QUADRUPLE SERIES EQUATIONS INVOLVING SERIES OF JACOBI POLYNOMIALS

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In this paper, we have solved some quadruple series equations involving series of Jacobi polynomials. The solution of four series equations is reduced to Fredholm integral equation of second kind in one independent variable.

1. INTRODUCTION

Recently, several authors (Dwivedi and Trivedi 1972, 1975, Singh 1974) have considered quadruple series equations which are extensions of the corresponding dual and triple series equations. Recently, Srivastava (1967) obtained the solution of a triple series equation involving Jacobi polynomials. In this paper, we shall be concerned mainly with quadruple series equations which are extensions of the triple series equation considered by Srivastava (1967), given by

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+3/2)} P_n^{(\alpha,\beta)}(\cos\theta)$$

$$= f_1(\theta),\ 0 \leq \theta < a \qquad \ldots(1.1)$$

$$= f_3(\theta),\ b < \theta < c \qquad \ldots(1.2)$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(n+\beta+1)\,\Gamma(n+\alpha+1/2)} P_n^{(\alpha,\beta)}(\cos\theta)$$

$$= f_2(\theta),\ a < \theta < b \qquad \ldots(1.3)$$

$$= f_4(\theta),\ c < \theta < \pi \qquad \ldots(1.4)$$

where α, $\beta > -1/2$ and f_1, f_2, f_3 and f_4 are known prescribed functions and equations are to be solved for an unknown coefficient A_n. It is assumed that series (1.1) to (1.4) are uniformly convergent and f_1, f_2, f_3, f_4 and their derivatives are continuous. It is worth noting that when $\alpha = \beta$, we have quadruple series equations in ultraspherical polynomials. By defining

$$A_n = B_n \Gamma(n+\alpha+1)\ \ \Gamma(n+\alpha+3/2)$$

Eqns. (1.1)–(1.4) have the simple forms

$$\sum_{n=0}^{\infty} (n + \alpha + 1/2) B_n P_n^{(\alpha, \alpha)} (\cos n\theta)$$

$$= f_1(\theta),\ 0 \leq \theta < a \qquad ...(1.5)$$

$$= f_3(\theta),\ b < \theta < c \qquad ...(1.6)$$

$$\sum_{n=0}^{\infty} B_n P_n^{(\alpha, \alpha)} (\cos n\theta)$$

$$= f_2(\theta),\ a < \theta < b \qquad ...(1.7)$$

$$= f_4(\theta),\ c < \theta < \pi \qquad ...(1.8)$$

where $P_n^{(\alpha, \alpha)} (\cos n\theta)$ is an ultraspherical polynomial. Further, when $\alpha = -1/2$, the above equations get converted to quadruple series equations involving Fourier cosine series,

$$\sum_{n=1}^{\infty} n B_n \cos(n\theta) = f_1(\theta),\ 0 \leq \theta < a \qquad ...(1.9)$$

$$= f_3(\theta),\ b < \theta < c \qquad ...(1.10)$$

$$B_0 + \sum_{n=1}^{\infty} B_n \cos(n\theta) = f_2(\theta),\ a < \theta < b$$

$$= f_4(\theta),\ c < \theta < \pi \qquad ...(1.11)$$

Eqns. (1.9)–(1.12) arise in the mixed boundary value problems, when we consider the distribution of stresses in the interior of an infinitely long strip containing three Griffith cracks situated on a line perpendicular to the boundary lines of the strip. It is, therefore, worthwhile to consider the more general equations (1.1)–(1.4). The method which we apply is similar to that of Cook (1963). The analysis is formal and no attempt has been made to justify various limiting processes.

2. Preliminary Results

(i) It is convenient to list here some results for ready reference. These results are of Srivastava (1967); which are valid for $\alpha > -1$, $\beta > -1$. The orthogonality condition for Jacobi polynomials is

$$\int_0^{\pi} \left(\sin \frac{\theta}{2}\right)^{2\alpha} \left(\cos \frac{\theta}{2}\right)^{2\beta} P_n^{(\alpha, \beta)} (\cos \theta)\, P_m^{(\alpha, \beta)} (\cos \theta) \sin(\theta)\, d\theta$$

$$= \frac{\Gamma(n + \alpha + 1)\,\Gamma(n + \beta + 1)}{q_n(\alpha, \beta)}\, \delta_{m, n} \qquad ...(2.1)$$

where $\delta_{m,n}$ is Kronecker delta and $q_n(\alpha, \beta) = \frac{1}{2} n!(\alpha + \beta + 2n + 1) \times \Gamma(\alpha + \beta + n + 1)$.

