EFFECTS OF COUPLE STRESSES ON ELASTIC WAVES AND VIBRATIONS

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The present paper is concerned with the investigation of the effect of couple stresses on the axisymmetric problems of propagation of elastic waves. It is found that the velocity of propagation of waves in the case of a circular cylinder increases under the influence of couple stresses.

1. INTRODUCTION

In recent years, increasing interest has been evidenced in problems of elastic waves and vibrations in the Cosserat medium. The concept of couple stress was originally introduced by Voigt (1887) and was amplified by Cosserat (1909). A modern derivation of the Cosserat equations has been given by Truesdell and Toupin (1960), Aero and Kuvshinski (1960, 1961), Grioli (1960), Mindlin and Tiersten (1962) and Mindlin (1963). Following the linearized theory of couple stress elasticity, as presented by Mindlin and Tiersten (1962), the authors have investigated the axisymmetric problems of propagation of elastic waves. The problem of propagation of elastic waves in a cylinder is considered in Section 3. In Section 5, axisymmetric Lamb's problem is considered.

2. THE BASIC EQUATIONS

We consider an elastic homogeneous centro-symmetric isotropic body. Then the states of stresses and couple stresses are given in terms of the cartesian tensor notation (Mindlin and Tiersten 1962).

$$\left.\begin{aligned} T_{ij}^S &= 2\mu\, e_{ij} + \lambda e_{kk}\,\delta_{ij} \\ \mu_{ii}^D &= 4\eta \chi_{ij} + 4\eta' \chi_{ji} \end{aligned}\right\} \quad \ldots(1)$$

where the local strain e_{ij}, rotation ω_i, and the curvature twist tensor χ_{ij} are expressed in terms of the displacement u_i as

$$\left.\begin{aligned} e_{ij} &= \tfrac{1}{2}(u_{i,j} + u_{j,i}), \qquad \omega_i = -\tfrac{1}{2}\varepsilon_{ijk}\, u_{,jk} \\ \chi_{ij} &= \omega_{j,i} \end{aligned}\right\} \quad \ldots(2)$$

and the deviation of the couple stress tensor μ_{ij}^D is given by

$$\mu_{ij}^D = \mu_{ij} - \tfrac{1}{3}\,\mu_{kk}\,\delta_{ij}$$

λ, μ are Lame's constants and η and η', the material constants associated with resistance to curvature. Stress equation of motion, couple-stress equation of motion and linearized stress equation of motion are given by the vector form (Mindlin and Tiersten 1962)

$$\left.\begin{aligned} &\nabla \cdot \tau + \rho f = \rho \ddot{u} \\ &\nabla \cdot \vec{\mu} + \rho c + \tau \times I = 0 \\ &\nabla \cdot \vec{\tau^s} + \tfrac{1}{2} \nabla \times \nabla \cdot \vec{\mu^D} + \rho f + \tfrac{1}{2} \rho \nabla \times c = \rho \ddot{u} \end{aligned}\right\} \quad ..(3)$$

respectively, where $I (\equiv \nabla r)$ is the unit spatial dyadic. Inserting (1) in the linearized stress equation of motion $(3)_3$, we get the displacement equation of motion

$$\left.\begin{aligned} \mu \nabla^2 u + (\lambda + \mu) \nabla \cdot \nabla u + \eta \nabla^2 \nabla \times \nabla \times u + \rho f + \tfrac{1}{2} \rho \nabla \times c \\ = \rho \ddot{u} \end{aligned}\right\} \quad ...(4)$$

where f and c are the body force and the body couple vector respectively.

3. Propagation of Elastic Waves in a Cylinder

3.1. *Statement of the problem and the boundary conditions*

Introducing the cylindrical coordinates (r, ϕ, z), we consider an infinite circular cylinder with free surface $r = a$, the axis of the cylinder coinciding with the z-axis. We consider the displacement equation of motion (4) in the absence of body forces and body couples

$$\mu \nabla^2 u + (\lambda + \mu) \nabla \nabla \cdot u + \eta \nabla^2 \times \nabla \times \nabla \times u = \rho \ddot{u} \quad ...(5)$$

Then the boundary conditions on $r = a$ are given by (Mindlin and Tiersten 1962)

$$\tau_{rr} = p_\alpha = p_\beta = \mu_{r\phi} = \mu_{rz} = 0 \quad ...(6)$$

where

$$p_\alpha = \tau^S_{r\phi} - \frac{1}{2} \left(\frac{\partial}{\partial r} \mu_{rz} + \frac{1}{r} \frac{\partial}{\partial \phi} \mu_{\phi z} + \frac{\partial}{\partial z} \mu^D_{zz} + \frac{\mu_{rz}}{r} - \frac{\partial}{\partial z} \mu^D_{rr} \right)$$

$$p_\beta = \tau^S_{rz} + \frac{1}{2} \left(\frac{\partial}{\partial r} \mu_{r\phi} + \frac{1}{r} \frac{\partial}{\partial \phi} \mu^D_{\phi\phi} + \frac{\partial}{\partial z} \mu_{z\phi} + \frac{\mu_{r\phi} + \mu_{\phi r}}{r} - \frac{1}{r} \frac{\partial}{\partial \phi} \mu^D_{rr} \right)$$

when the cylinder is of finite length, the frequency of free vibration would be determined by the condition that the plane ends are free from stresses and couple stresses. We shall find that, in general, these conditions are not satisfied exactly by modes of vibration of the kind described, but they are approximately satisfied when the radius of the cylinder is small compared to its length.

3.2. *Methods of solution*

We express the displacement into its lamellar and solenoidal components.

$$u = \nabla\phi + \nabla \times H, \qquad \nabla \cdot H = 0. \qquad \ldots(7)$$

Then, the equation of motion (5) yields the following two equations:

$$\left.\begin{aligned} c_1^2\,\Delta^2\,\phi &= \ddot{\phi} \\ c_2^2\,(1 - l^2\nabla^2)\,\Delta^2\,H &= \ddot{H} \end{aligned}\right\} \qquad \ldots(8)$$

where

$$l^2 = \frac{\eta}{\mu}, \quad c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}.$$

For harmonic waves $\phi = \bar{\phi}\, e^{i\omega t}$, $H = \bar{H}e^{i\omega t}$, eqns. (8) become

$$(\nabla^2 + \sigma_1^2)\bar{\phi} = 0, \qquad (\nabla^2 + \beta_1^2)(\nabla^2 - \beta_2^2)\,\bar{H} = 0 \qquad \ldots(9)$$

where

$$\sigma_1 = \omega/c_1, \qquad \beta_1 = 2^{-1/2}\,l^{-1}\,[(1 + 4l^2\sigma_2^2)^{1/2} - 1]^{1/2}$$

$$\sigma_2 = \omega/c_2, \qquad \beta_2 = 2^{-1/2}\,l^{-1}\,[(1 + 4l^2\sigma_2^2)^{1/2} + 1]^{1/2}.$$

Then, with $\bar{H} = \bar{H}' + \bar{H}''$, the complete solution is

$$\bar{u} = \nabla\bar{\phi} + \nabla \times \bar{H}' + \nabla \times \bar{H}'' \qquad \ldots(10)$$

where $\bar{\phi}$, $\bar{H}'$ and $\bar{H}''$ are governed by the Helmholtz equations

$$(\nabla^2 + \alpha^2)\bar{\phi} = 0, \quad (\nabla^2 + \beta_1^2)\,\bar{H}', \qquad (\nabla^2 - \beta_2^2)\,\bar{H}'' = 0. \qquad \ldots(11)$$

We consider the following types of vibrations of a cylinder and investigate the effect of couple stresses on the propagation of waves in a cylinder.

