A NOTE ON CONTINUABILITY AND BOUNDEDNESS OF SOLUTIONS OF CERTAIN SECOND ORDER INTEGRODIFFERENTIAL EQUATIONS

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In this note, results concerning continuability and boundedness of the solutions of the integrodifferential equation $(a(t)x')' + q(t)f(x)g(x') = h\left(t, x', \int_{t_0}^{t} k(t,s,x')\,ds\right)$ are obtained. The construction of a suitable energy function coupled with the integral inequality recently established by the author (Pachpatte 1975, 1976) is the main technique employed.

1. INTRODUCTION

In this paper we consider the second order integrodifferential equation

$$(a(t)x')' + q(t)f(x)g(x') = h\left(t, x', \int_{t_0}^{t} k(t,s,x')\,ds\right), \qquad \ldots(1)$$

where $a,\ q : I = [t_0, \infty) \to R,\ f,\ g : R \to R;\ k : I \times I \times R \to R,\ h : I \times R \times R \to R,\ a(t) > 0,\ q(t) > 0,\ g(x') > 0$, and a, q, f, g, h and k are continuous. Equations of this type have been studied by many authors, particularly when the right-hand side in (1) is zero. A good account of the earlier developments in this area has been given by Sansone and Conti (1964) and Wong (1968). More recent contributions in this area are by Burton and Grimmer (1970), Chang (1970), Graef and Spikes (1974, 1975), Lalli (1969), Pachpatte (1975) and Zarghamee and Mehri (1970). The present work was strongly inspired by the recent work of Graef and Spikes (1974, 1975). In fact, the present results are generalizations of some of their results.

2. RESULTS

In this section, we state and prove our main results on the continuability and boundedness of the solutions of integrodifferential eqn. (1) under some suitable conditions on the functions involved in (1). It will be convenient to write eqn. (1) as the equivalent system

$$x' = y$$

$$y' = \frac{1}{a(t)}\left[-a'(t)y - q(t)f(x)g(y) + h\left(t, y, \int_{t_0}^{t} k(t,s,y)\,ds\right)\right]. \qquad \ldots(2)$$

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We define $q'(t)_+ = \max\{q'(t), 0\}$ and $q'(t)_- = \max\{-q'(t), 0\}$, so that we have $q'(t) = q'(t)_+ - q'(t)_-$. A similar decomposition holds for $a(t)$. Define

$$F(x) = \int_0^x f(s)\,ds, \quad G(y) = \int_0^y \left[\frac{s}{g(s)}\right] ds,$$

$$p(t) = \exp\left(-\int_{t_0}^t \left[\frac{q'(s)_-}{q(s)}\right] ds\right),$$

and

$$b(t) = \exp\left(-\int_{t_0}^t \left[\frac{a'(s)_-}{a(s)}\right] ds\right).$$

To establish our main results, we require the following integral inequality recently established by the author (Pachpatte 1976, Theorem 2).

Lemma (Pachpatte 1976)—Let $u(t)$, $a(t)$, and $b(t)$ be real-valued non-negative continuous functions defined on I, for which the inequality

$$u(t) \leq u_0 + \int_{t_0}^t a(s)\,u(s)\left(u(s) + \int_{t_0}^s b(\tau)\,u(\tau)\,d\tau\right) ds, \quad t \in I,$$

holds, where u_0 is a positive constant. Then

$$u(t) \leq u_0 \exp\left(\int_{t_0}^t a(s)\,E(s)\,ds\right), \quad t \in I,$$

where $E(t)$ is defined by

$$E(t) = \frac{u_0 \exp\left(\int_{t_0}^t b(\tau)\,d\tau\right)}{1 - u_0 \int_{t_0}^t a(\tau)\exp\left(\int_{t_0}^\tau b(n)\,dn\right) d\tau}, \quad t \in I,$$

in which

$$\int_{t_0}^t a(\tau)\exp\left(\int_{t_0}^\tau b(n)\,dn\right) d\tau < u_0^{-1} \text{ for all } t \in I.$$

Our first theorem deals with the continuability of every solution of (2) under some appropriate restrictions on the functions involved in (2).

