ON SOME NEW INTEGRAL INEQUALITIES AND THEIR DISCRETE ANALOGUES

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This paper presents some new integral inequalities and their discrete analogues which can be used in the analysis of integral and finite difference equations of the more general type.

1. Introduction

The theory of integral and discrete inequalities plays a vital role in the study of integral and finite difference equations. A large number of papers dealing with integral and discrete inequalities have appeared which attribute their origin to the Gronwall-Bellman inequality (1919, 1953). Notable among these is the survey paper of Chandra and Fleishman (1970) and the references given therein. The integral and discrete inequalities recently established by Pachpatte (1973*a–c*, 1974*a-c*, 1976, 1977) have been applied with considerable success to the study of many problems in the theory of integral and finite difference equations of the more general type. In this paper, we wish to establish some new integral inequalities and their discrete analogues that have a wide range of applications in the analysis of a class of integral and finite difference equations. To the author's knowledge, the inequalities proved in this paper have not been reported in literature so far.

2. Integral Inequalities

In this section, we state and prove some interesting and useful integral inequalities which play a vital role in the study of many problems concerning the behaviour of solutions of a class of differential and integral equations.

Theorem 1—Let $x(t)$, $f(t)$, $g(t)$ and $h(t)$ be real-valued non-negative continuous functions defined on $I = [0, \infty)$, for which the inequality

$$x(t) \leqslant x_0 + \int_0^t f(s) \Big(\int_0^s g(\tau) \Big(\int_0^\tau h(k)\, x(k)\, dk \Big) d\tau \Big) ds, \qquad t \in I, \qquad \ldots(1)$$

holds, where x_0 is a non-negative constant. Then

$$x(t) \leqslant x_0 \exp\Big(-\int_0^t f(s)\, ds\Big) \Big[1 + \int_0^t f(s) \exp\Big(\int_0^s (2f(\tau) - g(\tau))\, d\tau \Big).$$

$$\times \Big\{ 1 + \int_0^s g(\tau) \exp\Big(\int_0^\tau (2g(k) + h(k))\, dk \Big) d\tau \Big\} ds \Big], \qquad \ldots(2)$$

for all $t \in I$.

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PROOF: Define a function $m(t)$ by the right member of (1). Then

$$m'(t) = f(t)\left(\int_0^t g(\tau)\left(\int_0^T h(k)\,x(k)\,dk\right)d\tau\right), \qquad m(0) = x_0,$$

which in view of (1) implies

$$m'(t) \leqslant f(t)\left(\int_0^t g(\tau)\left(\int_0^T h(k)\;m(k)\,dk\right)d\tau\right).$$

Adding $f(t)\,m(t)$ to both sides of the above inequality, we have

$$m'(t) + f(t)\,m(t) \leqslant f(t)\left[m(t) + \int_0^t g(\tau)\left(\int_0^T h(k)\,m(k)\,dk\right)d\tau\right]. \qquad \ldots(3)$$

If we put

$$v(t) = m(t) + \int_0^t g(\tau)\left(\int_0^T h(k)\,m(k)\,dk\right)d\tau, \qquad v(0) = m(0) = x_0, \qquad \ldots(4)$$

it follows by differentiating (4) and using the facts that $m'(t) \leqslant f(t)\,v(t)$ and $m(t) \leqslant v(t)$ from (3) and (4), the inequality

$$v'(t) \leqslant f(t)\,v(t) + g(t)\left(\int_0^t h(k)\,v(k)\,dk\right),$$

is satisfied. Adding $g(t)\,v(t)$ to both sides of the above inequality, we have

$$v'(t) + g(t)\,v(t) \leqslant f(t)\,v(t) + g(t)\left[v(t) + \int_0^t h(k)\,v(k)\,dk\right]. \qquad \ldots(5)$$

Put

$$u(t) = v(t) + \int_0^t h(k)\,v(k)\;dk, \qquad u(0) = v(0) = x_0, \qquad \ldots(6)$$

then differentiating (6) and using $v(t) \leqslant u(t)$ and hence $v'(t) \leqslant [f(t) + g(t)]$. $u(t)$ from (6) and (5), the inequality

$$u'(t) \leqslant [f(t) + g(t) + h(t)]\,u(t)$$

is satisfied, which implies the estimation for $u(t)$ such that

$$u(t) \leqslant x_0 \exp\left(\int_0^t [f(s) + g(s) + h(s)]\,ds\right),$$

since $u(0) = x_0$. Substituting this value of $u(t)$ in (5), we have

$$v'(t) + [g(t) - f(t)]\,v(t) \leqslant x_0\,g(t) \exp\left(\int_0^t [f(s) + g(s) + h(s)]\,ds\right).$$

