ALGEBRA OF SPACE-MATTER TENSOR IN GENERAL RELATIVITY*

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In the present paper, some algebraic properties of the space-matter tensor have been discussed in the bivector formalism within the framework of general relativity, and an attempt has been made to calculate the invariants of space-matter tensor in empty space-time. Finally, a covariant criterion for the existence of gravitational radiation is given.

1. INTRODUCTION

We assume that the metric

$$ds^2 = g_{ab}\, dx^a\, dx^b$$

of the space-time V_4 is reducible at a point to the Galilian form

$$ds^2 = -(dx_1)^2 - (dx_2)^2 - (dx_3)^2 + (dx_4)^2.$$

Let Einstein's field equations be

$$R_{ab} - \tfrac{1}{2} Rg_{ab} = \lambda\, T_{ab} \qquad ...(1.1)$$

where $\lambda = -8\pi G/c^4$ is a constant and T_{ab} is the energy-momentum tensor. On contraction, (1.1) yields

$$\lambda T = -R. \qquad ...(1.2)$$

Introduce a fourth order tensor (Petrov 1969)

$$H_{abcd} = \lambda/2\,(g_{ac}\, T_{bd} + g_{bd}\, T_{ac} - g_{ad}\, T_{bc} - g_{bc}\, T_{ad}). \qquad ...(1.3)$$

From the definition, this tensor has the following properties:

$$H_{abcd} = -H_{bacd} = -H_{abdc} = H_{cdab} \qquad ...(1.4)$$

$$H_{abcd} + H_{acbd} + H_{adbc} = 0. \qquad ...(1.5)$$

Contraction (1.3) over b and d yields

$$H_{ac} = \lambda\, T_{ac} + \lambda/2\, Tg_{ac} = \lambda T_{ac} - R/2\; g_{ac}. \qquad ...(1.6)$$

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Define a new fourth order tensor (Petrov 1969)

$$P_{abcd} = R_{abcd} - H_{abcd} + \sigma (g_{ac} g_{bd} - g_{ad} g_{bc}). \quad ...(1.7)$$

This tensor is known as space-matter [tensor. The first part of this tensor represents the curvature of the space and the second part, the distribution and motion of the matter. This tensor has the following properties:

(i) $P_{abcd} = - P_{bacd} = - P_{abdc} = P_{cdab}$.

(ii) $P_{abcd} + P_{acdb} + P_{adbc} = 0$.

(iii) $P_{ac} = R_{ac} - \lambda T_{ac} + \frac{1}{2} R g_{ac} + 3\sigma g_{ac} = (R + 3\sigma) g_{ac}$.

(iv) If the distribution and motion of the matter, i.e., T_{ab} and the space-matter tensor P_{abcd} are given, then R_{abcd}, the curvature of the space, is determined within the scalar σ.

(v) If $T_{ab} = 0$ and $\sigma = 0$, then P_{abcd} is the curvature of the empty space-time.

(vi) If g_{ab}, the metric tensor, σ, the scaler and P_{abcd} are known, then T_{ab} can be determined uniquely.

Recently, it has been shown by Ahsan (1977) that the space-matter tensor may be decomposed in the following form:

$$P_{abcd} = C_{abcd} + (g_{ad} R_{bc} + g_{bc} R_{ad} - g_{ac} R_{bd} - g_{bd} R_{ac}) + (\tfrac{2}{3} R + \sigma)(g_{ac} g_{bd} - g_{ad} g_{bc}). \quad ...(1.8)$$

In the present paper, some algebraic properties of the space-matter tensor have been discussed in the bivector formalism and an attempt has been made to calculate the invariants of the space-matter tensor in empty space. Finally, a covariant criterion for the existence of the gravitational radiation in empty space-time has been formulated.

