix A and obtain

$$h_0(x_0^2) = \frac{4Q}{\pi T} x_0^2 - \frac{2bC_1}{\pi\lambda T} \quad \ldots(36)$$

$$h_8(x_0^2) = \frac{2b}{\pi\lambda T}\left\{2A_8 - \frac{1+b^2}{2} C_2\right\} + \frac{2C_2}{\pi T}\left(x_0^2 - \frac{1+b^2}{2}\right) x_0^2 \quad \ldots(37)$$

$$h_{16}(x_0^2) = \frac{1}{\pi T}\left\{C_4 x_0^6 + \left[C_3 - \frac{1+b^2}{2} C_4\right] x_0^4 - \left[\frac{1+b^2}{2} C_3 + \frac{(1-b^2)^2}{8} C_4\right] x_0^2\right\} + \frac{2b}{\pi\lambda T}\left\{2A_{16} - \frac{\mu}{16} C_4 - \frac{1+b^2}{4} C_3\right\} \quad \ldots(38)$$

where

$$\left.\begin{aligned} &T = \sqrt{(1-x_0^2)(x_0^2-b^2)}, \qquad \lambda = \log[(1-b)/(1+b)] \\ &C_1 = 2Q\log[(1-b^2)/4c^4] - 2A_0, \\ &C_2 = Q(1+b^2) - (b\,C_1/\lambda) \\ &C_3 = \frac{2b}{\lambda}\left(2A_8 - \frac{1+b^2}{2} C_2\right) + \frac{1}{4}(1-b^2)^2 C_2 \\ &2\mu = 3(1+b^4) + 2b^2, \qquad C_4 = \mu Q - b(1+b^2)\,C_1\,\lambda^{-1} \end{aligned}\right\} \quad \ldots(39)$$

The constants A_0, A_8, A_{16} can be readily determined from the relations

$$\int_a^1 \sigma(r_0, 0)\, dr_0 + \int_a^1 \sigma\left(r_0, \frac{\pi}{2}\right) dr_0 = Q_1 + Q_2 = Q. \quad \ldots(40)$$

Using eqns. (36)–(38) and (23) in eqn. (40), we readily obtain the values of the constants:

$$\left.\begin{aligned} A_0 &= Q \log\left(\frac{1-b^2}{4c^4}\right), \qquad A_8 = \frac{1}{4} Q (1+b^2)^2 \\ A_{16} &= \frac{1}{32} Q [\mu^2 + (1-b^4)^2] \end{aligned}\right\} \qquad \text{...(41)}$$

where we have used some definite integral formulae given in Appendix B.

Now, we revert to the original variables r, r_0 and obtain from eqns. (23), (36)–(41), the approximate expression for the sum-density

$$\begin{aligned} \sigma_1(r_0) = \frac{4(Q_1+Q_2)}{\pi T} r_0^3 \Big\{ 1 &+ \frac{1+a^4}{2c^8}\left(r_0^4 - \frac{1+a^4}{2}\right) \\ &+ \frac{1}{8c^{16}}\Big[(3(1+a^8) + 2a^4) r_0^8 - (1+a^4)^2 r_0^4 \\ &- \frac{1}{8}(1-a^4)^2(5+5a^8+6a^4)\Big] + o(c^{-24})\Big\} \end{aligned} \qquad \text{...(42)}$$

where now

$$T = \sqrt{(1-r_0^4)(r_0^4-a^4)} \qquad \text{...(43)}$$

Similarly, eqns. (25) and (41) lead to a relation between $V = V_1 + V_2$ and $Q = Q_1 + Q_2$.

$$\begin{aligned} -2\pi(V_1+V_2) = (Q_1+Q_2)\Big\{\log\Big(\frac{1-a^4}{4c^4} &+ \frac{(1+a^4)^2}{4c^8} \\ &+ \frac{\mu^2+(1-a^8)^2}{32c^{16}} + o(c^{-24})\Big)\Big\} \end{aligned} \qquad \text{...(44)}$$