Now, multiplying both sides of the above inequality by $\exp\left(\int_0^t [g(s)-f(s)]\,ds\right)$ and integrating from 0 to t, we have

$$v(t) \leqslant x_0 \exp\left(-\int_0^t [g(s)-f(s)]\,ds\right) \left[1+\int_0^t g(s) \exp\left(\int_0^s [2g(\tau) + h(\tau)]\,d\tau\right) ds\right].$$

Substituting this value of $v(t)$ in (3), we have

$$m'(t) + f(t)m(t) \leqslant x_0 f(t) \exp\left(-\int_0^t [g(s)-f(s)]\,ds\right) \times \left[1+\int_0^t g(s) \exp\left(\int_0^s [2g(\tau)+h(\tau)]\,d\tau\right) ds\right]. \quad \ldots(7)$$

Now, multiplying both sides of (7) by $\exp\left(\int_0^t f(s)\,ds\right)$, integrating from 0 to t and substituting the value of $m(t)$ in (1), we obtain the desired bound in (2).

A useful general version of Theorem 1 may be stated as follows.

Theorem 2—Let $x(t)$, $f(t)$, $g(t)$, $h(t)$ and $p(t)$ be real-valued non-negative continuous functions defined on I, for which the inequality

$$x(t) \leqslant x_0 + \int_0^t f(s) \left(\int_0^s g(\tau) \left(\int_0^\tau [h(k)x(k) + p(k)]\,dk\right) d\tau\right) ds, \quad t \in I,$$

holds, where x_0 is a non-negative constant. Then

$$x(t) \leqslant \exp\left(-\int_0^t f(s)\,ds\right) \Big[x_0 + \int_0^t f(s) \exp\left(\int_0^s [2f(\tau) - g(\tau)]\,d\tau\right). \times \Big[x_0 + \int_0^s g(\tau) \Big\{x_0 \exp\left(\int_0^\tau [2g(k) + h(k)]\,dk\right) + \exp\left(\int_0^\tau [g(k)-f(k)]\,dk\right) \int_0^\tau p(k) \exp\left(\int_0^\tau [f(\rho) + g(\rho) + h(\rho)]\,d\rho\right) dk\Big\}\,d\tau\Big]\,ds\Big],$$

for all $t \in I$.

The proof of this theorem follows by a similar argument as in the proof of Theorem 1 with suitable modifications. We omit the details.

We next state a slightly more general form of Theorem 1 which can be used in some applications.

Theorem 3—Let $x(t)$, $f(t)$, $g(t)$, $h(t)$, $p(t)$ and $q(t)$ be real-valued non-negative continuous functions defined on I, for which the inequality

$$x(t) \leqslant p(t) + q(t)\left[\int_0^t f(s) \left(\int_0^s g(\tau) \left(\int_0^\tau h(k)x(k)\,dk\right) d\tau\right) ds\right], \quad t \in I, \quad \ldots(8)$$

holds. Then

$$x(t) \leqslant p(t) + q(t)\Big[\int_0^t f(s)\exp\Big(\int_t^s f(\tau)\,d\tau\Big)\Big\{\int_0^s g(\tau)\exp\Big(\int_s^T [g(k)$$

$$-f(k)]\,dk\Big)\Big[\int_0^T h(k)p(k)\exp\Big(\int_k^T [f(\rho)+g(\rho)$$

$$+h(\rho)q(\rho)]\,d\rho\Big)\,dk\Big]\,d\tau\Big\}\,ds\Big], \qquad \ldots(9)$$

for all $t \in I$.

By setting, $m(t)$ is equal to the expression in the parentheses [ ] given in (8) and following a similar argument as in the proof of Theorem 1, we obtain the desired bound in (9).

Another interesting and useful integral inequality is embodied in the following theorem.

Theorem 4—Let $x(t)$, $f(t)$, $g(t)$ and $h(t)$ be real-valued non-negative continuous functions defined on I, for which the inequality

$$x(t) \leqslant x_0 + \int_0^t f(s)\Big(\int_0^s g(\tau)\Big(\int_0^T h(k)x^{\alpha}(k)\,dk\Big)\,d\tau\Big)\,ds, \qquad t \in I,$$

holds, where x_0 is a non-negative constant and $0 < \alpha < 1$. Then

$$x(t) \leqslant \exp\Big(-\int_0^t f(s)\,ds\Big)\Big[x_0 + \int_0^t f(s)\exp\Big(\int_0^t [2f(\tau)$$

$$-g(\tau)]\,d\tau\Big)\Big\{x_0 + \int_0^s g(\tau)\exp\Big(\int_0^T 2g(k)\,dk\Big)\Big[x_0^{1-\alpha} + (1-\alpha)$$

$$\times \int_0^T h(k)\exp\Big(-(1-\alpha)\int_0^k [f(\rho)+g(\rho)]\,d\rho\Big)\,dk\Big]^{1/(1-\alpha)}\,d\tau\Big\}\,ds\Big],$$

for all $t \in I$.

The proof of this theorem follows by an argument similar to that used in the proof of Theorem 1. We invite the reader to accomplish the proof.