Longitudinal vibrations—We assume that $u_\phi = 0$ and the displacement components u_r, u_z are independent of ϕ. It is observed in this case that $\omega_r = \omega_z = 0$ and $\omega_\phi \neq 0$.

Hence, the boundary conditions (6) on $r = a$ reduce to

$$\left.\begin{aligned} &\tau_{rz}^S + \frac{1}{2}\left(\frac{\partial}{\partial r}\,\mu_{r\phi} + \frac{\partial}{\partial z}\,\mu_{z\phi} + \frac{\mu_{r\phi} + \mu_{\phi r}}{r}\right) = 0 \\ &\tau_{rr} = \mu_{r\phi} = 0. \end{aligned}\right\} \qquad \ldots(12)$$

In this particular case, $\bar{H}_r = \bar{H}_z = 0$. Since the wave is propagating along the z-axis, we assume

$$(\bar{\phi}\,(r, z),\ \bar{H}_\phi\,(r, z)) = (\phi^*\,(r),\ H_\phi^*\,(r))\; e^{i\alpha z}. \qquad \ldots(13)$$

Further, assuming

$$H_\phi^*\,(r) = -\frac{\partial}{\partial r}\,\psi_\phi^*$$

we obtain the solution of the differential equations (11) for a complete cylinder

$$\phi^* = AJ_0(\sigma r), \quad \psi_\phi^* = BJ_0(v_1 r), \quad \psi_\phi^{**} = CI_0(v_2 r) \qquad \text{...(14)}$$

where

$$\sigma = (\sigma_1^2 - \alpha^2)^{1/2}, \qquad v_1 = (\beta_1^2 - \alpha^2)^{1/2}, \qquad v_2 = (\beta_2^2 + \alpha^2)^{1/2},$$

J_0 and I_0 are Bessel's function and Bessel's modified function of first kind and order zero.

In view of (1), (2), (7) and (14), we obtain from the boundary conditions (12)

$$\begin{aligned}
&2i\,\alpha A \frac{d}{da} J_0(\sigma a) + \{(v_1^2 - \alpha^2) + l^2 (v_1^2 + \alpha^2)^2\} B \frac{d}{da} J_0(v_1 a) \\
&\quad + \{l^2 (v_2^2 - \alpha^2)^2 - (v_2^2 + \alpha^2)\} C \frac{d}{da} I_0(v_2 a) = 0 \\
&\left\{2\mu \frac{d^2}{da^2} J_0(\sigma a) - \sigma_1^2 \lambda J_0(\sigma a)\right\} A + 2i\alpha \left\{B \frac{d^2}{da^2} J_0(v_1 a)\right. \\
&\quad \left. + C \frac{d^2}{da^2} I_0(v_2 a)\right\} = 0 \qquad \text{...(15)} \\
&\beta_1^2 \left\{\frac{d^2}{da^2} J_0(v_1 a) - \frac{\eta'}{\eta} \frac{1}{a} \frac{d}{da} J_0(v_1 a)\right\} B - \beta_2^2 \left\{\frac{d^2}{da^2} I_0(v_2 a)\right. \\
&\quad \left. - \frac{\eta'}{\eta} \frac{1}{a} \frac{d}{da} I_0(v_2 a)\right\} C = 0 .
\end{aligned}$$

Eliminating A, B and C from (15) and rearranging, we obtain the transcendental equation permitting the determination of the phase velocity of propagation of waves in an infinite cylinder, when the effects of couple stresses are taken into account.

$$0 = \left| \begin{matrix}
2\mu \dfrac{d^2}{da^2} J_0(\sigma a) - \sigma_1^2 \lambda J_0(\sigma a), \quad 2i\mu\alpha \left\{ \dfrac{d^2}{da^2} J_0(v_1 a) \right. \\
\left. + \dfrac{\beta^2}{\beta_2^2} \left(\dfrac{\dfrac{d^2}{da^2} J_0(v_1 a) - \dfrac{\eta'}{\eta} \dfrac{1}{a} \dfrac{d}{da} J_0(v_1 a)}{\dfrac{d^2}{da^2} I_0(v_2 a) - \dfrac{\eta'}{\eta} \dfrac{1}{a} \dfrac{d}{da} I_0(v_2 a)} \right) \dfrac{d^2}{da^2} I_0(v_2 a) \right\} \\
2\,i\alpha \dfrac{d}{da} J_0(\sigma a), \quad (\sigma_2^2 - 2\alpha^2) \left\{ \dfrac{d}{da} J_0(v_1 a) \right. \\
\left. + \dfrac{\beta_1^2}{\beta_2^2} \left(\dfrac{\dfrac{d^2}{da^2} J_0(v_1 a) - \dfrac{\eta'}{\eta} \dfrac{1}{a} \dfrac{d}{da} J_0(v_1 a)}{\dfrac{d^2}{da^2} I_0(v_2 a) - \dfrac{\eta'}{\eta} \dfrac{1}{a} \dfrac{d}{da} I_0(v_2 a)} \right) \dfrac{d}{da} I_0(v_2 a) \right\}
\end{matrix} \right| \qquad \text{...(16)}$$

An investigation of eqn. (16) is very complicated. We shall confine ourselves to two limiting cases.

Case I—If the radius of the rod is small, σa, $v_1 a$ are small and we may neglect the terms of order a^2. Regarding $v_2 a$, the following discussion is made.