2. Bivector Formalism

From the symmetry properties of the space-matter tensor P_{abcd} we may introduce the notion of a bivector. A bivector is a second rank antisymmetric tensor

$$w_{ab} = - w_{ba}. \quad ...(2.1)$$

[For a more detailed properties of bivectors, *see* Greenberg (1972)]. Now let $l^a, m^a, n^a, \bar{n}^a$ be the four null vectors at a point of the space-time such that

$$l^a m_a = n^a \bar{n}_a = 1$$

$$l^a l_a = m^a m_a = n^a n_a = \bar{n}^a \bar{n}_a = 0$$

$$l^a n_a = l^a \bar{n}_a = m^a n_a = m^a \bar{n}_a = 0. \quad ...(2.2)$$

Here, l^a and m^a are real, while n^a and $\bar{n}^a$ are complex vectors defined by

$$n^a = 1/\sqrt{2}\,(p^a + iq^a)$$
$$\bar{n}^a = 1/\sqrt{2}\,(p^a - iq^a) \qquad \text{...(2.3)}$$

where p^a and q^a are real space-like vectors, such that

$$p^a p_a = q^a q_a = 1$$
$$p^a q_a = p^a l_a = p^a m_a = q^a l_a = q^a m_a = 0. \qquad \text{...(2.4)}$$

The vectors l^a, m^a, n^a, $\bar{n}^a$ are linearly independent and thus form a basis for the Minkowskian space at a point.

At a point of the space-time, introduce three complex bivectors, A_{ab}, B_{ab} and C_{ab} as follows (Sachs 1961).:

$$A_{ab} = l_a \bar{n}_b - \bar{n}_a l_b,$$
$$B_{ab} = m_a n_b - m_b n_a,$$
$$C_{ab} = l_a m_b - m_a l_b + \bar{n}_a n_b - n_a \bar{n}_b. \qquad \text{...(2.5)}$$

These bivectors have the following properties:

$$A^{ab} B_{ab} = 2, \qquad C^{ab} C_{ab} = -4,$$
$$A^{ab} A_{ab} = B^{ab} B_{ab} = C^{ab} A_{ab} = C^{ab} B_{ab} = 0. \qquad \text{...(2.6)}$$

3. Invariants

In this section, our aim is to find out the invariants of the space-matter tensor P_{abcd} in empty space-time in terms of the bivector formalism developed in Section 2.

In eqn. (1.8), if the Ricci tensor and the scalar σ are zero, then the space-matter tensor P_{abcb} reduces to the Weyl tensor.

Now, using the properties of the space-matter tensor and those of bivectors, we obtain, after lengthy calculations

$$\begin{aligned} P_{abcd} &+ iP^*_{abcd} \\ &= P_1 A_{ab} A_{cd} + P_2 (A_{ab} C_{cd} + C_{ab} A_{cd}) + P_3 (C_{ab} C_{cd} \\ &\quad + B_{ab} A_{cd} + A_{ab} B_{cd}) + P_4 (B_{ab} C_{cd} + C_{ab} B_{cd}) \\ &\quad + P_5\, B_{ab} B_{cd} \end{aligned} \qquad \text{...(3.1)}$$

where '*' denotes the dual of the space-matter tensor and

$$P_1 = 1/2\, P_{abcd} B^{ab} B^{cd},$$
$$P_2 = -1/4\, P_{abcd} B^{ab} C^{cd},$$

$$P_3 = \frac{1}{12} P_{abcd} (C^{ab} C^{cd} + 2B^{ab} A^{cd}),$$

$$P_4 = -\tfrac{1}{4} P_{abcd} A^{ab} C^{cd},$$

$$P_5 = \tfrac{1}{2} P_{abcd} A^{ab} A^{cd}. \qquad ...(3.2)$$

It may be noticed that, in empty space-time, space-matter tensor has five complex (ten real) independent components.

The component form for the invariants of space-matter tensor in empty space-time is derived by considering the characteristic value equation

$$(P_{abcd} + iP^{*}_{abcd})\, w^{cd} = \lambda\, w_{ab} = \lambda I_{abcd}\, w^{cd} \qquad ...(3.3)$$

where λ is the characteristic value, w^{cd} is a complex characteristic bivector, such that $w^{*}_{ab} = -iw_{ab}$, and

$$I_{abcd} = \tfrac{1}{2}(A_{ab} B_{cd} + B_{ab} A_{cd} - \tfrac{1}{2} C_{ab} C_{cd}) \qquad ...(3.4)$$

is the identity bivector (Greenberg 1972).