We next state and prove the following integral inequality which may be convenient in some applications.

Theorem 5—Let $x(t)$, $f(t)$, $g(t)$ and $h(t)$ be real-valued non-negative continuous functions defined on I, for which the inequality

$$x(t) \leqslant x_0 + \int_0^t f(s)\Big(\int_0^s g(\tau)x(\tau)\Big(\int_0^T h(k)x(k)\,dk\Big)\,d\tau\Big)\,ds, \quad t \in I, \quad \ldots(10)$$

holds, where x_0 is a positive constant. Then

$$x(t) \leqslant x_0\exp\Big(-\int_0^t f(s)\,ds\Big) + \int_0^t f(s)\exp\Big(\int_t^s f(\tau)\,d\tau\Big)$$

$$\times \left[\frac{x_0\exp\Big(\int_0^s [f(\tau)+g(\tau)Q(\tau)]\,d\tau\Big)}{1 + x_0\int_0^s g(\tau)\exp\Big(\int_0^T [f(k)+g(k)Q(k)]\,dk\Big)\,d\tau}\right]ds, \quad \ldots(11)$$

for all $t \in I$, where

$$Q(t) = \frac{x_0 \exp\left(\int_0^t [f(s) + g(s)]\, ds\right)}{1 - x_0 \int_0^t g(s) \exp\left(\int_0^s [f(\tau) + g(\tau)]\, d\tau\right) ds}, \quad t \in I, \quad \ldots(12)$$

in which

$$\int_0^t g(s) \exp\left(\int_0^s [f(\tau) + g(\tau)]\, d\tau\right) ds < x_0^{-1}, \quad \text{for all } t \in I.$$

PROOF: Define a function $m(t)$ by the right member of (1). Then

$$m'(t) = f(t)\left(\int_0^t g(\tau)\, x(\tau)\left(\int_0^\tau h(k)\, x(k)\, dk\right) d\tau\right), \quad m(0) = x_0,$$

which in view of (10) implies

$$m'(t) \leq f(t)\left(\int_0^t g(\tau)\, m(\tau)\left(\int_0^\tau h(k)\, m(k)\, dk\right) d\tau\right).$$

Adding $f(t)\, m(t)$ to both sides of the above inequality, we have

$$m'(t) + f(t)\, m(t) \leqslant f(t)\left[m(t) + \int_0^t g(\tau)\, m(\tau)\left(\int_0^\tau h(k)\, m(k)\, dk\right) d\tau\right]. \quad \ldots(13)$$

If we put

$$v(t) = m(t) + \int_0^t g(\tau)\, m(\tau)\left(\int_0^\tau h(k)\, m(k)\, dk\right) d\tau, \quad v(0) = m(0) = x_0, \quad \ldots(14)$$

it follows by differentiating (14) and using the facts that $m'(t) \leqslant f(t)\, v(t)$ and $m(t) \leqslant v(t)$ from (13) and (14), the inequality

$$v'(t) \leqslant f(t)\, v(t) + g(t)\, v(t)\left(\int_0^t h(k)\, v(k)\, dk\right),$$

is satisfied. Adding $g(t)\, v^2(t)$ to both sides of the above inequality, we have

$$v'(t) + g(t)\, v^2(t) \leqslant f(t)\, v(t) + g(t)\, v(t)\left[v(t) + \int_0^t h(k)\, v(k)\, dk\right]. \quad \ldots(15)$$

Put

$$u(t) = v(t) + \int_0^t h(k)\, v(k)\, dk, \qquad u(0) = v(0) = x_0, \quad \ldots(16)$$

then differentiating (16) and using $v(t) \leqslant u(t)$ and hence $v'(t) \leqslant f(t)\, u(t) + g(t)\, u^2(t)$ from (16) and (15), the inequality

$$u'(t) \leqslant f(t)\, u(t) + h(t)\, u(t) + g(t)\, u^2(t), \quad \ldots(17)$$

is satisfied. The inequality (17) can be written as

$$u^{-2}(t)\, u'(t) - [f(t) + h(t)]\, u^{-1}(t) \leqslant g(t). \quad \ldots(18)$$

Put $u^{-1}(t) = r(t)$, so that $-u^{-2}(t)\,u'(t) = r'(t)$ and $r(0) = x_0^{-1}$, then, we obtain

$$r'(t) + [f(t) + h(t)]\,r(t) \geqslant -g(t), \qquad \ldots(19)$$

which implies the estimate for $r(t)$ such that

$$r(t)\exp\Big(\int_0^t [f(s) + h(s)]\,ds\Big) \geqslant r(0) - \int_0^t g(s)\exp\Big(\int_0^s [f(\tau) + h(\tau)]\,d\tau\Big)\,ds.$$