Since σa, $v_1 a$ are small, making use of the expansion of $J_0(av)$ into series

$$J_0(av) = 1 - \frac{1}{4}(av)^2 + \frac{1}{64}(av)^4 \qquad ...(17)$$

and neglecting the terms of order a^2, we obtain from (16)

$$\begin{vmatrix} (\mu + \lambda)\dfrac{\omega^2}{c_1^2} - \mu\alpha^2, & i\mu\alpha(1 + \beta^2) \\ 2i\alpha\sigma^2 & \sigma_2^2 - 2\alpha^2 \end{vmatrix} = 0 \qquad ...(18)$$

which gives the phase velocity of propagation of waves in the simple formula

$$c = \frac{\omega}{\alpha} \approx \left\{\frac{(2\mu + 3\lambda)\mu}{(\lambda + \mu)\rho} + 2\alpha^2 l^2 \beta^2\right\}^{1/2} \qquad ...(19)$$

where

$$\beta^2 = \delta^2 \left\{\frac{1}{1 - \dfrac{\sigma'}{a}\left(\dfrac{d}{da} I_0(v_2 a) \Big/ \dfrac{d^2}{da^2} I_0(v_2 a)\right)} + \frac{1}{\sigma' - \left(a\dfrac{d^2}{da^2} I_0(v_2 a) \Big/ \dfrac{d}{da} I_0(v_2 a)\right)}\right\}$$

$$\sigma' = \frac{\eta'}{\eta}, \quad \delta^2 = \frac{1}{\rho(\mu + \lambda)}\left\{\mu(2\mu + \lambda) + \frac{(2\mu + 3\lambda)\mu^2}{\mu + \lambda}\right\}(1 - \sigma')$$

neglecting cubes and higher powers of 'l', the parameter of couple stress, which is generally small.

Now, (i) if a/l is small, $v_2 a$ is also small. Making use of the expansion of $I_0(av)$ into series

$$I_0(av) = 1 + \frac{1}{4}(av)^2 + \frac{1}{64}(av)^4 + \dots \qquad ...(20)$$

and neglecting the terms of order a^2, we obtain from (19) the phase velocity of propagation of the longitudinal waves

$$c = \frac{\omega}{\alpha} \approx \sqrt{\frac{\mu(3\lambda + 2\mu)}{\rho(\lambda + \mu)}} = \sqrt{\frac{E}{\rho}} = c^0. \qquad ...(21)$$

which is same as that without couple stress (Nowacki 1961) designated by c^0. (ii) If a/l is large, $v_2 a$ is large. Making use of the following formula for large $v_2 a$

$$I_1(va) = \frac{e^{va}}{\sqrt{(2\pi va)}} \sum_{m=0}^{\infty} \left(\frac{1}{2} - 1\right)_m \frac{\gamma(1 + m + \frac{1}{2}, 2va)}{\Gamma\frac{3}{2}\, m!\,(2va)^m} \qquad ...(22)$$

where γ denotes the 'incomplete gamma-function' of Legendre and for large values of va,

$$\frac{\gamma(1+m+\frac{1}{2},\, 2va)}{\Gamma(1+\frac{1}{2})} \sim \left(\frac{1}{2}+1\right)_m \text{ is } o\ (1) \quad \text{for each integral value of } m,$$

we obtain from (19), the approximate form of phase velocity in the simple equation

$$c = \frac{\omega}{a} \approx \left\{\frac{(2\mu + 3\lambda)\,\mu}{(\lambda+\mu)\,\rho} + 2\alpha^2\, l^2\, \delta^2\right\}. \qquad \text{...(23)}$$

Equation (23) shows that the phase velocity increases under the influence of couple stresses.

4. Numerical Results

In Table I, the percentage increase in phase velocity C has been computed for different values of $v_2 a$ in eqn. (19). From the numerical results, it is found that the phase velocity increases with increase in the values of $v_2\,a$ or in other words of a/l and attains the maximum increase of about 4% over its classical value for large $v_2 a$. In the numerical computation, we have assumed

$$\sigma' = 0{\cdot}05, \qquad \alpha^2\, l^2 = 0{\cdot}05,$$

and

$$\mu = 11{\cdot}5 \times 10^6 \text{ lb/cu in}, \qquad E = 30 \times 10^6 \text{ lb/cu in},$$

$$\rho = 7{\cdot}8 \times 62{\cdot}5 \text{ lb/cu ft}$$

Table I

Computation of the percentage increase of the phase velocity for different $v_2 a$

$v_2 a$	0·2	0·5	1·0	2·0	4·00	6·00	8·00	10·0	15·0
% increase in c	0·0507	0·2684	0·8795	2·0115	3·1748	3·5716	3·7596	3·8756	3·9908

Case II—If the radius of the rod is large, σa, $v_1 a$ and $v_2 a$ are large. Making use of the relation for large va

$$J_n(va) = \sqrt{\left(\frac{2}{\pi va}\right)} \cos\left(va - \frac{\pi}{4} - \frac{n\pi}{2}\right) \qquad \text{...(24)}$$

and (18) the transcendental equation (16) reduces to

$$\begin{vmatrix} \sigma_2{}^2 - 2\alpha^2 & 2i\alpha v_1\left(1 + \frac{\beta_1^2}{\beta_2^2}\right) \\ 2i\alpha\sigma & (\sigma_2^2 - 2\alpha^2)(1 + \beta_1^2\, v_1/\beta_2^2\, v_2) \end{vmatrix} = 0 \qquad \text{...(25)}$$

which determines the velocity of propagation of Rayleigh's surface waves under the influence of couple stresses (Sengupta and Ghosh 1975). It has been observed that the velocity of propagation of Rayleigh's surface waves increases under the influence of couple stresses (Sengupta and Ghosh 1975). It is evident that the velocity c which increases under the influence of couple stresses over its classical value varies in a wide range $c^\circ < c < c_R$, where c_R is the velocity of Rayleigh's surface waves with variation in the radius of the cylinder.

When the cylinder is terminated by two plane sections $z = 0$, and $z = l_1$ and these sections are free from stresses and couple stresses, τ_{zz}, τ_{zr} and $\mu_{z\phi}$ must vanish at $z = 0$ and $z = l_1$. Therefore, we can have free longitudinal vibrations of a circular cylinder of length l_1, in which the displacements u_r, u_z are expressed as

$$u_r = \left[A_n \frac{\partial}{\partial r} J_0(\sigma r) + \frac{in\pi}{l_1} \left\{ \frac{\partial}{\partial r} J_0(v_1^* r) B_n + c_n \frac{\partial}{\partial r} I_0(v_2^* r) \right\} \right] \sin \frac{n\pi z}{l_1} \cos(p_n t + \varepsilon)$$

$$u_z = \left[\frac{n\pi}{l_1} A_n J_0(\sigma r) - i \{ B_n v_1^{*2} J_0(v_1^* r) - c_n v_2^{*2} I_0(v_2^* r) \} \right] \cos \frac{n\pi z}{l_1} \cos(p_n t + \varepsilon) \quad \ldots(26)$$

where

$$\sigma^* = \left(\sigma_1^2 - \frac{n^2 \pi^2}{l_1^2} \right)^{1/2}, \quad v_1^* = \left(\beta_1^2 - \frac{n^2 \pi^2}{l_1^2} \right)^{1/2}, \quad v_2^* = \left(\beta_2^2 + \frac{n^2 \pi^2}{l_1^2} \right)^{1/2}$$

p_n is approximately equal to

$$\frac{n\pi}{l_1} \left\{ \frac{E}{\rho} + 2 \frac{n^2 \pi^2}{l_1^2} l^2 \beta^2 \right\}^{1/2}, \qquad \beta^2 \text{ being given by (19).}$$

The ratio of the constants A_n: B_n: C_n in (26) is known from the boundary conditions, which hold at $r = a$.