$$\delta(u, \theta) = \sum_{n=0}^{\infty} \frac{q_n(\alpha, \beta) \Gamma(n + \alpha + 1/2)}{(\Gamma(n + \alpha + 1))^2 \Gamma(n + \beta + 3/2)} \left(\sin \frac{u}{2}\right)^{2\alpha} P_n^{(\alpha, \beta)}$$

$$\times (\cos u) P_n^{(\alpha, \beta)} (\cos \theta)$$

$$= \frac{\left(\sin \frac{\rho}{2}\right)^{-2\alpha}}{\pi} \int_0^{\min(u, \theta)} \frac{E(y)\, dy}{(\cos y - \cos u)^{1/2} (\cos y - \cos \theta)^{1/2}} \qquad \ldots(2.2)$$

where

$$E(t) = \left(\sin \frac{t}{2}\right)^{2\alpha} \left(\cos \frac{t}{2}\right)^{-2\beta}, \; t = \min(u, \theta)$$

The proof of the result (2.2) is given in the paper of Srivastava (1967).

(ii) We shall also use the following two forms of Schlomilch's integral equations. If $f(\theta)$ and $f'(\theta)$ are continuous in $a \leq \theta \leq b$, then the solutions of the integral equations are:

$$f(\theta) = \int_a^\theta (\cos u - \cos \theta)^{-1/2} g(u)\, du, \qquad \ldots(2.3)$$

$$f'(\theta) = \int_\theta^b (\cos \theta - \cos u)^{-1/2} g'(u)\, du, \qquad \ldots(2.4)$$

and

$$g(u) = \frac{1}{\pi} \frac{d}{du} \int_a^u (\cos \theta - \cos u)^{-1/2} f(\theta) \sin(\theta)\, d\theta, \qquad \ldots(2.5)$$

$$g'(u) = -\frac{1}{\pi} \frac{d}{du} \int_u^b (\cos u - \cos \theta)^{-1/2} f'(\theta) \sin(\theta)\, d\theta, \qquad \ldots(2.6)$$

respectively.

3. Solution of Quadruple Equations

Let us suppose that

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(n + \beta + 1) \Gamma(n + \alpha + 1/2)} P_n^{(\alpha, \beta)} (\cos \theta)$$

$$= h(\theta) \quad 0 \leq \theta < a \qquad \ldots(3.1)$$

$$= g(\theta) \quad b < \theta < c. \qquad \ldots(3.2)$$

Using orthogonal relation (2.1), we get A_n from (3.1) and (3.2)

$$A_n = \frac{\Gamma(n+\alpha+1/2)}{\Gamma(n+\alpha+1)} q_n^{(\alpha,\beta)} [\int_0^a h_1(u) + \int_a^b f'_2(u) + \int_b^c g_1(u) + \int_c^\pi f'_4(u)]$$

$$\times \left(\sin\frac{u}{2}\right)^{2\alpha} P_n^{(\alpha,\beta)}(\cos\theta) P_n^{(\alpha,\beta)}(\cos u)\, du \qquad \ldots(3.3)$$

where

$$h_1(u) = \left(\cos\frac{u}{2}\right)^{2\beta} \sin u h(u), \qquad g_1(u) = \left(\cos\frac{u}{2}\right)^{2\beta} \sin u g(u),$$

$$f'_2(u) = \left(\cos\frac{u}{2}\right)^{2\beta} \sin(u) f_2(u), \qquad f'_4(u) = \left(\cos\frac{u}{2}\right)^{2\beta} \sin(u) f_4(u).$$

Substituting this expression for A_n from (3.3) in eqns. (1.1) and (1.3), applying the summation result (2.2), and interchanging the order of integration and summation, we obtain

$$[\int_0^a h_1(u) + \int_b^c g_1(u)] S(u,\theta)\, du = M(\theta), \qquad 0 \le \theta < a \qquad \ldots(3.4)$$

$$[\int_0^a h_1(u) + \int_b^c g_1(u)] S(u,\theta)\, du = N(\theta), \qquad b < \theta < c \qquad \ldots(3.5)$$

where

$$M(\theta) = f_1(\theta) - [\int_a^b f'_2(u) + \int_c^\pi f'_4(u)] S(u,\theta)\, du,$$

$$N(\theta) = f_3(\theta) - [\int_a^b f'_2(u) + \int_c^\pi f'_4(u)] S(u,\theta)\, du.$$