Now, substituting $r(t) = u^{-1}(t)$ in the above inequality, we have

$$u(t) \leqslant \frac{x_0\exp\Big(\int_0^t [f(s) + h(s)]\,ds\Big)}{1 - x_0\int_0^t g(s)\exp\Big(\int_0^s [f(\tau) + g(\tau)]\,d\tau\Big)\,ds} = Q(t),$$

since $r(0) = 1/x_0$. Substituting this value of $u(t)$ in (15), we have

$$v'(t) + g(t)\,v^2(t) \leqslant [f(t) + g(t)\,Q(t)]\,v(t).$$

Again, following the same argument as above with suitable modifications, we obtain the estimate for $v(t)$ such that

$$v(t) \leqslant \frac{x_0\exp\Big(\int_0^t [f(s) + g(s)\,Q(s)]\,ds\Big)}{1 + x_0\int_0^t g(s)\exp\Big(\int_0^s [f(\tau) + g(\tau)\,Q(\tau)]\,d\tau\Big)\,ds}$$

Now, substituting this value of $v(t)$ in (13), we have

$$m'(t) + f(t)\,m(t) \leqslant f(t)\left[\frac{x_0\exp\Big(\int_0^t [f(s) + g(s)\,Q(s)]\,ds\Big)}{1 + x_0\int_0^t g(s)\exp\Big(\int_0^s [f(\tau) + g(\tau)\,Q(\tau)]\,d\tau\Big)\,ds}\right]. \qquad \ldots(20)$$

Multiplying both sides of (20) by $\exp\Big(\int_0^t f(s)\,ds\Big)$, integrating from 0 to t and substituting the value of $m(t)$ in (10), we obtain the desired bound in (11).

We note that the integral inequalities established in this paper allow us to study the stability, boundedness and asymptotic behaviour of the solutions of a class of differential and integral equations similar to those obtained by Pachpatte (1973, 1975). To be more specific, the inequalities obtained in Theorems 1 and 4 can be used to establish similar results as in Pachpatte (1975, Theorems 1–3) and Pachpatte (1973, Theorems 1, 2) on the behaviour of solutions of non-linear perturbed differential equations of the form

$$x'(t) = f\big(t, x(t)\big) + H\Big[t, \int_0^t g\Big(t, s, \int_0^s k\big(s, \tau, x(\tau)\big)\,d\tau\Big)\,ds\Big],$$

and

$$x'(t) = f\,(t, x(t)) + H[t, \int_0^t g\,(t, s \int_0^s k\,(s, \tau, x^{\alpha}(t))\, d\tau)\, ds], 0 < \alpha < 1,$$

as a perturbation of the nonlinear differential system

$$y'(t) = f\,(t, y(t)),$$

under some suitable conditions on the functions involved. Since the details of these results are similar to those given in Pachpatte (1973, 1975), we do not discuss them here.

3. Discrete Analogues

Discrete inequalities involving sequences of real numbers, which may be considered as discrete analogues of the Gronwall-Bellman inequality or its variants, have been used extensively in the analysis of finite difference equations. In this section, we wish to establish the discrete analogues of the integral inequalities established in Section 2 that have a wide range of applications in the theory of finite difference equations of the more general type.

Before giving the main results in this section, we first recollect a few of the basic notions and definitions from Sugiyama (1969) (*see* also Pachpatte 1973*b*, 1973*c*). Let N be a set of points $n_0 + k\,(k=0, 1, 2, \ldots)$, where $n_0 \geqslant 0$ is a given integer. The expression $\sum_{s=n_0}^{n-1} b(s)$ represents a solution of the linear difference equation

$$\Delta\, x(n) = b\,(n)$$

for all $n \in N$ under the initial condition $x(n_0) = 0$, where Δ is the operator defined by $\Delta x(n) = x(n+1) - x(n)$. It is supposed that $\sum_{s=n_0}^{n_0-1} b(s) = 0$. The expression $\prod_{s=n_0}^{n-1} c(s)$ represents a solution of the linear difference equation

$$x(n+1) = c(n)\, x(n)$$

for all $n \in N$ under the initial condition $x(n_0) = 1$. It is supposed that $\prod_{s=n_0}^{n_0-1} c(s) = 1$.

We now state and prove the discrete versions of Theorems 1–5.

Theorem 6—Let $x(n)$, $f(n)$, $g(n)$ and $h(n)$ be real-valued non-negative functions defined on N, for which the inequality

$$x(n) \leqslant x_0 + \sum_{s=n_0}^{n-1} f(s) \Big(\sum_{t=n_0}^{s-1} g(t) \Big(\sum_{k=n_0}^{t-1} h(k)\, x(k) \Big)\Big), \qquad \ldots(21)$$

holds for all $n \in N$, where x_0 is a non-negative constant. If $1 - f(n) \geqslant 0$ and $1 + f(n) - g(n) \geqslant 0$, for all $n \in N$, then

$$x(n) \leqslant x_0 \Big[\prod_{s=n_0}^{n-1} (1 - f(s)) + \sum_{s=n_0}^{n-1} f(s) \prod_{t=s+1}^{n-1} (1 - f(t)) \Big\{ \prod_{t=n_0}^{s-1} (1 + f(t) - g(t)) + \sum_{t=n_0}^{s-1} g(t) \prod_{\tau=t+1}^{s-1} (1 + f(\tau) - g(\tau)) \cdot \prod_{\tau=n_0}^{t-1} [1 + f(\tau) + g(\tau) + h(\tau)] \Big\} \Big], \quad \ldots(22)$$

for all $n \in N$.