This solution (26) satisfies the condition $\tau_{zz}=0$ and $\mu_{z\phi}=0$ at $z=0$ and at $z=l_1$ but it does not satisfy the condition $\tau_{zr} = 0$ at these surfaces. However, since $\tau_{zr} = 0$ at the surface $r = a$ for all values of z, the stress τ_{zr} is very small at all points on the terminal sections $z = 0$ and $z = l_1$, when 'a' is small compared to l_1.

Torsional vibrations—Mindlin and Tiersten (1962) discussed the effect of couple stresses on the torsional vibrations of a circular cylinder. In this paper, we have investigated briefly the effect of couple stresses on the velocity of propagation of the torsional waves of a circular cylinder. In this case, we assume that $u_r = u_z = 0$ and u_ϕ is independent of ϕ. Then the solution of (5) is given by

$$u_\phi = \{ B\eta_1 J_1(\eta_1 r) - c\eta_2 I_1(\eta_2 r) \} e^{i\alpha z + i\omega t} \quad \ldots(27)$$

where

$$\eta_1 = (\beta_1^2 - \alpha^2)^{1/2}, \qquad \eta_2 = (\beta_2^2 + \alpha^2)^{1/2}.$$

From the boundary conditions (6) and (27), we obtain the frequency equation

$$\left\{\eta_1 a \frac{p}{2} \mathcal{J}_1(\eta_1 a) - 2\right\} - \frac{\eta_1^2}{\eta_2^2}\left\{(\eta_2 a)\frac{q}{2} f_1(\eta_2 a) + 2\right\} = 0 \qquad \text{...(28)}$$

where

$$p = 2(1 + \eta_1^2 l^2), \quad q = 2(\eta_2^2 l^2 - 1), \qquad \mathcal{J}_1(\eta_1 a) = \frac{J_0(\eta_1 a)}{J_1(\eta_1 a)}.$$

$$f_1(\eta_2 a) = I_0(\eta_2 a)/I_1(\eta_2 a).$$

One solution of eqn. (28) is $\eta_1 = 0$ and the corresponding form of u_ϕ is

$$u_\phi = \{Br + CI_1(\eta_2^0 r)\}\, e^{i\alpha z + i\omega t}$$

where B and C are constants and

$$\eta_2^0 = \frac{1}{l}\left(1 + 4\frac{\omega^2 l^2}{c_2^2}\right)^{1/2}.$$

Hence, we have simple harmonic motion of the type

$$u_r = 0, \qquad u_\phi = \{B \,.\, r + CI_1(\eta_0^2 r)\}\, e^{i(\alpha z + \omega t)}, \qquad u_z = 0. \qquad \text{...(29)}$$

These waves are propagated along the cylinder with velocity $(\mu/\rho)^{1/2}(1 + \alpha^2 l^2)$, neglecting higher powers of 'l' which is greater than that of the classical waves by $\alpha^2 l^2 \sqrt{\mu/\rho}$. In other words, the velocity of the torsional waves along the circular cylinder increases under the influence of couple stresses. The stress $\tau_{z\phi}$ across a normal section $z =$ constant vanishes if $\frac{\partial}{\partial z} u_\phi$ vanishes, and hence we can have free torsional vibrations of a circular cylinder of length l_1, under the influence of couple stresses in which the displacement is expressed as

$$u_\phi = \{B_n\, r + C_n I_1(\eta_2^0 r)\} \cos\frac{n\pi z}{l_1} \cos\left\{\frac{n\pi t}{l_1}\sqrt{\frac{\mu}{\rho}}\left(1 + \frac{n^2\pi^2}{l_1^2} l^2\right) + \varepsilon\right\} \qquad \text{...(30)}$$

n being any integer, and the origin being at one end.

Transverse vibrations—Another interesting solution of the displacement equation (5) in the absence of body forces and body couples is obtained by assuming that all the three components of $\bar{H}(\bar{H}_r, \bar{H}_\phi, \bar{H}_z)$ vector do not vanish and vary in accordance with the simple relations

$$\bar{H}_r = H_r^* \sin\phi\, e^{i\alpha z}, \quad \bar{H}_\phi = H_\phi^* \cos\phi\, e^{i\alpha z}, \quad \bar{H}_z = H_z^* \sin\phi\, e^{i\alpha z} \qquad \text{...(31a)}$$

and

$$\bar{\phi} = \phi^* \cos\phi\, e^{i\alpha z} \qquad \text{...(31b)}$$

In view of (31a) and (31b), we can write the solution of (11) in the form

$$\left.\begin{aligned}\bar{H}'_r &= \left\{\frac{iB}{r} J_1(\eta_1 r) + \frac{i\alpha}{\beta_1^3} C \frac{\partial}{\partial r} J_1(\eta_1 r)\right\} \sin\phi\, e^{i\alpha s} \\ \bar{H}'_\phi &= \left\{iB \frac{\partial}{\partial r} J_1(\eta_1 r) + \frac{i\alpha}{\beta_1^3} C \frac{J_1(\eta_1 r)}{r}\right\} \cos\phi\, e^{i\alpha s} \\ \bar{H}'_z &= \left\{\frac{\eta_1^2}{\beta_1^2} C J_1(\eta_1 r) \sin\phi\, e^{i\alpha s}\right\} \\ \bar{H}''_r &= \left\{\frac{iB_1}{r} I_1(\eta_2 r) + \frac{i\alpha}{\beta_2^2} C_1\, \partial/\partial r\, I_1(\eta_2 r)\right\} \sin\phi\, e^{i\alpha s} \\ \bar{H}''_\phi &= \left\{iB_1 \frac{\partial}{\partial r} I_1(\eta_2 r) + \frac{i\alpha}{\beta_2^2 \,.\, r} C_1 I_1(\eta_2 r)\right\} \cos\phi\, e^{i\alpha s} \\ \bar{H}''_z &= -\frac{\eta_2^2}{\beta_2^2} C_1 I_1(\eta_2 r) \sin\phi\, e^{i\alpha s}\end{aligned}\right\} \quad \ldots(32)$$

