ON MULTIVARIATE FRACTIONAL INTEGRATION OPERATORS

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The object of the present paper is to introduce two multivariate fractional integration operators associated with a generalized function $\Phi(Z_1, Z_2, \ldots, Z_n)$ as kernel. Two theorems on the Mellin transforms of these operators are established. The third theorem gives the property of fractional integration by parts. Certain important properties of the operators are also discussed. By giving specific values to the kernel, various integral operators of one and more variables may be deduced.

1. INTRODUCTION

Various definitions of operators of fractional integration of one variable have been given from time to time due to their usefulness in physical sciences. Recently, Munot and Mathur (1976) and Mourya (1970) have defined bivariate fractional integration operators. In the present paper, we introduce fractional integration operators in N-variables, which are associated with a most generalized kernel. These operators also serve as key formulae for several important integral operators, notably those by Kober (1940), Erdélyi (1950–51), Saxena (1967), Parashar (1968), Kalla and Saxena (1969), Kalla (1969), Lowndes (1970), Mourya (1970), Saxena and Kumbhat (1973*a*, 1973*b*, 1974) and Munot and Mathur (1976). Three important properties of these operators are given in the form of three theorems which provide expressions for their Mellin transforms and integration by parts.

In what follows i will take the value from 1 to N.

2. DEFINITIONS

We define fractional integration operators in N-variables by means of the following integral equations:

$$B_{\delta_1,\delta_2,\ldots\delta_N}\, f(x_1, x_2, \ldots, x_N)$$

$$= \prod_{i=1}^{N} (r_i x_i^{-\delta_i - r_i\beta_i - 1}) \int_0^{x_1}\int_0^{x_2} \ldots \int_0^{x_N} t_1^{\delta_1} t_2^{\delta_2} \ldots t_N^{\delta_N} (x_1^{r_1} - t_1^{r_1})^{\beta_1}$$

$$\times (x_2^{r_2} - t_2^{r_2})^{\beta_2} \ldots (x_N^{r_N} - t_N^{r_N})^{\beta_N}\, \Phi\left(\frac{t_1}{x_1}, \frac{t_2}{x_2}, \ldots, \frac{t_N}{x_N}\right)$$

$$\times f(t_1, t_2, \ldots, t_N)\, dt_1 dt_2 \ldots dt_N \qquad \ldots(2.1)$$

$$L_{\alpha_1, \alpha_2, \ldots, \alpha_N} f(x_1, x_2, \ldots, x_N)$$

$$= \prod_{i=1}^{N} (r_i x_i^{\alpha_i}) \int_{x_1}^{\infty} \int_{x_2}^{\infty} \ldots \int_{x_N}^{\infty} t_1^{-\alpha_1-\beta_1 r_1-1} t_2^{-\alpha_2-\beta_2 r_2-1} \ldots t_N^{-\alpha_N-\beta_N r_N-1}$$

$$\times (t_1^{r_1} - x_1^{r_1})^{\beta_1} (t_2^{r_2} - x_2^{r_2})^{\beta_2} \ldots (t_N^{r_N} - x_N^{r_N})^{\beta_N}$$

$$\times \Phi\left(\frac{x_1}{t_1}, \frac{x_2}{t_2}, \ldots, \frac{x_N}{t_N}\right) f(t_1, t_2, \ldots, t_N)\, dt_1 dt_2 \ldots dt_N \quad \ldots(2.2)$$

where the kernel $\Phi(z_1, z_2, \ldots, z_N)$ is such that the integrals make sense.

Here, α's, β's and δ's are complex parameters and r's are positive numbers. The conditions of validity of these operators are as follows:

(i) $1 \leqslant p_i, \quad q_i < \infty, \quad p_i^{-1} + q_i^{-1} = 1$

(ii) $\operatorname{Re} \delta_i > -q_i^{-1}, \quad \operatorname{Re} (\alpha_i + \beta_i r_i) > -p_i^{-1},$

$\operatorname{Re} \beta_i > -(q_i r_i)^{-1}$

(iii) $f \in L_{p_i}(0, \infty)$.

The last condition ensures that both Bf and Lf exist and also that both belong to $L_{p_i}(0, \infty)$.

3. Mellin Transform

We denote the Mellin transform of a multivariate function f by Mf or $F(s_1, s_2, \ldots, s_N)$.

The N-dimensional Mellin-transform pair may be defined as

$$F(s_1, s_2, \ldots, s_N) = \int_0^{\infty} \ldots \int_0^{\infty} f(t_1, t_2, \ldots, t_N) \prod_{i=1}^{N} t_i^{s_i-1}\, dt_1 \ldots dt_N \quad \ldots(3.1)$$

and

$$f(t_1, t_2, \ldots, t_N) = \frac{1}{(2i\pi)^N} \int_{c_1-i\infty}^{c_1+i\infty} \ldots \int_{c_N-i\infty}^{c_N+i\infty} F(s_1, s_2, \ldots, s_N)$$

$$\times \prod_{i=1}^{N} t_i^{-s_i}\, ds_1\, ds_2 \ldots ds_N. \quad \ldots(3.2)$$

under suitable conditions.

In the theorems that follow, $M_{p_i}(0, \infty)$ will denote the class of all functions f of $L_{p_i}(0, \infty)$ with $p_i > 2$, which are inverse Mellin transforms of the functions of $L_{q_i}(-\infty, \infty)$.