Proof: Define a function $m(n)$ by the right member of (21). Then

$$\Delta m(n) = f(n) \Big(\sum_{t=n_0}^{n-1} g(t) \Big(\sum_{k=n_0}^{t-1} h(k) x(k) \Big) \Big), \qquad m(n_0) = x_0,$$

which, in view of (21), implies

$$\Delta m(n) \leqslant f(n) \Big(\sum_{t=n_0}^{n-1} g(t) \Big(\sum_{k=n_0}^{t-1} h(k) m(k) \Big) \Big).$$

Adding $f(n) m(n)$ to both sides of the above inequality, we have

$$\Delta m(n) + f(n) m(n) \leqslant f(n) \Big[m(n) + \sum_{t=n_0}^{n-1} g(t) \Big(\sum_{k=n_0}^{t-1} h(k) m(k) \Big) \Big]. \quad \ldots(23)$$

If we put

$$v(n) = m(n) + \sum_{t=n_0}^{n-1} g(t) \Big(\sum_{k=n_0}^{t-1} h(k) m(k) \Big), \quad v(n_0) = m(n_0) = x_0, \quad \ldots(24)$$

it follows that

$$\Delta v(n) = \Delta m(n) + g(n) \Big(\sum_{k=n_0}^{n-1} h(k) m(k) \Big). \quad \ldots(25)$$

Using the facts that $\Delta m(n) \leqslant f(n) v(n)$ and $m(n) \leqslant v(n)$ from (23) and (24), we observe that the inequality

$$\Delta v(n) \leqslant f(n) v(n) + g(n) \Big(\sum_{k=n_0}^{n-1} h(k) v(k) \Big),$$

is satisfied. Adding $g(n) v(n)$ to both sides of the above inequality, we have

$$\Delta v(n) + g(n) v(n) \leqslant f(n) v(n) + g(n) \Big[v(n) + \sum_{k=n_0}^{n-1} h(k) v(k) \Big]. \quad \ldots(26)$$

Put

$$u(n) = v(n) + \sum_{k=n_0}^{n-1} h(k) v(k), \qquad u(n_0) = v(n_0) = x_0, \quad \ldots(27)$$

then

$$\Delta u(n) = \Delta v(n) + h(n) v(n). \quad \ldots(28)$$

Using $v(n) \leqslant u(n)$ and hence $\Delta v(n) \leqslant [f(n) + g(n)]u(n)$ from (27) and (26), the inequality

$$u(n+1) \leqslant [1 + f(n) + g(n) + h(n)]\, u(n),$$

is satisfied, which implies the estimation for $u(n)$ such that

$$u(n) \leqslant x_0 \prod_{s=n_0}^{n-1} [1 + f(s) + g(s) + h(s)],$$

since $u(n_0) = x_0$. Substituting this value of $u(n)$ in (26), we have

$$v(n+1) - [1 + f(n) - g(n)]\, v(n) \leqslant x_0 g(n) \prod_{s=n_0}^{n-1} [1 + f(s) + g(s) + h(s)].$$

Multiplying by $\prod_{s=n_0}^{n} [1 + f(s) - g(s)]^{-1}$ to both sides of the above inequality and summing up both sides from n_0 to $n-1$, it follows that

$$v(n) \prod_{s=n_0}^{n-1} [1 + f(s) - g(s)]^{-1} - x_0 \leqslant x_0 \sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^{s} [1 + f(t) - g(t)]^{-1} \times \prod_{t=n_0}^{s-1} [1 + f(t) + g(t) + h(t)]$$

which implies

$$v(n) \leqslant x_0 \Big[\prod_{s=n_0}^{n-1} [1 + f(s) - g(s)] + \sum_{s=n_0}^{n-1} g(s) \prod_{t=s+1}^{n-1} [1 + f(t) - g(t)] \times \prod_{t=n_0}^{s-1} [1 + f(t) + g(t) + h(t)].$$

Substituting this value of $v(n)$ in (23), we have

$$m(n+1) - (1 - f(n))\, m(n) \leqslant x_0 f(n) \Big[\prod_{s=n_0}^{n-1} [1 + f(s) - g(s)] + \sum_{s=n_0}^{n-1} g(s) \prod_{t=s+1}^{n-1} [1 + f(t) - g(t)] \times \prod_{t=n_0}^{s-1} [1 + f(t) + g(t) + h(t)].$$