and

$$\bar{\phi} = A \cos\phi\, J_1(\sigma r)\, e^{i\alpha s}. \quad \ldots(33)$$

Then, we obtain from (32), (33), (9) and (10)

$$u_r = U_r \cos\phi\, e^{iv}, \qquad u_\phi = U_\phi \sin\phi\, e^{iv}, \qquad u_z = U_z \cos\phi\, e^{iv} \quad \ldots(34)$$

where

$$\begin{aligned}U_r &= \Big[A \frac{\partial}{\partial r} J_1(\sigma r) + B\alpha \frac{\partial}{\partial r} J_1(\eta_1 r) + C \frac{J_1(\eta_1 r)}{r} \\ &\quad + B_1 \alpha \frac{\partial}{\partial r} I_1(\eta_2 r) - C_1 \frac{I_1(\eta_2 r)}{r}\Big] \\ U_\phi &= -A \frac{J_1(\sigma r)}{r} - B\alpha \frac{J_1(\eta_1 r)}{r} - C \frac{\partial}{\partial r} J_1(\eta_1 r) - B_1 \alpha \frac{I_1(\eta_2 r)}{r} \\ &\quad + C_1 \frac{\partial}{\partial r} I_1(\eta_2 r) \\ U_z &= i\alpha A J_1(\sigma r) - iB\eta_1^2 J_1(\eta_1 r) + iB_1 \eta_2^2 I_1(\eta_2 r)\end{aligned} \quad \ldots(35)$$

and

$$v = \alpha z + \omega t.$$

Since $u_r \sin\phi + u_\phi \cos\phi$ vanishes when $r = 0$, the motion of the points on the axis of the cylinder takes place in the plane containing the unstrained position of that axis and line from which ϕ is measured; and, since u_z vanishes when $r = 0$, the motion of these points is at right angles to the axis of the cylinder. Hence, the vibrations are of a "transverse" or "flexural" type.

From the boundary conditions (6), we obtain

$$\left| \beta^2 \frac{\partial}{\partial r} U_r + (\beta^2 - 2)\left(\frac{U_r}{r} + \frac{U_\phi}{r} + i\alpha U_z\right)\right|_{r=a} = 0$$

$$\Bigg| \left\{\frac{\partial}{\partial r} U_\phi - \frac{1}{r}(U_r + U_\phi)\right\} - l^2\Bigg[\left(\frac{\partial^3}{\partial r^3} + \frac{2}{r}\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\frac{\partial}{\partial r} - 2\alpha^2\frac{\partial}{\partial r} - \frac{\alpha^2}{r}\right) U_\phi$$

$$+ \left(\frac{1}{r}\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\frac{\partial}{\partial r} - \frac{\alpha^2}{r}\right) U_r + i\alpha\left(\frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) U_z$$

$$- \sigma'\left\{\alpha^2 \frac{\partial}{\partial r} U_\phi + i\alpha\left(\frac{1}{r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right) U_z\right\}\Bigg]\Bigg|_{r=a} = 0$$

$$\Bigg|\left(i\alpha U_r + \frac{\partial}{\partial r} U_z\right) + l^2\Bigg[i\alpha\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{2}{r^2} - \alpha^2\right) U_r$$

$$- \left(\frac{\partial^3}{\partial r^3} - \frac{3}{r^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial r^2} + \frac{3}{r^3} - \alpha^2\frac{\partial}{\partial r}\right) U_z - i\alpha\left(\frac{2}{r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right) U_\phi$$

$$+ \sigma'\left\{\left(\frac{1}{r^2}\frac{\partial}{\partial r} - \frac{1}{r^3}\right) U_z + \frac{i\alpha}{r}\frac{\partial}{\partial r} U_\phi\right\}\Bigg]\Bigg|_{r=a} = 0 \qquad \ldots(36)$$

$$\left| i\alpha \frac{\partial}{\partial r} U_r - \frac{\partial^2}{\partial r^2} U_z - \sigma'\left\{\frac{i\alpha}{r} U_r + \frac{i\alpha}{r} U_\phi - \left(\frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) U_z\right\}\right|_{r=a} = 0$$

$$\left| \frac{\partial}{\partial r}\left(\frac{U_r}{r}\right) + \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) U_\phi + \sigma'\left(\alpha^2 U_\phi - \frac{i\alpha}{r} U_z\right)\right|_{r=a} = 0$$

where

U_r, U_ϕ, U_z are given by (35) and $\beta = \frac{c_1}{c_2}$.

Substituting the value of U_r, U_ϕ, U_z and expressing B_1 and C_1 in terms of B and C with the help of the two equations obtained from the last two equations of (36) and then equating to zero the determinant of the system of equations, we obtain the transcendental equation determining the velocities of propagation of bending waves $c = \omega/\alpha$ under the influence of couple stresses

$$\begin{vmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} \end{vmatrix} = 0 \qquad \ldots(37)$$

where

$$\Delta_{11} = \frac{J_1(\sigma a)}{a^2} \left[\{\beta^2(\sigma^2 + \alpha^2) - 2\alpha^2\} a^2 + 2(\gamma(\sigma a) - 2)\right],$$

$$\Delta_{12} = 2\alpha \frac{J_1(\eta_1 a)}{a^2}\{\gamma(\eta_1 a) - 2 + \eta_1^2 a^2\} + D_1\Delta_{14} + D_2\Delta_{15},$$

$$\Delta_{13} = -\left[2\frac{J_1(\eta_1 a)}{a^2}(\gamma(\eta_1 a) - 2) - D_3\Delta_{14} + D_4\Delta_{15}\right] ,$$

$$\Delta_{14} = \frac{2\alpha}{a^2} I_1(\eta_2 a)\{\gamma(\eta_2 a) - 2 - \eta_2^2 a^2\},$$

$$\Delta_{15} = \frac{2\alpha}{a^2} I_1(\eta_2 a)\{\gamma(\eta_2 a) - 2\}$$

$$\Delta_{21} = \frac{2}{a^2} J_1(\sigma a)\{\gamma(\sigma a) - 2\}$$

$$\Delta_{22} = \frac{2\alpha}{a^2} J_1(\eta_1 a)\{\gamma(\eta_1 a) - 2\} + l^2 \left[\frac{\alpha(\eta_1^3 + \alpha^2)}{a^2} J_1(\eta_1 a)\{\gamma(\eta_1 a) - 2\}\right.$$
$$\left. - \sigma'(\eta_1^2 + \alpha^2)\frac{\alpha}{a^2} J_1(\eta_1 a)\{\gamma(\eta_1 a) - 2\}\right] + D_1\Delta_{24} + D_2\Delta_{25}$$

