SOME MAPPINGS ON METRIC SPACES

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It is proved that if T is a mapping of a compact metric space X into itself satisfying the inequality $d(Tx, Ty) \geqslant \frac{1}{2}\{d(x, Ty) + d(y, Tx)\}$ for all distinct x, y in X and such that TX, the range of T, is closed, then T is pointwise periodic.

In a recent paper (Fisher 1975), the following theorem was proved.

Theorem 1—If T is a mapping of the complete metric space X into itself satisfying the inequality

$$d(Tx, Ty) \leqslant c\{d(x, Ty) + d(y, Tx)\}$$

for all x, y in X, where $0 \leqslant c < \frac{1}{2}$, then T has a unique fixed point.

Later the following theorem was proved (see Fisher 1976).

Theorem 2—If T is a continuous mapping of a compact metric space X into itself satisfying the inequality

$$d(Tx, Ty) < \tfrac{1}{2}\{d(x, Ty) + d(y, Tx)\}$$

for all distinct x, y in X, then T has a unique fixed point.

We will now prove the following theorem.

Theorem 3—If T is a mapping of a metric space X into itself satisfying the inequality

$$d(Tx, Ty) \geqslant \tfrac{1}{2}\{d(x, Ty) + d(y, Tx)\} \qquad \ldots(1)$$

for all x, y in X, then T is the identity mapping on X and so every point is a fixed point.

Proof: Let x be an arbitrary point in X. Then from inequality (1), we have

$$0 = d(Tx, Tx) \geqslant \tfrac{1}{2}\{d(x, Tx) + d(x, Tx)\}.$$

It follows that

$$d(x, Tx) = 0$$

and so $Tx = x$ for all x in X, completing the proof of the theorem.

We note that the proof of Theorem 3 required that inequality (1) held when $x = y$. This condition is essential, as can be seen by considering the set

$$X = \{1, 2, \ldots, n, \ldots\}$$

with metric

$$d(m, n) = |m - n|$$

and mapping T defined by

$$T(n) = n + 1.$$

Inequality (1) holds for $m \neq n$, but T has no fixed point.

However, we can prove the following theorem.

Theorem 4—If T is a mapping of a metric space X into itself satisfying the inequality

$$d(Tx, Ty) \geqslant \tfrac{1}{2}\{d(x, Ty) + d(y, Tx)\}$$

for all distinct x, y in X, then $Tx = Ty$ implies $x = y$.

PROOF: Suppose we have $Tx = Ty$ with $x \neq y$. Then

$$0 = d(Tx, Ty) \geqslant \tfrac{1}{2}\{d(x, Ty) + d(y, Tx)\}$$

and so

$$d(x, Ty) = 0 = d(y, Tx),$$

from which it follows that

$$x = Ty = Tx = y,$$

giving a contradiction. Thus, $Tx = Ty$ implies $x = y$, completing the proof of the theorem.

We can also prove the following theorem.

Theorem 5—If T is a mapping of a compact metric space X into itself satisfying the inequality

$$d(Tx, Ty) \geqslant \tfrac{1}{2}\{d(x, Ty) + d(y, Tx)\} \qquad \ldots(2)$$

for all distinct x, y in X and such that TX, the range of T, is closed, then T is pointwise periodic.

PROOF: Let x be an arbitrary point in X and assume that the sequence $\{T^n x : n = 1, 2, \ldots\}$ in TX is a sequence of distinct points. Then, since X is compact, it has a convergent subsequence $\{T^{n(r)} x : r = 1, 2, \ldots\}$, converging to a point z' in X. Since TX is closed, z' must be in TX and so $z' = Tz$ for some z in X. From our assumption, either

$$T^n x \neq z$$

for $n = 1, 2, \ldots,$ or

$$T^m x = z$$

for at most one m. Then, with $n(r) > m$, if such an m exists

$$d(Tz, T^{n(r)} x) \geqslant \tfrac{1}{2} \{d(z, T^{n(r)} x) + d(T^{n(r)-1} x, Tz)\}$$
$$\geqslant \tfrac{1}{2} d(z, T^{n(r)} x).$$

On letting r tend to infinity, we have

$$d(Tz, z') \geqslant \tfrac{1}{2} d(z, z')$$

and since $z' = Tz$, it follows that

$$z' = z = Tz.$$

We now have

$$d(z, T^{n(r)} x) = d(Tz, T^{n(r)} x)$$
$$\geqslant \tfrac{1}{2} \{d(z, T^{n(r)} x) + d(T^{n(r)-1} x, Tz)\}$$

and so

$$d(z, T^{n(r)} x) \geqslant d(z, T^{n(r)-1} x)$$
$$\geqslant d(z, T^{n(r)-2} x)$$
$$\geqslant \ldots$$
$$\geqslant d(z, Tx)$$
$$\geqslant d(z, x).$$

Letting r tend to infinity, it follows that

$$z = x = Tx = T^2 x = \ldots,$$

contradicting our assumption that the sequence $\{T^n x\}$ was a sequence of distinct points. This assumption must, therefore, be false, and so

$$T^n x = T^{n+r} x$$

for some $n \geqslant 0$ and $r \geqslant 1$. It follows that $T^n x$ is a fixed point of T^r.

We will now suppose that n is the smallest such n and then that r is the smallest such r. It follows from Theorem 4 that if $n \geqslant 1$, then

$$T^n x = T^{n+r} x$$

implies

$$T^{n-1} x = T^{n+r-1} x,$$

contradicting the minimality of n. Thus, $n = 0$ and so $\{x, Tx, \ldots, T^{r-1} x\}$ is a set of distinct points with

$$T^r(T^i x) = T^i x$$

for $i = 0, 1, 2, \ldots, r-1$. Since this is true for any point x and some r, depending on x, it follows that T is pointwise periodic, completing the proof of the theorem.

We finally note the following two corollaries.

Corollary 1—If T is a mapping of a compact metric space X onto itself satisfying the inequality

$$d(Tx, Ty) \geqslant \tfrac{1}{2}\{d(x, Ty) + d(y, Tx)\}$$

for all distinct x, y in X, then T is pointwise periodic.

PROOF: Since T is a mapping of X onto X, the range of T is closed and the result follows from Theorem 5.

Corollary 2—If T is a continuous mapping of a compact metric space X into itself satisying the inequality

$$d(Tx, Ty) \geqslant \tfrac{1}{2}\{d(x, Ty) + d(y, Tx)\}$$

for all distinct x, y in X, then T is pointwise periodic.

PROOF: Since T is continuous, the range of T must be closed and the result follows from Theorem 5.

REFERENCES

Fisher, B. (1975). A fixed point theorem. *Math. Mag.*, **48**, 223-25.

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