Theorem 1—Assume that there exist non-negative constants m and n such that

$$\frac{|y|}{g(y)} \leq m + n\,G(y). \qquad \ldots(3)$$

Suppose that the functions h and k in (2) satisfy

$$|h(t,y,u)| \leq c(t)\left[\frac{p(t)|y|}{q(t)g(y)} + |u|\right], \quad t \in I, \qquad \ldots(4)$$

$$|k(t,s,y)| \leq w(s)\frac{p(s)|y|}{q(s)g(y)}, \quad t, \; s \in I, \qquad \ldots(5)$$

where p, q and g are as defined above and $c(t)$, $w(t)$ are real-valued non-negative continuous functions defined on I such that

$$\int_{t_0}^{\infty} \frac{nc(s)}{a(s)} E_0(s)\, ds < \infty, \qquad \ldots(6)$$

in which

$$E_0(t) = \frac{M_1 \exp\left(\int_{t_0}^{t} w(\tau)\, d\tau\right)}{1 - M_1 \int_{t_0}^{t} n\frac{c(\tau)}{a(\tau)} \exp\left(\int_{t_0}^{\tau} w(\rho)\, d\rho\right) d\tau}, \quad t \in I \qquad \ldots(7)$$

where $M_1 > 0$ is a constant and

$$\int_{t_0}^{t} n\frac{c(\tau)}{a(\tau)} \exp\left(\int_{t_0}^{\tau} w(\rho)\, d\rho\right) d\tau < M_1^{-1}.$$

Further, suppose $a'(t) \geqslant 0$, $F(x)$ is bounded from below, and $G(y) \to \infty$ as $|y| \to \infty$. Then all solutions of (2) can be defined for all $t \geqslant t_0$.

PROOF: Suppose there is a solution $(x(t), y(t))$ of (2) and $T > t_0$, such that $\lim_{t \to T^-} [|x(t)| + |y(t)|] = +\infty$. Since $F(x)$ is bounded from below, there exists a real number $M > 0$, such that $F(x) \geqslant -M$ for all x. Consider the function $V(t)$ defined by

$$V(x,y,t) = p(t)\left[\frac{(F(x)+M)}{a(t)} + \frac{G(y)}{q(t)}\right],$$

then

$$V'(t) = p(t)\left[-\frac{(F(x)+M)\,a'(t)}{a^2(t)} + \frac{f(x)\,x'}{a(t)} - \frac{G(y)q'(t)}{q^2(t)} + \frac{yy'}{g(y)\,q(t)} - \frac{(F(x)+M)\,q'(t)_-}{a(t)q(t)} - \frac{G(y)q'(t)_-}{q^2(t)}\right]$$

$$\leqslant p(t)\left[-\frac{G(y)[q'(t)+q'(t)_-]}{q^2(t)} + \frac{y}{g(y)q(t)a(t)} \times h\left(t, y, \int_{t_0}^{t} k(t,s,y)\, ds\right)\right]$$

$$\leqslant \frac{p(t)\,y}{g(y)\,q(t)\,a(t)}\, h\left(t, y, \int_{t_0}^{t} k(t,s,y)\, ds\right).$$

Integrating both sides of the above inequality from t_0 to t, we obtain

$$V(t) \leqslant V(t_0) + \int_{t_0}^{t} \frac{p(s)\, y(s)}{g(y(s))\, q(s)\, a(s)}\, h\Big(s, y(s), \int_{t_0}^{s} k(s, \tau, y(\tau))\, d\tau\Big)\, ds.$$

By (3) and the fact that

$$\frac{p(t)\, G(y(t))}{q(t)} \leqslant V(t),$$

we have

$$\frac{p(t)\, |y(t)|}{g(y(t))\, q(t)} \leqslant m \frac{p(t)}{q(t)} + nV(t_0) + n \int_{t_0}^{t} \frac{p(s)\, y(s)}{g(y(s))\, q(s)\, a(s)}$$

$$\times\, h\Big(s,\ y(s), \int_{t_0}^{s} k(s, \tau, y(\tau))\, d\tau\Big)\, ds.$$

Using (4) and (5) and the fact that on $[t_0, T]$, $\dfrac{p(t)}{q(t)}$

is bounded, we have

$$\frac{p(t)\, |y(t)|}{g(y(t))\, q(t)} \leqslant M_1 + \int_{t_0}^{t} n \frac{c(s)}{a(s)} \frac{p(s)\, |y(s)|}{g(y(s))\, q(s)}$$

$$\times \Big[\frac{p(s)\, |y(s)|}{g(y(s))\, q(s)} + \int_{t_0}^{s} w(\tau) \frac{p(\tau)\, |y(\tau)|}{g(y(\tau))\, q(\tau)}\, d\tau\Big]\, ds,$$