Using the summation result in terms of integral (2.2), eqn. (3.4) can be written as

$$\int_0^a h_1(u)\, du \int_0^{\min(u,\theta)} \frac{E(y)\, dy}{(\cos y - \cos u)^{1/2} (\cos y - \cos\theta)^{1/2}}$$

$$= M(\theta)\, \pi \left(\sin\frac{\theta}{2}\right)^{2\alpha}$$

$$- \int_b^c g_1(u)\, du \int_0^{\min(u,\theta)} \frac{E(y)\, dy}{(\cos y - \cos u)^{1/2} (\cos y - \cos\theta)^{1/2}},$$

$$0 \le \theta < a. \qquad \ldots(3.6)$$

Changing the order of integration, we get from (3.6)

$$\int_0^\theta \frac{E(y)\,dy}{(\cos y - \cos\theta)^{1/2}} \int_y^a \frac{h_1(u)\,du}{(\cos y - \cos u)^{1/2}}$$

$$= M(\theta)\,\pi\left(\sin\frac{\theta}{2}\right)^{2\alpha} - \int_0^\theta \frac{E(y)\,dy}{(\cos y - \cos\theta)^{1/2}} \int_b^c \frac{g_1(u)\,du}{(\cos y - \cos u)^{1/2}},$$

$$0 \le \theta < a. \qquad \text{...(3.7)}$$

Using results (2.3) and (2.5), we have from (3.7)

$$E(y)\int_y^a \frac{h_1(u)\,du}{(\cos y - \cos u)^{1/2}}$$

$$= \frac{d}{dy}\int_0^y \frac{\left(\sin\frac{\theta}{2}\right)^{2\alpha} M(\theta)\sin\theta\,d\theta}{(\cos\theta - \cos y)^{1/2}} - \frac{1}{\pi}\frac{d}{dy}\int_0^y \frac{\sin\theta\,d\theta}{(\cos\theta - \cos y)^{1/2}}$$

$$\times \int_0^\theta \frac{E(y)\,dy}{(\cos y - \cos\theta)^{1/2}} \int_b^c \frac{g_1(u)\,du}{(\cos y - \cos u)^{1/2}},\ 0 \le \theta < a$$

$$\text{...(3.8)}$$

The second integral on the right-hand side of (2.8) can be written in the form

$$-\frac{1}{\pi}\int_b^c \frac{g_1(u)\,du}{(\cos y - \cos u)^{1/2}}\frac{d}{dy}\int_0^y E(y)\,dy$$

$$\times \int_t^y \frac{\sin(\theta)\,d\theta}{(\cos\theta - \cos y)^{1/2}(\cos t - \cos\theta)^{1/2}} = -\int_b^c \frac{g_1(u)\,E(y)\,du}{(\cos y - \cos u)^{1/2}}$$

$$\text{...(3.9)}$$

Since

$$\int_t^y \frac{\sin\theta\,d\theta}{(\cos\theta - \cos y)^{1/2}(\cos t - \cos\theta)^{1/2}} = \pi$$

Hence, from (3.8) and (3.9), we have

$$\int_y^a \frac{h_1(u)\,du}{(\cos y - \cos u)^{1/2}} = \frac{M_1(y)}{E(y)} - \int_b^c \frac{g_1(u)\,du}{(\cos y - \cos u)^{1/2}} \qquad \text{...(3.10)}$$

where

$$M_1(y) = \frac{d}{dy} \int_0^y \frac{\left(\sin \frac{\theta}{2}\right)^{2\alpha} \sin(\theta)\, M(\theta)\, d\theta}{(\cos\theta - \cos y)^{1/2}}$$

Again, using result (2.4) and (2.6), we get from (3.10)

$$h_1(u) = -\frac{1}{\pi} \frac{d}{du} \int_u^a \frac{\sin(y)\, M_1(y)\, dy}{E(y)(\cos u - \cos y)^{1/2}}$$

$$+ \frac{1}{\pi} \frac{d}{du} \int_u^a \frac{\sin y\, dy}{(\cos u - \cos y)^{1/2}} \int_b^c \frac{g_1(s)\, ds}{(\cos y - \cos s)^{1/2}},$$

$$0 \leq u < a. \qquad \ldots(3.11)$$

Now changing the order of integration and using result

$$\frac{d}{dy} \int_a^y \frac{\sin\theta\, d\theta}{(\cos\theta - \cos y)^{1/2}(\cos t - \cos\theta)^{1/2}}$$

$$= \frac{\sin y\,(\cos t - \cos a)^{1/2}}{(\cos a - \cos y)^{1/2}(\cos t - \cos y)}$$