From eqns. (3.1), (3.3) and (3.4), after rearrangement of terms, we get

$$[P_1 A_{ab} A_{cd} + P_2 (A_{ab} C_{cd} + C_{ab} A_{cd}) + (P_3 + \tfrac{1}{4}\lambda) C_{ab} C_{cd} + (P_3 - \tfrac{1}{2}\lambda)$$
$$(A_{ab} B_{cd} + B_{ab} A_{cd}) + P_4 (B_{ab} C_{cd} + C_{ab} B_{cd}) + P_5 B_{ab} B_{cd}]\, w^{cd} = 0. \qquad ...(3.5)$$

Multiplying eqn. (3.5) successively by A^{ab}, B^{ab}, and C^{ab}, we obtain

$$P_1 (A_{cd} w^{cd}) + (P_3 - \tfrac{1}{2}\lambda)(B_{cd} w^{cd}) + P_2 (C_{cd} w^{cd}) = 0.$$

$$(P_3 - \tfrac{1}{2}\lambda)(A_{cd} w^{cd}) + P_5 (B_{cd} w^{cd}) + P_4 (C_{cd} w^{cd}) = 0.$$

$$P_2 (A_{cd} w^{cd}) + P_4 (B_{cd} w^{cd}) + (P_3 + \tfrac{1}{4}\lambda)(C_{cd} w^{cd}) = 0. \qquad ...(3.6)$$

For a non-trivial solution of w^{cd}, we must have

$$\begin{bmatrix} P_1 & P_3 - \tfrac{1}{2}\lambda & P_2 \\ P_3 - \tfrac{1}{2}\lambda & P_5 & P_4 \\ P_2 & P_4 & P_3 + \tfrac{1}{4}\lambda \end{bmatrix} = 0. \qquad ...(3.7)$$

Interchanging the first two columns and multiplying the first and second row by 2 and the third row by -4 and expanding the resulting determinant, we obtain

$$\lambda^3 - L\lambda - M = 0 \qquad ...(3.8)$$

where L and M are the invariants (complex) of the space-matter tensor for empty space-time and are as follows:

$$L = 4(3P_3^2 - 4P_2P_4 + P_1P_5) \qquad ...(3.9)$$

$$M = 16(-P_3^3 + 2P_2P_3P_4 + P_1P_3P_5 - P_1P_4^2 - P_2^2P_5). \qquad ...(3.10)$$

In general, there are three solutions of eqn. (3.8). Denote them by

$$\lambda = (\lambda_1, \lambda_2, \lambda_3)$$

then

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0.$$

Expanding and comparing the coefficients of λ, we get

$$\lambda_1 + \lambda_2 + \lambda_3 = 0$$
$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = L.$$
$$\lambda_1\lambda_2\lambda_3 = -M.$$

We now derive the associated Cayley-Hamilton equation in order to obtain the covariant form of the invariants. Suppose B_{abcd} is any bivector of rank 2, such that

$$B_{ab\overset{*}{cd}} = -iB_{abcd} \quad \text{and} \quad B^{*}_{abcd} = -iB_{abcd}$$

and has the characteristic values as λ_1, λ_2 and λ_3. Then we may write

$$(B - \lambda_1 I)(B - \lambda_2 I)(B - \lambda_3 I) = 0. \qquad ...(3.11)$$

Expanding eqn. (3.11), we obtain

$$B^3 - (\lambda_1 + \lambda_2 + \lambda_3) B^2 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3) B - \lambda_1\lambda_2\lambda_3 I = 0. \qquad ...(3.12)$$

In the case of space-matter tensor P_{abcd} in empty space-time, eqn. (3.12) becomes

$$(P_{abmn} + iP^{*}_{abmn})(P^{mnrs} + i\overset{*}{P}{}^{mnrs})(P_{rscd} + iP^{*}_{rscd})$$
$$- L(P_{abcd} + iP^{*}_{abcd}) - MI_{abcd} = 0. \qquad ...(3.13)$$