Theorem 1—If $f \in L_{p_i}(0, \infty)$, $1 \leqslant p_i \leqslant 2$ [or $f \in M_{p_i}(0, \infty)$ with $p_i > 2$], $p_i^{-1} + q_i^{-1} = 1$, $\operatorname{Re} \delta_i > -q_i^{-1}$, $\operatorname{Re} \beta_i > -(q_i r_i)^{-1}$ and the integrals involved are absolutely convergent, then

$$MB_{\delta_i} f(x_1, x_2, \ldots, x_N) = Mf(x_1, x_2, \ldots, x_N) L_{\delta_i - s_i + 1} 1. \qquad \ldots(3.3)$$

PROOF: From (2.1), it follows that

$$MBf(x_1, x_2, \ldots, x_N)$$

$$= \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^N (x_i)^{s_i - 1} \Bigg[\prod_{i=1}^N r_i x_i^{-\delta_i - r_i \beta_i - 1}$$

$$\times \int_0^{x_1} \cdots \int_0^{x_N} \prod_{i=1}^N \{t_i^{\delta_i} (x_i^{r_i} - t_i^{r_i})^{\beta_i}\} \; \Phi\left(\frac{t_1}{x_1}, \ldots, \frac{t_N}{x_N}\right)$$

$$\times f(t_1, t_2, \ldots, t_N)\, dt_1\, dt_2 \ldots dt_N \Bigg] dx_1 dx_2, \ldots dx_N$$

$$= \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^N t_i^{\delta_i} f(t_1, t_2, \ldots, t_N)\, dt_1\, dt_2 \ldots dt_N$$

$$\times \int_{t_1}^\infty \cdots \int_{t_N}^\infty \prod_{t=1}^N [(r_i x_i^{-\delta_i - \beta_i r_i + s_i - 2} (x_i^{r_i} - t_i^{r_i})^{\beta_i}]$$

$$\times \Phi\left(\frac{t_1}{x_1}, \frac{t_2}{x_2}, \ldots, \frac{t_N}{x_N}\right) dx_1, dx_2 \ldots dx_N.$$

The change of order of integration is permissible by virtue of de la Vallée Poussin's theorem (Bromwich 1954) under the conditions stated with the theorem. The theorem follows readily on interpreting the results with (2.2) and (3.1).

In a similar manner, the following theorems can be established:

Theorem 2—If $f \in L_{p_i}(0, \infty)$, $1 \leqslant p_i \leqslant 2$ [or $f \in M_{p_i}(0, \infty), P_i > 2$], $\operatorname{Re} \beta_i r_i > -q_i^{-1}$, $\operatorname{Re}(\alpha_i + \beta_i r_i) > -p_i^{-1}$, $p_i^{-1} + q_i^{-1} = 1$ and the integrals involved are absolutely convergent, then

$$ML_{\alpha_i} f(x_1, x_2, \ldots, x_N) = Mf(x_1\, x_2, \ldots, x_N)\, B_{\alpha_i + s_i - 1}. \qquad \ldots(3 \cdot 4)$$

Theorem 3—If $f \in L_{p_i}(0, \infty)$, $p_i^{-1} + q_i^{-1} = 1$, $g \in L_{p_i}(0, \infty)$, $\operatorname{Re} \delta_i > \max(p_i^{-1}, q_i^{-1})$, $\operatorname{Re} \beta_i > 0$, then

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty g(x_1, x_2, \ldots, x_N)\, B_{\delta_i} f(x_1, x_2, \ldots, x_N)\, dx_1\, dx_2 \ldots dx_N$$

$$= \int_0^\infty \int_0^\infty \cdots \int_0^\infty f(x_1, x_2, \ldots, x_N)\, L_{\delta_i} g(x_1, x_2, \ldots, x_N)\, dx_1\, dx_2 \ldots dx_N \qquad \ldots(3.5)$$

The above theorem gives the property of fractional integration by parts and it follows immediately on interpreting it with the operators (2.1) and (2.2).

4. Formal Properties of the Operators

We give here some formal properties of the operators which follow as consequences of the definitions (2.1) and (2.2)

$$\text{If } g(X_1, X_2, \ldots, X_N) = (X_1 X_2, \ldots, X_N)^{-1} f\left(\frac{1}{X_1}, \frac{1}{X_2}, \ldots, \frac{1}{X_N}\right)$$

then

$$Bf\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_N}\right) = x_1, x_2, \ldots x_N\, L_g(x_1, x_2, \ldots, x_N) \qquad \ldots(4.1)$$

$$\text{If } g(X_1, X_2, \ldots, X_N) = X_1^{a_1} X_2^{a_2} \ldots X_N^{a_N}\; f(X_1, X_2, \ldots, X_N),$$

then

$$x_1^{a_1} x_2^{a_2} \ldots x_N^{a_N}\; B_{\delta_1, \delta_2 \ldots, \delta_N}\, f(x_1, x_2, \ldots, x_N)$$

$$= B_{\delta_1 - a_1, \delta_2 - a_2, \ldots, \delta_N - a_N}\, g(x_1, x_2, \ldots, x_N). \qquad \ldots(4.2)$$

If

$$g(X_1, X_2, \ldots, X_N) = f(cX_1, cX_2, \ldots, cX_N),$$

then

$$Bf(cx_1, cx_2, \ldots, cx_N) = Bg(x_1, x_2, \ldots, x_N). \qquad \ldots(4.3)$$

Similar properties hold for other operators also.

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