We have from (3.11)

$$h_1(u) = M_2(u) + \frac{\sin u}{\pi(\cos a - \cos u)^{1/2}}$$

$$\times \int_b^c \frac{(\cos s - \cos a)^{1/2}\, g_1(s)\, ds}{(\cos s - \cos u)} \qquad \ldots(3.12)$$

where

$$M_2(u) = -\frac{1}{\pi} \frac{d}{du} \int_u^a \frac{\sin y\, M_1(y)\, dy}{E(y)(\cos u - \cos y)^{1/2}}. \qquad \ldots(3.13)$$

Again, using summation result in terms of integral (2.2), we get from eqn. (3.5)

$$\int_b^\theta \frac{E(y)\, G(y)\, dy}{(\cos y - \cos\theta)^{1/2}}$$

$$= \left(\sin\frac{\theta}{2}\right)^{2\alpha} \pi N(\theta) - \int_0^a h_1(u)\, du$$

$$\times \int_b^{\min(u,\,\theta)} \frac{E(y)\, dy}{(\cos y - \cos\theta)^{1/2}(\cos y - \cos u)^{1/2}} -$$

(*equation continued on p.* 1074)

$$- \int_0^b \frac{E(y)\, dy}{(\cos y - \cos\theta)^{1/2}} \int_b^c \frac{g_1(u)\, du}{(\cos y - \cos u)^{1/2}} ,$$

$$b < \theta < c \qquad \ldots(3.14)$$

where

$$G(y) = \int_y^c \frac{g_1(u)\, du}{(\cos y - \cos u)^{1/2}} . \qquad \ldots(3.15)$$

Applying result (2.4) and (2.6), we have

$$g_1(u) = -\frac{1}{\pi} \frac{d}{du} \int_u^c \frac{G_1(y) \sin y\, dy}{(\cos u - \cos y)^{1/2}} . \qquad \ldots(3.16)$$

Again applying (2.3) and (2.5) to eqn. (3.14), we have

$$E(y)\, G(y) = N_1(y) + I_1 + I_2, \qquad b < y < c \qquad \ldots(3.17)$$

where

$$N_1(y) = \frac{d}{dy} \int_b^y \frac{\left(\sin\frac{\theta}{2}\right) N(\theta) \sin\theta\, d\theta}{(\cos\theta - \cos y)^{1/2}} , \qquad \ldots(3.18)$$

$$I_1 = -\frac{1}{\pi} \frac{d}{dy} \int_b^y \frac{\sin\theta\, d\theta}{(\cos\theta - \cos y)^{1/2}} \int_0^a h_1(u)\, du$$

$$\times \int_0^{\min(u,\theta)} \frac{E(y)\, dy}{(\cos y - \cos\theta)^{1/2} (\cos y - \cos u)^{1/2}} , \qquad \ldots(3.19)$$

$$I_2 = -\frac{1}{\pi} \frac{d}{dy} \int_b^y \frac{\sin\theta\, d\theta}{(\cos\theta - \cos y)^{1/2}} \int_0^b \frac{E(y)\, dy}{(\cos y - \cos\theta)^{1/2}}$$

$$\times \int_b^c \frac{g_1(u)\, du}{(\cos y - \cos u)^{1/2}} . \qquad \ldots(3.20)$$

Changing the order of integration and applying the result

$$\frac{d}{dy} \int_b^y \frac{\sin\theta\, d\theta}{(\cos\theta - \cos y)^{1/2} (\cos t - \cos\theta)^{1/2}} .$$

$$= \frac{\sin y (\cos t - \cos a)^{1/2}}{(\cos a - \cos y)^{1/2} (\cos t - \cos y)} \qquad \ldots(3.21)$$

Hence, we have from (3.19)

$$I_1 = -\frac{1}{\pi} \frac{\sin y}{(\cos b - \cos y)^{1/2}} \int_0^a \frac{E(t)(\cos t - \cos b)^{1/2}\, dt}{(\cos t - \cos y)}$$

$$\times \int_t^a \frac{h_1(u)\, du}{(\cos t - \cos u)^{1/2}} \qquad \ldots(3.22)$$

Putting the value of the last integral of (3.22) from (3.10)

$$I_1 = M_3(y) + \frac{1}{\pi} \frac{\sin y}{(\cos b - \cos y)^{1/2}} \int_t^a \frac{E(t)(\cos t - \cos b)^{1/2}\, dt}{(\cos t - \cos y)}$$

$$\times \int_b^c \frac{g_1(u)\, du}{(\cos t - \cos u)^{1/2}} \qquad \ldots(3.23)$$

where

$$M_3(y) = -\frac{1}{\pi} \frac{\sin y}{(\cos b - \cos y)^{1/2}} \int_0^a \frac{M_1(t)(\cos t - \cos b)^{1/2}\, dt}{(\cos t - \cos y)} \qquad \ldots(3.24)$$