Now, multiplying by $\prod_{s=n_0}^{n} (1 - f(s))^{-1}$ to both sides of the above inequality and summing up both sides from n_0 to $n-1$, we obtain the estimate for $m(n)$ such that

$$m(n) \leqslant x_0 \Big[\prod_{s=n_0}^{n-1} (1 - f(s)) + \sum_{s=n_0}^{n-1} f(s) \prod_{t=s+1}^{n-1} (1 - f(t)) \Big\{ \prod_{t=n_0}^{s-1} [1 + f(t) - g(t)] + \sum_{t=n_0}^{s-1} g(t) \prod_{\tau=t+1}^{s-1} [1 + f(\tau) - g(\tau)] \cdot \prod_{\tau=n_0}^{t-1} [1 + f(\tau) + g(\tau) + h(\tau)] \Big\} \Big].$$

Now, substituting the value of $m(n)$ in (21), we obtain the desired bound in (22).

Theorem 7—Let $x(n)$, $f(n)$, $g(n)$, $h(n)$ and $p(n)$ be real-valued nonnegative functions defined on N, for which the inequality

$$x(n) \leqslant x_0 + \sum_{s=n_0}^{n-1} f(s) \Big(\sum_{t=n_0}^{s-1} g(t) \Big(\sum_{k=n_0}^{t-1} [h(k)\, x(k) + p(k)] \Big)\Big),$$

holds for all $n \in N$, where x_0 is a non-negative constant. If $1 - f(n) \geqslant 0$ and $1 + f(n) - g(n) \geqslant 0$, for all $n \in N$, then

$$\begin{aligned} x(n) \leqslant{} & x_0 \prod_{s=n_0}^{n-1} (1 - f(s)) + \sum_{s=n_0}^{n-1} f(s) \prod_{t=s+1}^{n-1} (1 - f(t)) \\ & \times x_0 \prod_{t=n_0}^{s-1} [1 + f(t) - g(t)] + \sum_{t=n_0}^{s-1} g(t) \prod_{\tau=t+1}^{s-1} [1 + f(\tau) - g(\tau)] \\ & \times \{ x_0 \prod_{\tau=n_0}^{t-1} [1 + f(\tau) + g(\tau) + h(\tau)] \\ & + \sum_{\tau=n_0}^{t-1} p(\tau) \prod_{k=\tau+1}^{t-1} [1 + f(k) + g(k) + h(k)\}] , \end{aligned}$$

for all $n \in N$.

The proof of this theorem follows by a similar argument as in the proof of Theorem 6 with suitable modifications. We omit the details.

Theorem 8—Let $x(n)$, $f(n)$, $g(n)$, $p(n)$ and $q(n)$ be real-valued non-negative functions defined on N, for which the inequality

$$x(n) \leqslant p(n) + q(n) \Big[\sum_{s=n_0}^{n-1} f(s) \Big(\sum_{t=n_0}^{s-1} g(t) \Big(\sum_{k=n_0}^{t-1} h(k)\, x(k) \Big)\Big)\Big], \qquad \ldots(29)$$

holds for all $n \in N$. If $1 - f(n) \geqslant 0$ and $1 + f(n) - g(n) \geqslant 0$ for all $n \in N$, then

$$\begin{aligned} x(n) \leqslant{} & p(n) + q(n) \Big[\sum_{s=n_0}^{n-1} f(s) \prod_{t=s+1}^{n-1} (1 - f(t)) \cdot \sum_{t=n_0}^{s-1} g(t) \\ & \times \prod_{\tau=t+1}^{s-1} (1 + f(\tau) - g(\tau)) \cdot \\ & \times \Big(\sum_{\tau=n_0}^{t-1} h(\tau) p(\tau) \prod_{k=\tau+1}^{t-1} [1 + f(k) + g(k) + h(k) q(k)) \Big], \qquad \ldots(30) \end{aligned}$$

for all $n \in N$.

By setting, $m(n)$ is equal to the expression in the parentheses [ ] given in (29) and following the same argument as in the proof of Theorem 6, we obtain the desired bound in (30).

Theorem 9—Let $x(n)$, $f(n)$, $g(n)$ and $h(n)$ be real-valued non-negative functions defined on N, for which the inequality

$$x(n) \leqslant x_0 + \sum_{s=n_0}^{n-1} f(s) \Big(\sum_{t=n_0}^{s-1} g(t) \Big(\sum_{k=n_0}^{t-1} h(k)\, x^{\alpha}(k) \Big)\Big), \qquad \ldots(31)$$

holds for all $n \in N$, where x_0 is a non-negative constant and $0 < \alpha < 1$. If $1 - f(n) \geqslant 0$ and $1 + f(n) - g(n) \geqslant 0$ for all $n \in N$, then

