# APPROXIMATE POINT SPECTRA OF POLAR FACTORS OF HYPO-NORMAL OPERATORS*

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An investigation is made of the interdependence and properties of the approximate point spectra of hyponormal operator $T$ and of the factors in a polar factorization of $T$, when the latter exists.

§1. Only bounded operators on a fixed separable Hilbert space $H$ are considered in this paper. An operator $T$ is said to have a polar factorization if $T = UP$, where $U$ is unitary and $P$ is a non-negative self-adjoint operator. Thus, if $T$ has a polar factorization $T = UP$, then $T^* = PU^*$ and $T^* T = P^2$; hence $P = (T^* T)^{1/2}$, so that

$$T = UP, U \text{ unitary and } P = (T^* T)^{1/2}. \qquad ...(1.1)$$

In general, the unitary factor is not unique. In case $T$ is non-singular, i.e., if 0 is not in its spectrum, the polar factorization exists and is unique. It was given by Wintner (1932) and a generalization was obtained by Von Neumann (1932, p. 307).

As noted above, if $T = UP$ where $U$ is unitary and $P$ is non-negative, then necessarily $P = (T^* T)^{1/2}$. Also,

$$TT^* = U(T^* T) U^* \text{ (equivalently, } (TT^*)^{1/2} = U(T^* T)^{1/2} U^*)$$
$$U \text{ unitary}. \qquad ...(1.2)$$

Conversely, it was shown by Hartmann (1935) using the above-mentioned result of Von Neumann, that if $T$ is arbitrary, then the non-zero spectra of $T^* T$ and $TT^*$ are identical, including multiplicities of both point and continuous spectra, while 0 may occur in the point spectra of $T^*T$ and $TT^*$ with different multiplicities.

Further, (1.2) holds for some unitary $U$ if and only if the multiplicities of 0 in the point spectra of $T^* T$ and $TT^*$ (equivalently, of $T$ and of $T^*$) are equal, i.e.,

$$\dim \{x: Tx = 0\} = \dim \{x : T^* x = 0\}. \qquad ...(1.3)$$

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In addition (cf. Von Neumann 1935, p. 234), $T$ has a polar factorization (1.1), for some unitary $U$, if and only if (1.2) holds for some (not necessarily the same) unitary $U$, or equivalently, if and only if (1.3) holds. In this case, the unitary operator $U$ of (1.1) [but, of course, not that of (1.2)] is uniquely determined, if 0 is not in the point spectrum of $T$ (and/or $T^*$), i.e., if the common dimension of (1.3) is 0.

Next, an operator is said to be hyponormal if

$$T^* T - TT^* = D \geqslant 0, \quad \text{...(1.4)}$$

and completely hyponormal if, in addition, there is no non-trivial subspace reducing $T$ on which $T$ is normal. It was shown by Putnam (1970) that if $T$ is completely hyponormal, then its spectrum, $sp(T)$, has positive planar measure and, in fact,

if $T$ is completely hyponormal, then $means_2$ $(sp(T) \cap \alpha)$ whenever

$$sp(T) \cap \alpha \neq \text{empty set}, \quad \text{...(1.5)}$$

where $\alpha$ denotes any open disk of the complex plane. Let $T_z = T - zI$ for any complex $z$. Then $T_z^* T_z - T_z T_z^* = T^* T - TT^*$ and hence

$$\{x : T_z x = 0\} \subset \{x : T_z^* x = 0\} \text{ if } T \text{ is hyponormal.} \quad \text{...(1.6)}$$

Hence, if $z$ is in the point spectrum of a hyponormal $T$, the corresponding eigen space is a reducing space of $T$ on which it is normal. It is also clear that if $T$ is hyponormal and if 0 is not in the point spectrum of $T^*$, then $T$ has a unique polar factorization (1.1). Of course, if $T$ is normal, and whether or not 0 is in the point spectrum of $T^*$, equality holds in (1.6) for all $z$, in particular, for $z = 0$, and it follows that $T$ must have a (i.e., at least one) polar factorization. Such a factorization is easily constructed, for instance, from the spectral resolution of the operator. The unilateral shift (cf. Halmos 1967, p. 40) is an example of a completely hyponormal operator which fails to have a polar factorization (1.1)

For use below, recall that a number $t$ is said to be in the essential spectrum of a self-adjoint operator $A$, $essp(A)$, if $t$ is either a limit point of $sp(A)$ or is an eigen value of infinite multiplicity. The point spectrum of any operator $T$ will be denoted by $ptsp(T)$.