$$\Delta_{23} = -\left[\{2(\gamma(\eta_1 a) - 2) + \eta_1^2 a^2\}\frac{J_1(\eta_1 a)}{a^2} + l^2\left\{\eta_1^2(\eta_1^2 + \alpha^2) J_1(\eta_1 a)\right.\right.$$
$$+ \alpha^2(\eta_1^2 a^2 + \gamma(\eta_1 a) - 2)\frac{J_1(\eta_1 a)}{a^2} - \sigma'\alpha^2\Big(\eta_1^2 J_1(\eta_1 a)$$
$$\left.\left. + \frac{J_1(\eta_1 a)}{a^2}\{\gamma(\eta_1 a) - 2\}\Big)\right\} - D_3\Delta_{24} + D_4\Delta_{25}\right]$$

$$\Delta_{24} = 2\alpha\frac{I_1(\eta_2 a)}{a^2}(\gamma(\eta_2 a) - 2) + l^2\left\{\frac{\alpha}{a^2}(\alpha^2 - \eta_2^2) I_1(\eta_2 a)(\gamma(\eta_2 a) - 2)\right.$$
$$\left. + \sigma'(\eta_2^2 - \alpha^2)\frac{\alpha}{a^2} I_1(\eta_2 a)(\gamma(\eta_2 a) - 2)\right\}$$

$$\Delta_{25} = \{2(\gamma(\eta_2 a) - 2) - \eta_2^2 a^2\}\frac{I_1(\eta_2 a)}{a^2} + l^2\Big[\eta_2^2(\eta_2^2 - \alpha^2) I_1(\eta_2 a)$$
$$- \alpha^2\{\eta_2^2 a^2 - (\gamma(\eta_2 a) - 2)\}\frac{I_1(\eta_2 a)}{a^2}$$
$$+ \sigma'\alpha^2\left\{\eta_2 I_1(\eta_2 a) - \frac{1}{a^2} I_1(\eta_2 a)(\gamma(\eta_2 a) - 2)\right\}\Big],$$

$$\Delta_{31} = \frac{2i\alpha}{a} J_1(\sigma a)(\gamma(\sigma a) - 1)$$

$$\Delta_{32} = -i(\eta_1^2 - \alpha^2)\frac{1}{a}(\gamma(\eta_1 a) - 1) J_1(\eta_1 a)$$
$$- il^2\Big[(\eta_1^2 + \alpha^2)\left\{\frac{1}{a}(\eta_1^2 + \alpha^2) J_1(\eta_1 a)(\gamma(\eta_1 a) - 1)\right.$$
$$\left. + \frac{1}{a^3}(\gamma(\eta_1 a) - 2)\right\} - \frac{\sigma'}{a^3}(\eta_1^2 - \alpha^2)(\gamma(\eta_1 a) - 2) J_1(\eta_1 a)\Big]$$
$$+ D_1\Delta_{34} - D_2\Delta_{35},$$

$$\Delta_{33} = \frac{i\alpha}{a} J_1(\eta_1 a) + il^2\Big[\alpha\Big\{\frac{J_1(\eta_1 a)}{a^3}(\gamma(\eta_1 a) - 2) - \frac{\alpha^2}{a} J_1(\eta_1 a)\Big\}$$
$$+ \sigma' \frac{\alpha}{a}\{\eta_1^2 a^2 + (\gamma(\eta_1 a) - 2)\}\frac{J_1(\eta_1 a)}{a^2}\Big] + D_3 \Delta_{34} + D_4 \Delta_{35}.$$

$$\Delta_{34} = i\Big[(\eta_2^2 + \alpha^2)\frac{I_1(\eta_2 a)}{a}(\gamma(\eta_2 a) - 1)$$
$$+ l^2\Big\{(\alpha^2 - \eta_2^2)(\eta_2^2 - \alpha^2)\frac{1}{a} I_1(\eta_2 a)(\gamma(\eta_2 a) - 1)$$
$$- \frac{1}{a^3}(\gamma(\eta_2 a) - 2) + \sigma' \frac{\eta_2^2 + \alpha^2}{a^3}(\gamma(\eta_2 a) - 2) I_1(\eta_2 a)\Big\}\Big],$$

$$\Delta_{35} = -i\Big[\frac{\alpha}{a} I_1(\eta_2 a) + l^2\Big\{\alpha\Big(\frac{I_1(\eta_2 a)}{a^3}(\gamma(\eta_2 a) - 2) - \frac{\alpha^2}{a} I_1(\eta_2 a)\Big)$$
$$+ \sigma' \frac{\alpha}{a}(\eta_2^2 a^2 - \{\gamma(\eta_2 a) - 2\})\frac{I_1(\eta_2 a)}{a^2}\Big\}\Big],$$

$$D_1 = (\Delta_{42} - \Delta_8)/\Delta_{44}, \qquad D_2 = \Delta_8/\Delta_{45}, \qquad D_3 = \Delta_7/\Delta_{44},$$

$$D_4 = \frac{1}{\Delta_{45}}(\Delta_{43} + \Delta_7), \qquad \Delta_7 = (\Delta_{53} - \Delta_{43}\Delta_{55})/\Delta \,.\, \Delta_{45}.$$

$$\Delta_8 = \sigma'\Big\{\alpha\beta_1^2 J_1(\eta_1 a) - \frac{\Delta_{42}}{\Delta_{44}}\alpha\beta_2^2 I_1(\eta_2 a)\Big\}\Big/\Delta$$

$$\Delta = \frac{\Delta_{55}}{\Delta_{45}} - \{\sigma' \alpha \beta_2^2 I_1(\eta_2 a)\}/\Delta_{44},$$

$$\Delta_{42} = [\beta_1^2\{\eta_1^2 a^2 + (\gamma(\eta_1 a) - 2)\} + \sigma'(\eta_1^2 + a^2)(\gamma(\eta_1 a) - 2)] J_1(\eta_1 a)/a^2$$

$$\Delta_{43} = \frac{\alpha}{a^2}\{(\gamma(\eta_1 a) - 2) - \sigma'(\gamma(\eta_1 a) - 2)\} J_1(\eta_1 a),$$

$$\Delta_{44} = \beta_2{}^2\{\eta_2^2 a^2 - (\gamma(\eta_2 a) - 2)\}\frac{I_1(\eta_2 a)}{a^2} - \sigma' \beta_2^2(\gamma(\eta_2 a) - 2)\frac{I_1(\eta_2 a)}{a^2}$$