We now attend to eqn. (18) briefly, since its analysis is exactly similar to that of eqn. (17) given above. When we substitute from eqns. (24) and (26) into eqn. (18), equate the coefficients of like powers of c on either side, transform to the new variables x, x_0 as defined in eqn. (31), use $Z(x, x_0)$, where

$$Z(x, x_0) = \frac{x+x_0}{x-x_0} \qquad \text{...(45)}$$

we have

$$\int_b^1 H_0(x_0^2) \log|Z|\, dx_0 = B_0, \qquad b < x < 1 \qquad \text{...(46)}$$

$$\int_b^1 H_4(x_0^2) \log|Z|\, dx_0 = B_4 + 2x \int_b^1 H_0(x_0^2)\, x_0\, dx_0, \qquad b < x < 1$$

...(47)

$$\int_b^1 H_8(x_0^2)\log|Z|\,dx_0 = B_8 + 2x\int_b^1 H_4(x_0^2)\,x_0\,dx_0, \qquad b < x < 1 \qquad \ldots(48)$$

$$\int_b^1 H_{12}(x_0^2)\log|Z|\,dx_0 = B_{12} + 2x\int_b^1 H_8(x_0^2)\,x_0\,dx_0 + \frac{2}{3}x^3\int_b^1 H_0(x_0^2)\,x_0^3\,dx_0, \; b < x < 1 \qquad \ldots(49)$$

We invert these integral equations by formula (A4) given in Appendix A, evaluate the unknown constants B_0, B_4, B_8, B_{12} from the consideration that

$$\int_a^1 \sigma_2(r_0)\,dr_0 = Q_1 - Q_2 = Q_0 \text{ (say)} \qquad \ldots(50)$$

use integral formulae from Appendix B, revert to the original variable r_0 and obtain an approximate expression for the difference density:

$$\sigma_2(r_0) = \frac{2(Q_1 - Q_2)\,r_0}{K'T}\Big\{1 - \frac{1}{c^4}\Big(\frac{E'}{K'} - r_0^4\Big) + \frac{1}{c^8 K}\Big(\frac{\pi}{2K'} - E + \frac{1-a^4}{2}K\Big)\Big(\frac{E'}{K'} - r_0^4\Big) - \frac{1}{4c^{12}}\Big[\frac{(1+a^4)^2}{3}\Big(\frac{E'}{K'} - \frac{2a^4}{1+a^4}\Big) + \frac{4E'}{K'}\Big(\frac{E'}{K'} - \frac{1+a^4}{2}\Big)^2 - \frac{4E'}{K'}\Big(\frac{E'}{K'} - 1 - a^4\Big)r_0^4 - 2(1+a^4)\,r_0^8\Big] + o(c^{-16})\Big\} \qquad \ldots(51)$$

where

$$\left.\begin{aligned} E &= \int_0^{\pi/2}\sqrt{(1 - b^2\sin^2\theta)}\,d\theta, \qquad K = \int_0^{\pi/2}\frac{d\theta}{\sqrt{(1 - b^2\sin^2\theta)}} \\ E' &= \int_0^{\pi/2}\sqrt{(1 - b'^2\sin^2\theta)}\,d\theta, \qquad K' = \int_0^{\pi/2}\frac{d\theta}{\sqrt{(1 - b'^2\sin^2\theta)}} \\ b'^2 &= 1 - b^2, \quad b = a^2 \end{aligned}\right\} \qquad \ldots(52)$$

are the complete elliptic integrals and we have also used Legendre's identity.