$$x(n) \leqslant x_0 \prod_{t=n_0}^{n-1} (1 - f(s)) + \sum_{s=n_0}^{n-1} f(s) \prod_{t=s+1}^{n-1} (1 - f(t))$$
$$\times [x_0 \prod_{t=n_0}^{s-1} (1 + f(t) - g(t)) + \sum_{t=n_0}^{s-1} g(t) \prod_{\tau=t+1}^{s-1} (1 + f(\tau) - g(\tau))$$
$$\times \prod_{\tau=n_0}^{t-1} (1 + f(\tau) + g(\tau)) \,.\, \{ x_0^{1-\alpha} + (1 - \alpha) \sum_{\tau=n_0}^{t-1} h(\tau)$$
$$\times \prod_{k=n_0}^{\tau} (1 + f(k) + g(k))^{\alpha-1}\}^{1/(1-\alpha)}], \qquad \ldots(32)$$

for all $n \in N$.

PROOF: Define a function $m(n)$ by the right member of (31). Then

$$\Delta m(n) = f(n) \Big(\sum_{t=n_0}^{n-1} g(t) \Big(\sum_{k=n_0}^{t-1} h(k) x^{\alpha}(k) \Big) \Big), \qquad m(n_0) = x_0,$$

which implies

$$\Delta m(n) \leqslant f(n) \Big(\sum_{t=n_0}^{n-1} g(t) \Big(\sum_{k=n_0}^{t-1} h(k) m^{\alpha}(k) \Big) \Big),$$

since $x^{\alpha}(n) \leqslant m^{\alpha}(n)$. Adding $f(n) m(n)$ to both sides of the above inequality, we have

$$\Delta m(n) + f(n) m(n) \leqslant f(n) [m(n) + \sum_{t=n_0}^{n-1} g(t) \Big(\sum_{k=n_0}^{t-1} h(k) m^{\alpha}(k) \Big)]. \qquad \ldots(33)$$

If we put

$$v(n) = m(n) + \sum_{t=n_0}^{n-1} g(t) \Big(\sum_{k=n_0}^{t-1} h(k) m^{\alpha}(k) \Big), \; v(n_0) = m(n_0) = x_0, \qquad \ldots(34)$$

it follows that

$$\Delta v(n) = \Delta m(n) + g(n) \Big(\sum_{k=n_0}^{n-1} h(k) m^{\alpha}(k) \Big). \qquad \ldots(35)$$

Using the facts $\Delta m(n) \leqslant f(n) v(n)$ and $m(n) \leqslant v(n)$ from (33) and (34), the inequality

$$\Delta v(n) \leqslant f(n) v(n) + g(n) \Big(\sum_{k=n_0}^{n-1} h(k) v^{\alpha}(k) \Big),$$

is satisfied. Adding $g(n) v(n)$ to both sides of the above inequality, we have

$$\Delta v(n) + g(n) v(n) \leqslant f(n) v(n) + g(n) [v(n) + \sum_{k=n_0}^{n-1} h(k) v^{\alpha}(k)]. \qquad \ldots(36)$$

Put

$$u(n) = v(n) + \sum_{k=n_0}^{n-1} h(k) v^{\alpha}(k), \qquad u(n_0) = v(n_0) = x_0, \qquad \ldots(37)$$

then

$$\Delta u(n) = \Delta v(n) + h(n) v^{\alpha}(n). \qquad \ldots(38)$$

Using the facts $v(n) \leqslant u(n)$ and hence $\Delta v(n) \leqslant [f(n) + g(n)] u(n)$ from (37) and (36), the inequality

$$u(n+1) - [1 + f(n) + g(n)] u(n) \leqslant h(n) u^{\alpha}(n), \qquad \ldots(39)$$

is satisfied. Define

$$e(n) = \prod_{s=n_0}^{n-1} [1 + f(s) + g(s)]^{-1}, \qquad e(n_0) = 1,$$

then

$$e(n+1) - e(n) = -[f(n) + g(n)] e(n+1). \qquad \ldots(40)$$

Multiplying by $e(n+1)$ to both sides of (39) and using (40), we obtain

$$u(n+1) e(n+1) - u(n) e(n) \leqslant h(n) e^{1-\alpha}(n+1) [u(n) e(n+1)]^{\alpha}. \qquad \ldots(41)$$

Because $u(n)$ is monotone increasing, $e(n)$ is monotone decreasing, and $-\alpha < 0$, we know that (Willet and Wong 1964)

$$[u(n) e(n+1)]^{-\alpha} \geqslant z^{-\alpha},$$

for all values of z between $u(n) e(n)$ and $u(n+1) e(n+1)$. So if we apply the mean value theorem to the function

$$F(z) = \frac{z^{1-\alpha}}{1-\alpha},$$

we see that

$$\frac{[u(n+1) e(n+1)]^{1-\alpha} - [u(n) e(n)]^{1-\alpha}}{1-\alpha}$$
$$\leqslant [u(n) e(n+1)]^{-\alpha} [u(n+1) e(n+1) - u(n) e(n)]. \qquad \ldots(42)$$