$$\Delta_{45} = \frac{\alpha}{a^2}(\gamma(\eta_2 a) - 2) I_1(\eta_2 a) + \sigma'(\gamma(\eta_2 a) - 2) I_1(\eta_2 a),$$

$$\Delta_{53} = (\eta_1^2 - \sigma' \alpha^2)(\gamma(\eta_1 a) - 1)\frac{J_1(\eta_1 a)}{a}$$

$$\Delta_{55} = (\eta_2^2 + \sigma' \alpha^2)(\gamma(\eta_2 a) - 1)\frac{I_1(\eta_2 a)}{a},$$

$$\gamma(\sigma a) = \sigma a \frac{J_0(\sigma a)}{J_1(\sigma a)},$$

$$\gamma(\eta_1 a) = \eta_1 a \frac{J_0(\eta_1 a)}{J_1(\eta_1 a)},$$

$$\gamma(\eta_2 a) = \eta_2 a \frac{I_0(\eta_2 a)}{I_1(\eta_2 a)}.$$

Investigation of eqn. (37) is very cumbersome, but this equation may be analysed to study the effect of couple stresses on the said classical problem by assuming the parameter of couple stress to be so small that its cubes and higher powers, and $\sigma' l^2$ are always neglected. By expanding the Bessel's equation in series, when the radius a of the cylinder is very small, we obtain, on simplification, the approximate form of eqn. (37),

$$\begin{vmatrix} \dfrac{c^2}{2c_2^2}-1 & -\left(\dfrac{c^2}{c_2^2}-1\right)\left(\dfrac{7}{8}\dfrac{c^2}{c_2^2}+\dfrac{3}{4}\right)\times\alpha^2 l^2 & \left(\dfrac{c^2}{c_2}-1\right)-\dfrac{l^2\alpha^2}{4}\times\left(\dfrac{c^2}{c_2^2}+3\right) \\ -1 & \left(\dfrac{c^2}{2c_1^2}+\dfrac{\alpha^2 a^2}{2}\dfrac{c^2}{c_2^2}-1\right)\times\left(1-\dfrac{c^2}{4c_2^2}\right)+p_1\alpha^2 l^2 & -\left\{\dfrac{\alpha^2 a^2}{2}\left(\dfrac{c^2}{c_2^2}-1\right)+1\right\}+p_2 l^2\alpha^2 \\ 1-\dfrac{1}{4}\alpha^2 a^2\left(\dfrac{c^2}{c_1^2}-1\right), & -\dfrac{c^2}{2c_2^2}+\dfrac{\alpha^2 a^2}{4}\left(\dfrac{c^2}{c_2^2}-1\right)\times\left(\dfrac{c^2}{2c_2^2}-1\right)+p_3\alpha^2 l^2 & 1-\dfrac{\alpha^2 l^2}{4}\left(\dfrac{c^2}{c_2^2}+3\right) \end{vmatrix}=0 \quad \ldots(38)$$

where

$$c=\omega/\alpha,$$

$$p_1=\left(\frac{c^6}{8c_2^6}+\frac{3c^4}{8c_2^4}-\frac{1}{8}\frac{c^2}{c_2}-\frac{3}{4}\right)\alpha^2 a^2-\frac{1}{2}\left(\frac{c^2}{c_2^2}+1\right),$$

$$p_2=\left(-\frac{1}{2}\frac{c^4}{c_2^4}+\frac{1}{4}\frac{c^2}{c_2^2}+\frac{3}{4}\right)\alpha^2 a^2+\frac{1}{4}\left(\frac{c^2}{c_2^2}+3\right)$$

$$p_3=\left(-\frac{c^6}{8c_2^6}+\frac{c^4}{4c_2^4}\right)\alpha^2 a^2+\frac{1}{2}\left(\frac{c^2}{c_2^2}+1\right).$$

It is found from the transcendental equation (38) that the velocity of propagation of transverse waves is nearly

$$c^2=\omega^2/\alpha^2\approx\frac{\alpha^2 a^2}{4}\frac{E}{\rho}+\frac{7}{4}\,v\,\frac{\mu}{\rho}\,\alpha^2 l^2 \quad \ldots(39)$$

where E is Young's modulus and v is the Poisson's ratio.

This velocity (39) is greater than the velocity of propagation of transverse waves $c=\omega/\alpha\approx(\frac{1}{4}\,\alpha^2 a^2 E/\rho)^{1/2}$ (Love 1952) in the absence of couple stress. In other words, the velocity of transverse waves increases under the influence of couple stresses.

When the cylinder is terminated by two normal sections $z=0$ and $z=l_1$, writing m/l_1 for the real positive fourth root of $\omega^2\rho/(1/4\,a^2 E+7/4\,v\mu l^2)$, we can obtain four forms of solutions by substituting the four quantities $\pm\, m/l_1$ and

$\pm\, im/l_1$ successively in (34). For the same value of ω, the ratios of the constants $A : B : C : B_1 : C_1$ in each solution can be calculated from the conditions which hold at $r = a$. As in the problem of longitudinal vibrations, in this case also, the condition that the stress τ_{zr} vanishes at the ends is satisfied approximately, when the cylinder is thin.

5. Axisymmetric Lamb's Problem

Statement of the Problem and the Boundary Conditions

We consider a time varying loading $z(r, t) = P(r) \exp(i\omega t)$ acting on the elastic semi-space bounded by the $z = 0$ plane, z-axis being pointed in the medium. The loading being axially symmetrical, it produces in the semi-space an axisymmetric state of stress and deformation, and the cylindrical coordinates (r, ϕ, z) is used to investigate the problem. The boundary conditions on $z = 0$,

$$\left.\begin{aligned} &\tau_{zz} = -P(r)\, e^{i\omega t} \\ &p_r = p_\phi = \mu_{z\phi} = \mu_{zr} = 0 \end{aligned}\right\} \qquad \ldots(40)$$

where

$$p_r = \tau^s_{zr} - \frac{1}{2}\left(\frac{\partial}{\partial r}\mu_{r\phi} + \frac{1}{r}\frac{\partial}{\partial \phi}\mu^D_{\phi\phi} + \frac{\partial}{\partial z}\mu_{z\phi} + \frac{\mu_{r\phi} + \mu_{\phi r}}{r} - \frac{1}{r}\frac{\partial}{\partial \phi}\mu^D_{zz}\right)$$

$$p_\phi = \tau^s_{z\phi} + \frac{1}{2}\left(\frac{\partial}{\partial r}\mu^D_{rr} + \frac{1}{r}\frac{\partial}{\partial \phi}\mu_{\phi r} + \frac{\partial}{\partial z}\mu_{zr} + \frac{\mu^D_{rr} - \mu^D_{\phi\phi}}{r} - \frac{\partial}{\partial r}\mu^D_{zz}\right).$$

We consider the particular case in which the external loading and the displacement vector $\boldsymbol{u}$ are independent of ϕ, the body forces and body couples being discarded.