A complex number $z$ is an approximate proper value of $T$, provided $z$ and $T$ satisfy

$$\| (T - zI) x_n \| \to 0 \, (n \to \infty).$$

The approximate spectrum denoted by $\Pi(T)$ and is defined to be the set of all approximate proper value of $T$.

A number $t$ is said to be in the essential approximate spectrum of a self-adjoint operator $A$, $es\Pi(A)$, if $t$ is either a limit point of $\Pi(A)$ or is an eigen value of infinite multiplicity.

The following lemma is well known (we quote it for easy reference):

*Lemma* 1—Let $T$ be an arbitrary operator and let $|z| = \| T \|$. If $(T - zI) x_n = 0$, then $(T^* - \bar{z}I) x_n = 0$, if $\lim_{n \to \infty} \| (T - zI) x_n \| = 0$, then $\lim_{n \to \infty} \| (T^* - \bar{z}I) x_n \| = 0$.

§2. *Theorem* 1—Let $T$ be hyponormal operator and

$$z \in \text{boundary of } sp(T). \quad \ldots(2.1)$$

Then

$$|z| \in \Pi (T^* T)^{1/2} \cap \Pi (T T^*)^{1/2} \quad \ldots(2.2)$$

Further, if $T$ is completely hyponormal, then

$$|z| \in essp (T^* T)^{1/2} \cap essp (T T^*)^{1/2} \quad \ldots(2.3)$$

PROOF: By hypothesis, there exists $z \in sp(T)$, such that $|z| = \| T \|$. Since $z \in$ boundary of $sp(T)$, there exist unit vectors $\{x_n\}$, such that $\| (T - zI) x_n \| \to 0$.

From Lemma 1, we have

$$\| (T^* - \bar{z}I) x_n \| \to 0,$$

And so $\| (T^* T - |z|^2 I) x_n \| \to 0$ and $\| (TT^* - |z|^2 I) x_n \| \to 0$ and $\| ((T^* T^*)^{1/2} - |z| I) x_n \| \to 0$ and $\| ((TT^*)^{1/2} - |z| I) x_n \| \to 0$, and so (2.2) follows.†

(2.3) follows from Putnam (1974)

*Theorem* 2—Let $T$ be hyponormal and suppose $z \in sp(T)$ and $\bar{z} \notin pt\, sp(T)$. Then $|z| \in es\, \Pi (T^* T)^{1/2} \cap es\, \Pi (TT^*)^{1/2}$.

PROOF: Since $T$ is hyponormal, $T_z^* T_z \geqslant T_z T_z^*$, where $T_z = T - zI$, and so, since $z \in sp(T), 0 \in sp(T_z T_z^*)$. Also, since $\bar{z} \notin pt\ sp(T^*)$, then, $z \notin pt\, sp(T)$. Consequently, 0 is in the essential spectrum of both $T_z^* T_z$ and $T_z T_z^*$. In view of the equality $T_z^* T_z T_z T_z^*$, there exists a sequence of unit vectors, $\{ x_n \}$, converging weakly to 0 for which both $\| (T - zI) x_n \| \to 0$ and from Lemma 1 $\| (T^* - \bar{z}I) x_n \| \to 0$ and hence $\| (T^* T - |z|^2 I) x_n \| \to 0$ and $\| (TT^* - |z|^2 I) x_n \| \to 0$ or $\| ((T^* T)^{1/2} - |z| I) x_n \| \to 0$ and $\| ((TT^*)^{1/2} - |z| I) x_n \| \to 0$. Thus, $|z|$ is in the essential approximate spectrum of both $T^* T$ and $TT^*$, and the assertion of the theorem follows.