$$EK' + E'K - KK' = \Big(\frac{\pi}{2}\Big). \qquad \ldots(53)$$

Finally, we substitute the values of the constants B_0, B_4, B_8, B_{12} in eqn. (26) and obtain a relation between $V_0 = V_1 - V_2$ and $Q_0 = Q_1 - Q_2$

$$2\pi (V_1 - V_2) = \frac{\pi (Q_1 - Q_2) K}{K'} \Big\{ 1 - \frac{\pi c^{-4}}{2KK'} + \frac{\pi c^{-8}}{2K^2 K'} \times \Big(\frac{\pi}{2K'} - E + \frac{1 - a^4}{2} K \Big) - \frac{\pi c^{-12}}{2K^3 K'} \Big[\frac{(1 + a^4)^2}{12} K^2 + \Big(\frac{\pi}{2K'} - E + \frac{1 - a^4}{2} K \Big)^2 \Big] + o(c^{-16}) \Big\} \qquad \ldots(54)$$

When we substitute the expressions for $\sigma_1(r_0)$ and $\sigma_2(r_0)$ from eqns. (42) and (51) in eqns. (21), we readily obtain the values of the total charge densities per unit of the four strips. Similarly, results (44) and (54) yield the values of the unknown constant potentials V_1 and V_2 of the two pairs of the strips in terms of known total charges Q_1 and Q_2 per unit length of the strips.

5. Special Cases

By taking appropriate limits in our results of Section 4, we present here solutions of problems I, II and III posed at the end of Section 1.

(I) If we let $a \to 0$, $Q_1 = Q_2 = q_0$ in eqns. (42), (51), (44) and (54), we get the following corresponding results for a cross lying inside a grounded circular cylinder of radius c, when the cross is at constant potential ϕ_0 and has a total charge $4q_0$ per unit length:

$$\sigma \Big(r_0, \frac{n\pi}{2} \Big) = \frac{4 q_0 r_0}{\pi \sqrt{(1 - r_0^4)}} \Big\{ 1 + \frac{1}{2c^8} \Big(r_0^4 - \frac{1}{2} \Big) + \frac{1}{8c^{16}} \times \Big[3r_0^8 - r_0^4 - \frac{5}{8} \Big] + o\Big(\frac{1}{c^{24}} \Big) \Big\} \; n = 0, 1, 2, 3 \qquad \ldots(55)$$

$$\phi_0 = \frac{q_0}{\pi} \Big\{ \log (2c^2) - \frac{1}{8c^8} - \frac{13}{256\, c^{16}} + o\Big(\frac{1}{c^{24}} \Big) \Big\} \qquad \ldots(56)$$

It follows from (56) that the capacity C of condenser formed by the cross and the surrounding circular cylinder is given by

$$C = \frac{4q_0}{\phi_0} = \frac{4\pi}{[\log (2c^2)]} \Big\{ 1 + \frac{1}{8c^8 [\log (2c^2)]} + o\Big(\frac{1}{c^{16} [\log (2c^2)]} \Big) \Big\} \qquad \ldots(57)$$

(II) If we let $c \to \infty$ in results (42) and (51), we obtain values of the charged densities of the four non-intersecting strips lying in an unbounded medium, when the alternate strips are having charges Q_1 and Q_2 per unit length. These values are:

$$\sigma (r_0, 0) = \sigma (r_0, \pi) = \frac{r_0}{T} \Big\{ \frac{2(Q_1 + Q_2)}{\pi} r_0^2 + \frac{Q_1 - Q_2}{K'} \Big\} \qquad \ldots(58)$$

$$\sigma\left(r_0, \frac{\pi}{2}\right) = \sigma\left(r_0, \frac{3\pi}{2}\right) = \frac{r_0}{T}\left\{\frac{2(Q_1 + Q_2)}{\pi} r_0^2 - \frac{Q_1 - Q_2}{K'}\right\} \qquad \ldots(59)$$

where we have used eqn. (21). Similarly, eqns. (44) and (54) yield in the limit when $c \to \infty$, the constant values of the potentials of the strips in terms of the known charges Q_1, Q_2. These values are

$$V_1 = \frac{1}{4\pi}\left\{(Q_1 + Q_2)\log\frac{4}{(1 - a^4)} + \frac{\pi K}{K'}(Q_1 - Q_2)\right\} \qquad \ldots(60)$$

$$V_2 = \frac{1}{4\pi}\left\{(Q_1 + Q_2)\log\frac{4}{(1 - a^4)} - \frac{\pi K}{K'}(Q_1 - Q_2)\right\}. \qquad \ldots(61)$$