for some $M_1 > 0$. Now, an application of Lemma yields

$$\frac{p(t)\, |y(t)|}{g(y(t))\, q(t)} \leqslant M_1 \exp\Big(\int_{t_0}^{t} n \frac{c(s)}{a(s)} E_0(s)\, ds\Big)$$

$$\leqslant M_1 \exp\Big(\int_{t_0}^{\infty} n \frac{c(s)}{a(s)} E_0(s)\, ds\Big) \leqslant M_2 < \infty,$$

where $E_0(t)$ is as defined in (7). Hence,

$$V(t) \leqslant V(t_0) + \int_{t_0}^{t} M_2 \frac{c(s)}{a(s)} \Big[M_2 + \int_{t_0}^{s} M_2\, w(\tau)\, d\tau\Big]\, ds \leqslant M_3 < \infty.$$

Thus,

$$\frac{p(t)\,G(y(t))}{q(t)} \leqslant M_3$$

on $[t_0, T)$ and so $G(y(t))$ is bounded on $[t_0, T)$. This implies that $y(t) = x'(t)$ is bounded on $[t_0, T)$, and an integration yields that $x(t)$ is also bounded on $[t_0, T)$, contradicting the assumption that $(x(t), y(t))$ was a solution of (2) with finite escape time, and the proof of the theorem is complete.

Graef and Spikes (1975) obtained a result on the continuability of every solution of (1), when the right side in (1) is equal to $r(t)$, a continuous function. Our Theorem 1 may be regarded as a generalization to second order integrodifferential equations of the classical result due to Graef and Spikes (1975).

Our next theorem deals with the boundedness of the solutions of (1) under some suitable conditions on the functions involved in (1).

Theorem 2—Suppose the hypotheses (3)–(6) of Theorem 1 hold. Further, suppose $a'(t) \geqslant 0$, $F(x) \to \infty$ as $|x| \to \infty$, $a(t)$ is bounded, and

$$\int_{t_0}^{\infty} \left[\frac{q'(s)_-}{q(s)}\right] ds < \infty. \qquad \ldots(8)$$

Then all solutions of eqn. (1) are bounded.

PROOF: Since $F(x) \to \infty$ as $|x| \to \infty$, $F(x)$ is bounded from below, say $F(x) \geqslant -M$ for some $M > 0$. Defining V as in the proof of Theorem 1, differentiating and integrating and using conditions (3)–(5), we obtain

$$\frac{p(t)\,|y(t)|}{g(y(t))\,q(t)} \leqslant m\frac{p(t)}{q(t)} + nV(t_0) + \int_{t_0}^{t} n\frac{c(s)}{a(s)}\,\frac{p(s)\,|y(s)|}{g(y(s))\,q(s)}$$

$$\times \left[\frac{p(s)\,|y(s)|}{g(y(s))\,q(s)} + \int_{t_0}^{s} w(\tau)\,\frac{p(\tau)\,|y(\tau)|}{g(y(\tau))\,q(\tau)}\,d\tau\right] ds.$$

Now, $p(t) \leqslant 1$ and (8) bounds $q(t)$ away from zero, so

$$\frac{mp(t)}{q(t)} + nV(t_0) \leqslant M_1.$$

Then, using Lemma, we have

$$\frac{p(t)\,|y(t)|}{g(y(t))\,q(t)} \leqslant M_1 \exp\left(\int_{t_0}^{t} n\frac{c(s)}{a(s)}\,E_0(s)\,ds\right)$$

$$\leqslant M_1 \exp\left(\int_{t_0}^{\infty} n\frac{c(s)}{a(s)}\,E_0(s)\,ds\right) \leqslant M_2 < \infty,$$

where $E_0(t)$ is as defined in Theorem 1. As in the proof of Theorem 1, it follows that $V(t)$ is bounded; so

$$\frac{p(t)\,F\big(x(t)\big)}{a(t)}$$

is bounded and hence $x(t)$ is bounded.

Corollary—If, in addition to the hypotheses of Theorem 2, we have $q(t)$ bounded from above and $G(y) \to \infty$ as $|y| \to \infty$, then all solutions of system (2) are bounded.

PROOF: Since $V(t)$ is bounded,

$$\frac{p(t)\,G\big(y(t)\big)}{q(t)}$$

is bounded, and the conclusion follows.

Recently, there have been a number of results on the continuability and boundedness of equations which are special cases of (1), *see References*. Although such problems have been considered by various investigators, the results in the present note are of interest because of the more general function on the right side of (1). In fact, the results obtained in this note are possible, since the integral inequality given in Lemma is available.

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