§3. *Theorem* 3—Let $T$ be hyponormal with a polar factorization (1.1). Suppose $z \neq 0$ and satisfies (2.4) and $z = |z| e^{i\theta}$. Then, for any $U$ of (1.1)

$$e^{i\theta} \in \Pi (U). \quad \ldots(3.1)$$

PROOF: Let $z_1 = re^{i\theta}$, where $r = \max \{|z| : z = |z| e^{i\theta}$ and $z \in sp\ (T)\}$ (hence $r > 0$). Clearly, $z_1$ is the boundary point of $sp(T)$ and, in as Theorem 1, there exists a sequence of unit vectors $\{x_n\}$, such that $\| (T - z_1 I) x_n \| \to 0$ and

†The above argument shows that if $T$ is normal, then (2.2) holds, if, instead of (2.1), it is supposed only that

$$z \in sp\,(T)$$

from Lemma 1 $\| (T^* - z_1 I) x_n \| \to 0$ and hence also $\| ((T^* T)^{1/2} - |z_1| I) x_n \| \to 0$ or $\| ((T^* T)^{1/2} - rI) x_n \| \to 0$. But $\| (T - z_1 I) x_n \| = \| U(T^* T)^{1/2} x_n - z_1 x_n \| \to 0.$

Since $r > 0$, this implies that $\| (U x_n - e^{i\theta} x_n) \| \to 0$ and hence, (3.1) holds.

§4. *Theorem* 4—Let $T$ be hyponormal and non-singular, so that $T$ has a (unique) polar factorization (1.1). Then if $e^{i\theta} \in \Pi\ (U)$, there exists a $z = |z| e^{i\theta} \neq 0$ satisfying (2.4).

PROOF: We have $T = UP$ and

$$P^2 - UP^2 U^* = T^* T - TT = D \geqslant 0. \qquad \ldots(4.1)$$

Since $e^{i\theta} \in \Pi\ (U)$, there exists a sequence of unit vectors, $\{x_n\}$, satisfying $\| (U - e^{i\theta} I) x_n \| \to 0$ and from Lemma 1 $\| (U^* - e^{-i\theta} I) x_n \| \to 0$. Clearly

$$\| Dx_n \|^2 = (Dx_n, Dx_n) = ((P^2 - UP^2 U^*) x_n, \ (P^2 - UP^2 U^*) x_n)$$

$$= (P^2 x_n, P^2 x_n) + (P^2 x_n, UP^2 U^* x_n)$$

$$- (UP^2 U^* x_n, P^2 x_n) - (UP^2 U^* x_n, \ UP^* U^* x_n)$$

$$= (P^2 x_n, P^2 x_n) + (P^2_{xn}, e^{i\theta} e^{-i\theta} P^2_{xn})$$

$$- (e^{i\theta} P^2 e^{-i\theta} x_n, P^2 x_n) - (e^{i\theta} P^2 e^{-i\theta} x_n, e^{i\theta} P^2 e^{-i\theta} x_n)$$

$$= (P^2 x_n, P^2 x_n) + (P^2 x_n, P^2 x_n) - (P^2 x_n, P^2 x_n)$$

$$- (P^2 x_n, P^2 x_n) \to 0$$

and so $\| Dx_n \| \to 0$. Hence, by (4.1),

$$\| (P^2 x_n - UP^2 U^* x_n) \| \to 0, \qquad \text{i.e.,} \qquad \| (U - e^{-i\theta} I) P^2 x_n \| \to 0.$$

A similar argument shows that

$$\| (U^* - e^{-i\theta} I) f\ (P^2)\, x_n \| \to 0,$$

where $f\ (t)$ is a polynomial. It then follows that there exists a number $s > 0$ and sequence of unit vectors $\{y_n\}$, such that

$$\| (P^2 - sI)\, y_n \| \to 0 \quad \text{and} \quad \| (U^* - e^{i\theta} I)\, y_n \| \to 0;$$

hence also, $\| (P - s^{1/2} I)\, y_n \| \to 0$ and $\| (U - e^{i\theta} I)\, y_n \| \to 0$. Consequently, if $z = s^{1/2} e^{i\theta}$, then $\| (T - zI)\, y_n \| \to 0$ (also $\| (T^* - \bar{z} I)\, y_n \| \to 0$) and so (2.4) holds.