Observe that the term $\log c$ occurring in eqn. (44) has been dropped before taking the limit, because the value of the Green's function $G(r, \theta \mid r_0, \theta_0)$ in an unbounded medium is

$$G(r, \theta \mid r_0, \theta) = -\frac{1}{4\pi}\log\left[r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)\right]. \qquad \ldots(62)$$

and, therefore, the kernel $M_1(r, r_0)$ of integral equation (12) in this limiting case shall be

$$\left\{-\log\left|1 - \frac{r_0^4}{r^4}\right| - 4\log r\right\}.$$

If we put $Q_1 = -Q_2 = q_1$ in eqns. (58)–(61), we obtain the following corresponding results for the non-intersecting cross in an unbounded medium, when the alternate strips are at constant potentials ϕ_1 and $-\phi_1$ carrying charges q_1 and $-q_1$ per unit length:

$$\sigma(r_0, 0) = \sigma(r_0, \pi) = -\sigma\left(r_0, \frac{\pi}{2}\right) = -\sigma\left(r_0, \frac{3\pi}{2}\right)$$

$$= 2q_1 r_0 / K' \sqrt{(1 - r_0^4)(r_0^4 - a^4)} \qquad \ldots(63)$$

$$\phi_1 = Kq_1/2K'. \qquad \ldots(64)$$

If we let $Q_1 = Q_2 = q_2$ in eqns. (58)–(61), we obtain the common values of the charge density of the non-intersecting cross in an unbounded medium, when each strip is having the total charge q_2 per unit length. This value is

$$\sigma\left(r_0, \frac{n\pi}{2}\right) = \frac{4q_2 r_0^3}{\pi T}, \quad n = 0, 1, 2, 3. \qquad \ldots(65)$$

The corresponding common constant value of the potential ϕ_2 is

$$\phi_2 = \left(\frac{q_2}{2\pi}\right)\log\left(\frac{4}{1 - a^4}\right). \qquad \ldots(66)$$

It follows that the capacity C of the conductor in the form of non-intersecting cross in an unbounded medium is

$$C = \frac{4q_2}{\phi_2} = \frac{8\pi}{\log[4/(1-a^4)]}. \quad \text{...(67)}$$

(III) When we allow $a \to 0$ in results (65), (66), (67), we obtain the following corresponding results for a cross in an unbounded medium, when each strip is at the same constant potential ϕ_2 and is having charge q_2 per unit length:

$$\sigma\left(r_0, \frac{n\pi}{2}\right) = \frac{4q_2 r_0}{\pi\sqrt{(1-r_0^4)}}, \qquad n = 0, 1, 2, 3. \quad \text{...(68)}$$

$$\phi_2 = \frac{q_2}{\pi}\log 2, \qquad C = \frac{4\pi}{\log 2}. \quad \text{...(69)}$$

To the authors' knowledge, even the limiting results (55)–(69) derived from their main analysis are new.

Appendix A

1. The solution of the integral equation

$$\int_a^1 r_0^{-1} h(r_0^2) \log\left|1 - \frac{r_0^2}{r^2}\right| dr_0 = f(r), \qquad a < r < 1 \quad \text{...(A1)}$$

is given by

$$h(r_0^2) = l(r_0^2) + (C/T) \quad \text{...(A2)}$$

where

$$l(r_0^2) = \frac{2}{\pi^2}\left(\frac{r_0^2 - a^2}{1 - r_0^2}\right)^{1/2} \int_a^1 \left(\frac{1-r^2}{r^2-a^2}\right)^{1/2} \frac{r^2 f'(r)}{r^2 - r_0^2}\, dr$$