Now, suppose $L = L_1 + iL_2$ and $M = M_1 + iM_2$. Using the properties of identity bivectors (Greenberg 1972) and equating the real and imaginary parts of eqn. (3.13) to zero, we get

$$P_{abmn} P^{mnrs} P_{rscd} - P^{*}_{abmn} \overset{*}{P}{}^{mnrs} P_{rscd} - P_{abmn} \overset{*}{P}{}^{mnrs} P^{*}_{rscd}$$
$$- P^{*}_{abmn} P^{mnrs} P^{*}_{rscd} - L_1 P_{abcd} + L_2 P^{*}_{abcd}$$
$$- 1/4 M_1 (g_{ac} g_{bd} - g_{ad} g_{bc}) + 1/4 M_2 \eta_{abcd} = 0. \qquad ...(3.14)$$

and

$$P_{abmn} P^{mnrs} P^{*}_{rscd} - P^{*}_{abmn} \overset{*}{P}{}^{mnrs} P^{*}_{rscd} + P_{abmn} \overset{*}{P}{}^{abrs} P_{rscd}$$
$$+ P^{*}_{abmn} P^{mnrs} P_{rscd} - L_1 P^{*}_{abcd} - L_2 P_{abcd} - 1/4 M_1 \eta_{abcd}$$
$$- 1/4 M_2 (g_{ac} g_{bd} - g_{ad} g_{bc}) = 0. \qquad ...(3.15)$$

Making use of the properties of the duals, eqns. (3.14) and (3.15) may be further simplified and we finally obtain the Cayley-Hamilton equations in the following form:

$$4 P_{abmn} P^{mnrs} P_{rscd} - L_1 P_{abcd} + L_2 P^{*}_{abcd}$$
$$- 1/4 M_1 (g_{ac} g_{bd} - g_{ad} g_{bc}) + 1/4 M_2 \eta_{abcd} = 0 \qquad ...(3.16)$$

and

$$4 P^{*}_{abmn} P^{mnrs} P_{rscd} - L_1 P^{*}_{abcd} - L_2 P_{abcd} - 1/4 M_1 \eta_{abcd} - 1/4 M_2 (g_{ac} g_{bd} - g_{ad} g_{bc}) = 0. \quad ...(3.17)$$

Eqns. (3.16) and (3.17) are duals of each other. Contraction of a with c and b with d in eqns. (3.16) and (3.17), we get

$$M_1 = 4/3 P_{abmn} P^{mnrs} P^{ab}_{rs} \quad ...(3.18)$$

$$M_2 = 4/3 P^{*}_{abmn} P^{mnrs} P^{ab}_{rs}. \quad ...(3.19)$$

For obtaining the expressions for L_1 and L_2, we consider the expression

$$(P_{abcd} + iP^{*}_{abcd})(P^{abcd} + i P^{*abcd}).$$

From eqn. (3.1), after some calculations, we obtain

$$L_1 = P_{abcd} P^{abcd} \quad ...(3.20)$$

$$L_2 = P^{*}_{abcd} P^{abcd}. \quad ...(3.21)$$

Eqns. (3.18)–(3.21) are the covariant expressions for the invariants of the space-matter tensor in empty space-time. Thus, we notice that there are only four invariants of the space-matter tensor in empty space-time.

4. Forms of Invariants

In this section, we shall find out the forms of these invariants in the classification recently given by Ahsan (1977). The results obtained from lengthy calculations have been summarized in Table I.

TABLE I

Case No.	L_1	L_2	M_1	M_2
I	$2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) - 2(\beta_1^2 + \beta_2^2 + \beta_3^2)$	0	$4/3 [6(\alpha_1\beta_1^2 + \alpha_2\beta_2^2 + \alpha_3\beta_3^2) - 2(\alpha_1^3 + \alpha_2^3 + \alpha_3^3)]$	0
II (*a*)	$12(\alpha_1^2 - \beta_1^2)$	0	$16(\alpha_1^2 - \beta_1^2)$	0
II (*b*)	$12(\alpha^2 - \beta^2)$	0	$4/3(12\alpha^3 - 36\alpha\beta^2 - 32\beta\gamma\delta)$	0
III (*a*)	0	0	0	0
III (*b*)	0	0	0	0

From Table I, we notice that for the cases III (*a*) and III (*b*), the invariants of the space-matter tensor are zero. But as the cases III (*a*) and III (*b*) correspond to the case of gravitational radiation (Ahsan 1977), we can thus say that "If the invariants of the space-matter tensor in empty space-time vanish, while

$P_{abcd} \neq 0$, then the gravitational radiation is present; otherwise there is no gravitational radiation."

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