§5. *Theorem* 5—Let $T$ be hyponormal and suppose

$$r \in \Pi\, (T^* T) (\text{hence } r > 0). \qquad \ldots(5.1)$$

Then there exists a $z \in \Pi\, (T)$ for which $|z| = r^{1/2}$.

PROOF: We first established the theorem under the added hypothesis that $T$ has a polar factorization (1.1) and that $sp(U)$ is not the entire circle $|z| = 1$. Thus,

$$T = UP \text{ and } means_1(sp(U)) < 2\Pi. \qquad ...(5.2)$$

Let $e^{i\theta} \notin \Pi(U)$ and define the unitary operator $U_\theta = e^{-i\theta} U$. Then $1 \notin \Pi(U_\theta)$ and relation (4.1) beomes $P^2 - U_\theta P^2 U^* = D$. Now, $U_\theta$ is the Cayley transform of a self-adjoint operator $A$, where

$$U = (A - iI)(A + iI)^{-1} (U_\theta = e^{-i\theta} U). \qquad ...(5.3)$$

*If* $C = \frac{1}{2}(A + iI) D (A + iI)$, it is seen that

$$AP^2 - P^2 A = iC, \qquad C \geqslant 0. \qquad ...(5.4)$$

For a similar argument, see Putnam (1967, pp. 16–21). Next, by (5.1), $r \in \Pi(P^2)$, so that $\| (P^2 - rI) x_n \| \to 0$ for some sequence of unit vectors $\{x_n\}$. But, by (5.4),

$$\begin{aligned} i \| Cx_n \|^2 &= (Cx_n Cx_n) \\ &= \big((AP^2 - P^2 A) x_n, \ (AP^2 - P^2 A) x_n\big) \\ &= \big(AP^2 x_n, \ (AP^2 - P^2 A) x_n\big) - (P^2 Ax_n, (AP^2 - P^2 A) x_n) \\ &= (AP^2 x_n, AP^2 x_n) - (AP^2 x_n, P^2 Ax_n) \\ &\quad - (P^2 Ax_n, AP^2 x_n) + (P^2 Ax_n, P^2 Ax_n) \\ &= (Ar\, x_n, Ax_n) - (Ax_n, rA\, x_n) - (rAx_n, Ax_n) \\ &\quad + (rA\, x_n, rAx_n) \\ &= \to 0. \end{aligned}$$

and so $\| Cx_n \| \to 0$, and hence by (5.4) again $\| (P^2 - rI) \ Ax_n \| \to 0$. Similarly, one obtains

$$\| (P^2 - rI) A^k x_n \| \to 0 \text{ as } n \to \infty (k = 0, 1, 2, \ldots). \qquad ...(5.5)$$

In view of (5.3) and the relation $U^* = (A - iI)^{-1}(A + iI)$ it follows from (5.5) that

$$\| (P^2 - rI) U^k x_n \| \to 0, \ n \to \infty (k = 0, \pm 1, \pm 2, \ldots). \qquad ...(5.6)$$

Hence, by an argument similar to that of Putnam (1967, p. 46) there exists some $e^{i\theta} \in \Pi(U)$ and a sequence of unit vectors, $\{y_n\}$, such that $\| (P^2 - rI) y_n \| \to 0$ (hence $\| (P - r^{1/2} I) y_n \| \to 0$) and $\| (U - e^{i\theta} I) y_n \| \to 0$. Thus, if $z = r^{1/2} e^{i\theta}$, then $\| (T - zI) y_n \| \to 0$; thus, $z \in \Pi(T)$ and so Theorem 5 is proved in the special case in which (5.2) is assumed.

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