In this case,

$$\omega_r = \omega_z = 0, \quad p_r = \mu_{zr} = 0.$$

Hence, the boundary conditions reduce to

$$\left.\begin{aligned} &\tau^s_{zr} - \frac{1}{2}\left(\frac{\partial}{\partial r}\mu_{r\phi} + \frac{\partial}{\partial z}\mu_{z\phi} + \frac{\mu_{r\phi} + \mu_{\phi r}}{r}\right) = 0 \\ &\tau_{zz} = -P(r)\, e^{i\omega t}, \\ &\mu_{z\phi} = 0. \end{aligned}\right\} \qquad \ldots(41)$$

With these boundary conditions (41), we seek solutions of (11)

$$(\nabla^2 + \sigma_1^2)\bar{\phi} = 0, \qquad (\nabla^2 + \beta_1^2)\bar{H}' = 0, \qquad (\nabla^2 - \beta_2^2)\bar{H}'' = 0.$$

In this particular case, $\bar{H}_r = \bar{H}_z = 0$. Hence, these equations take the form

$$\left.\begin{aligned} &(\nabla^2 + \sigma_1^2)\bar{\phi} = 0 \\ &\left(\nabla^2 - \frac{1}{r^2} + \beta_1^2\right)\bar{H}'_\phi = 0 \\ &\left(\nabla^2 - \frac{1}{r^2} - \beta_2^2\right)\bar{H}''_\phi = 0 \end{aligned}\right\} \qquad \ldots(42)$$

Applying in eqn. (42), the Hankel transformation defined by

$$\left.\begin{aligned} \phi^*(\alpha, z) &= \int_0^\infty r\bar{\phi}(r, z) J_0(\alpha r)\, dr \\ \psi^*(\alpha, z) &= \int_0^\infty r\bar{\psi}(r, z) J_0(\alpha r)\, dr \end{aligned}\right\} \qquad \ldots(43)$$

where

$$\bar{H}'_\phi = -\frac{\partial}{\partial r}\bar{\psi}_1, \qquad \bar{H}''_\phi = -\frac{\partial}{\partial r}\bar{\psi}_2,$$

we obtain

$$\phi^* = Ae^{-\sigma z}, \qquad \psi_1^* = Be^{-v_1 z}, \qquad \psi_2{}^* = Ce^{-v_2 z} \qquad \ldots(44)$$

where

$$\sigma = (\alpha^2 - \sigma_1^2)^{1/2}, \qquad v_1 = (\alpha^2 - \beta_1^2)^{1/2}, \qquad v_2 = (\alpha^2 + \beta_2^2)^{1/2}.$$

Hence, in view of (7), (10) and (44), we obtain

$$u_r = e^{i\omega t}\int_0^\infty \{-\alpha Ae^{-\sigma z} + \alpha(v_1 Be^{-v_1 z} + v_2 Ce^{-v_2 z})\}\, \alpha J_1(\alpha r)\, d\alpha \qquad \ldots(45)$$

$$u_z = e^{i\omega t}\int_0^\infty \{-\sigma Ae^{-\sigma z} + \alpha^2(Be^{-v_1 z} + Ce^{-v_2 z})\}\, \alpha J_0(\alpha r)\, d\alpha. \qquad \ldots(46)$$

When (45) and (46) are substituted in the boundary conditions (41), there results

$$\left.\begin{aligned} &2\sigma A - \{(v_1^2 + \alpha^2) - l^2(v_1^2 - \alpha^2)^2\}\,\alpha\beta - \{(v_1^2 + \alpha^2) - l^2(v_2^2 - \alpha^2)^2\} \\ &\qquad\qquad \alpha C = 0 \\ &(2\alpha^2 - \sigma_2^1)A - 2\alpha^2(v_1 B + v_2 C) = -P^*(\alpha)/\mu \\ &\alpha v_1(\alpha^2 - v_1^2)B + \alpha v_2)(\alpha^2 - v_2^2)C = 0 \end{aligned}\right\} \ldots(47)$$

where we have expressed the loading $P(r)$ by the Hankel integral

$$\left.\begin{aligned} P(r) &= \int_0^\infty P^*(\alpha)\,\alpha J_0(\alpha r)\,d\alpha \\ P^*(\alpha) &= \int_0^\infty r\,P(r)\,J_0(\alpha r)\,dr \end{aligned}\right\} . \qquad \ldots(48)$$

Solving the system of equations (47), we obtain

$$A(\alpha) = -\frac{KP^*}{R(\alpha)\,.\,\mu}, \qquad B(\alpha) = -\frac{2\sigma}{R(\alpha)}\cdot\frac{P^*}{\mu},$$

$$C(\alpha) = \frac{v_1\beta_1^2}{v_2\beta_2^2}\,B(\alpha) \qquad \ldots(49)$$

where

$$R(\alpha) = K(2\alpha^2 - \sigma_2^2) - 4\alpha^2 v_1 \sigma \left(1 + \frac{\beta_1^2}{\beta_2^2}\right),$$

$$K = (2\alpha^2 - \sigma_2^2)\left(1 + \frac{v_1\beta_1^2}{v_2\beta_2^2}\right).$$

Hence, for a concentrated force $z(r, t)$

$$\left.\begin{aligned} u_r &= \frac{P_0 e^{i\omega t}}{2\pi\mu}\int_0^\infty \left\{K\alpha e^{-\sigma z} - 2\sigma\, v_1\, \alpha\left(e^{-v_1 z} + \frac{\beta_1^2}{\beta_2^2}\, e^{-v_2 z}\right)\right\}\frac{\alpha J_1(\alpha r)}{R(\alpha)}\,d\alpha \\ u_z &= \frac{P_0 e^{i\omega t}}{2\pi\mu}\int_0^\infty \left\{K\sigma e^{-\sigma z} - 2\sigma\,\alpha^2\left(e^{-v_1 z} + \frac{v_1}{v_2}\frac{\beta_1^2}{\beta_2^2}\, e^{-v_2 z}\right)\right\}\frac{\alpha J_0(\alpha r)}{R(\alpha)}\,d\alpha \end{aligned}\right\} \qquad \ldots(50)$$

$$z(r, t) = \frac{P_0\,\delta(r)}{2\pi r}\, e^{i\omega t}, \qquad P^*(\alpha) = \frac{P_0}{2\pi}.$$

Thus, the displacement field given by (50) under the influence of couple stresses being known, we can determine the stresses and the couple stresses from eqn. (1). If l goes to zero (absence of couple stress), the classical results follow (Nowacki 1961).

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