$$C = \frac{2a}{\pi\lambda}\left\{f(r) - \int_a^1 r_0^{-1} l(r_0^2) \log\left|1 - \frac{r_0^2}{r^2}\right| dr_0\right\}$$

$$T = \sqrt{(r_0^2 - a^2)(1 - r_0^2)}, \qquad \lambda = \log[(1-a)/(1+a)]$$

2. The solution of the integral equation

$$\int_a^1 H(r_0^2) \log\left|\frac{r + r_0}{r - r_0}\right| dr_0 = F(r), \qquad a < r < 1 \quad \text{...(A3)}$$

is given by

$$H(r_0^2) = m(r_0^2) + (C/T) \quad \text{...(A4)}$$

where

$$m(r_0^2) = -\frac{2}{\pi^2}\left(\frac{r_0^2-a^2}{1-r_0^2}\right)^{1/2}\int_a^1 \left(\frac{1-r^2}{r^2-a^2}\right)^{1/2}\frac{rF'(r)}{r^2-r_0^2}\,dr$$

$$C = \frac{1}{\pi K}\left\{F(r) - \int_a^1 m(r_0^2)\log\left|\frac{r+r_0}{r-r_0}\right| dr_0\right\}$$

$$T = \sqrt{(r_0^2-a^2)(1-r_0^2)}, \qquad K \equiv F\left(\frac{\pi}{2}, a\right), \qquad \text{elliptic integral.}$$

We have used the inversion formula of Srivastava and Lowengrub (1970) to obtain solutions of integral equations (A1) and (A3).

Appendix B

We give here some standard formulae of definite integrals, which have been used in our analysis.

$$\int_a^1 \frac{r_0}{T}\log|Y|\,dr_0 = \pi\left[\log(1-a^2)^{1/2} - \log 2r\right], \qquad a<r<1 \qquad \text{...(B1)}$$

$$\int_a^1 \frac{r_0^3}{T}\log|Y|\,dr_0 = \frac{\pi}{2}\left[(1+a^2)\left\{\frac{1}{2}+\log(1-a^2)^{1/2}-\log 2r\right\} - r^2\right], \qquad a<r<1 \qquad \text{...(B2)}$$

$$\int_a^1 \frac{1}{r_0 T}\log|Y|\,dr_0 = \frac{\pi}{2a}\lambda, \qquad a<r<1 \qquad \text{...(B3)}$$

$$\int_a^1 \frac{1}{r_0^3 T}\log|Y|\,dr_0 = \frac{\pi}{2a^2}\left[1+\frac{1+a^2}{2a}\lambda - \frac{a}{r^2}\right], \qquad a<r<1 \qquad \text{...(B4)}$$

$$\int_a^1 \frac{1}{T}\log|Z|\,dr_0 = \pi K(a), \qquad a<r<1 \qquad \text{...(B5)}$$

$$\int_a^1 \frac{r_0^2}{T}\log|Z|\,dr_0 = \pi\left[r + K(a) - E(a)\right], \qquad a<r<1 \qquad \text{...(B6)}$$

$$\int_a^1 \frac{r_0^4}{T} \log |Z| \, dr_0 = \pi \left[\frac{1+a^2}{2} r + \frac{1}{3} r^3 + \frac{(2+a^2)}{3} K(a) - \frac{2(1+a^2)}{3} E(a) \right], \quad a < r < 1 \quad \text{...(B7)}$$

$$\int_a^1 \frac{1}{Tr_0^2} \log |Z| \, dr_0 = \frac{\pi}{a^2} \left[\frac{a}{r} - E(a) + K(a) \right], \quad a < r < 1 \quad \text{...(B8)}$$

where

$$Y = 1 - \frac{r_0^2}{r^2}, \qquad Z = \frac{r + r_0}{r - r_0}$$

$$T = \sqrt{(1 - r_0^2)(r_0^2 - a^2)}, \qquad \lambda = \log \left(\frac{1 - a}{1 + a} \right)$$

$K(a)$ and $E(a)$ are complete elliptic integrals of the first and second kind respectively.

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