Similarly, eqn. (3.20) can be written as

$$I_2 = -\frac{1}{\pi} \frac{\sin y}{(\cos b - \cos y)^{1/2}} \int_0^b \frac{E(t)(\cos t - \cos b)^{1/2}\, dt}{(\cos t - \cos y)}$$

$$\times \int_b^c \frac{g_1(u)\, du}{(\cos t - \cos u)^{1/2}} \,. \qquad \ldots(3.25)$$

Hence,

$$I_1 + I_2 = M_3(y) - \frac{1}{\pi} \frac{\sin y}{(\cos b - \cos y)^{1/2}} \int_a^b \frac{E(t)(\cos t - \cos b)^{1/2}\, dt}{(\cos t - \cos y)} \times R \qquad \ldots(3.26)$$

where

$$R = \int_b^c \frac{g_1(u)\, du}{(\cos t - \cos u)^{1/2}} \qquad \ldots(3.27)$$

Putting the value of $g_1(u)$ from (3.16), we have

$$R = \int_b^c \frac{1}{(\cos t - \cos u)^{1/2}} \left(-\frac{1}{\pi} \frac{d}{du} \int_u^c \frac{G(y) \sin(y)\, dy}{(\cos u - \cos y)^{1/2}} \right)$$

$$= -\frac{1}{\pi} \int_b^c (\cos t - \cos u)^{-1/2} \frac{dJ}{du} \qquad \text{...(3.28)}$$

where

$$J = \int_u^c \frac{G(y) \sin(y)\, dy}{(\cos u - \cos y)^{1/2}} \qquad \text{...(3.29)}$$

Integrating by parts, we have

$$R_1 = \frac{1}{\pi} \frac{1}{(\cos t - \cos b)^{1/2}} \int_b^c \frac{G(s) \sin s\, ds}{(\cos b - \cos s)^{1/2}}$$

$$- \frac{1}{2\pi} \int_b^c \frac{\sin(u)\, du}{(\cos t - \cos u)^{3/2}} \times \int_u^c \frac{G(s) \sin(s)\, ds}{(\cos u - \cos s)^{1/2}} \qquad \text{...(3.30)}$$

Changing the order of integration of (3.30)

$$R = \frac{1}{\pi} \frac{1}{(\cos t - \cos s)^{1/2}} \int_b^c \frac{G(s) \sin(s)\, ds}{(\cos b - \cos s)^{1/2}}$$

$$- \frac{1}{\pi} \int_b^s \frac{G(s) \sin(s)}{(\cos t - \cos s)} \frac{(\cos b - \cos s)^{1/2}}{(\cos t - \cos b)^{1/2}} ds\,. \qquad \text{...(3.31)}$$

Since

$$\int_b^s \frac{\sin(u)\, du}{(\cos t - \cos u)^{1/2} (\cos u - \cos s)^{1/2}}$$

$$= \frac{2 (\cos b - \cos s)^{1/2}}{(\cos t - \cos s)(\cos t - \cos b)^{1/2}}$$

Hence,

$$R = \frac{1}{\pi} \frac{1}{(\cos t - \cos b)^{1/2}} \int_b^c \frac{G(s) \sin s (\cos t - \cos b)\, ds}{(\cos b - \cos s)^{1/2} (\cos t - \cos s)} \qquad \text{...(3.32)}$$

Putting the value R from (3.32) in eqn. (3.26)

$$I_1 + I_2 = M_3(y) - \frac{1}{2} \int_b^c G(s) B(s, y) \, ds \qquad \ldots(3.33)$$

where

$$B(s, y) = \frac{\sin(s)\sin(y)}{(\cos b - \cos y)^{1/2} (\cos b - \cos s)^{1/2}} R(s, y), \qquad \ldots(3.34)$$

$$R(s, y) = \int_a^b \frac{E(t)(\cos t - \cos b)}{(\cos t - \cos y)(\cos t - \cos s)} \, dt. \qquad \ldots(3.35)$$

Hence

$$E(y) G(y) = N_1(y) + M_3(y) - \frac{1}{\pi^2} \int_b^c G(s) B(s, y) \, ds \quad b < y < c \qquad \ldots(3.36)$$

Now, eqn. (3.36) is a Fredholm integral equation of the second kind and $B(s, y)$ is a symmetric kernel, which determines $G(y)$. Then $g_1(u)$ is determined from (3.16) and $h_1(u)$ from (3.10). Knowing the values of $h_1(u)$ and $g_1(u)$, the value of coefficient A_n can be obtained from (3.3).

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