From (41) and (42), we obtain

$$[u(n+1) e(n+1)]^{1-\alpha} - [u(n) e(n)]^{1-\alpha} \leqslant (1-\alpha) h(n) e^{1-\alpha}(n+1). \qquad \ldots(43)$$

Summing up both sides of (43) from n_0 to $n-1$, we obtain the estimate for $u(n)$ such that

$$u(n) \leqslant \prod_{s=n_0}^{n-1} [1 + f(s) + g(s)] [x_0^{1-\alpha} + (1-\alpha) \sum_{s=n_0}^{n-1} h(s)$$
$$\times \prod_{t=n_0}^{s} [1 + f(t) + g(t)]^{\alpha-1}]^{1/(1-\alpha)}.$$

Substituting this value of $u(n)$ in (36), we obtain

$$v(n+1)-[1+f(n)-g(n)]\,v(n)\leqslant g(n)\prod_{s=n_0}^{n-1}[1+f(s)+g(s)]$$
$$\times\,[x_0^{1-\alpha}+(1-\alpha)\sum_{s=n_0}^{n-1}h(s)\prod_{t=n_0}^{s}[1+f(t)+g(t)]^{\alpha-1}]^{1/(1-\alpha)}.$$

Multiplying by $\prod_{s=n_0}^{n}[1+f(s)-g(s)]^{-1}$ to both sides of the above inequality and summing up both sides from n_0 to $n-1$, we obtain the estimate for $v(n)$ such that

$$v(n)\leqslant x_0\prod_{s=n_0}^{n-1}[1+f(s)-g(s)]+\sum_{s=n_0}^{n-1}g(s)\prod_{t=s+1}^{n-1}[1+f(t)-g(t)]$$
$$\times\prod_{t=n_0}^{s-1}[1+f(t)+g(t)]\,.\,[x_0^{1-\alpha}+(1-\alpha)\sum_{t=n_0}^{s-1}h(t)$$
$$\times\prod_{\tau=n_0}^{t}[1+f(\tau)+g(\tau)]^{\alpha-1}]^{1/(1-\alpha)}.$$

Substituting this value of $v(n)$ in (33), we obtain

$$m(n+1)-[1-f(n)]\,m(n)\leqslant f(n)\,[x_0\prod_{s=n_0}^{n-1}[1+f(s)-g(s)]$$
$$+\sum_{s=n_0}^{n-1}g(s)\prod_{t=s+1}^{n-1}[1+f(t)-g(t)]\,.\prod_{t=n_0}^{s-1}[1+f(t)+g(t)]$$
$$\times\,[x_0^{1-\alpha}+(1-\alpha)\sum_{t=n_0}^{s-1}h(t)\prod_{\tau=n_0}^{t}[1+f(\tau)+g(\tau)]^{\alpha-1}]^{1/(1-\alpha)}].$$

Multiplying by $\prod_{s=n_0}^{n}[1-f(s)]^{-1}$ to both sides of the above inequality and summing up both sides from n_0 to $n-1$, we obtain the estimate for $m(n)$ such that

$$m(n)\leqslant x_0\prod_{s=n_0}^{n-1}[1-f(s)]+\sum_{s=n_0}^{n-1}f(s)\prod_{t=s+1}^{n-1}[1-f(t)]$$
$$\times\,[x_0\prod_{t=n_0}^{s-1}[1+f(t)+g(t)]+\sum_{t=n_0}^{s-1}g(t)\prod_{\tau=t+1}^{s-1}[1+f(\tau)-g(\tau)]$$
$$\times\prod_{\tau=n_0}^{t-1}[1+f(\tau)+g(\tau)]\,.\,[x_0^{1-\alpha}+(1-\alpha)\sum_{\tau=n_0}^{t-1}h(\tau)$$
$$\times\prod_{k=n_0}^{\tau}[1+f(k)+g(k)]^{\alpha-1}]^{1/(1-\alpha)}].$$

Now, substituting the value of $m(n)$ in (31), we obtain the desired bound in (32).

To this end, we state the discrete analogue of the integral inequality established in Theorem 5.

Theorem 10—Let $x(n)$, $f(n)$, $g(n)$ and $h(n)$ be real-valued non-negative functions defined on N, for which the inequality

$$x(n) \leqslant x_0 + \sum_{s=n_0}^{n-1} f(s)\Big(\sum_{t=n_0}^{s-1} g(t)\, x(t)\Big(\sum_{k=n_0}^{t-1} h(k)\, x(k)\Big)\Big), \qquad \ldots(44)$$

holds for all $n \in N$, where x_0 is a positive constant. If $1 - f(n) \geqslant 0$ for all $n \in N$, then

$$x(n) \leqslant x_0 \prod_{s=n_0}^{n-1} [1 - f(s)] + \sum_{s=n_0}^{n-1} f(s) \prod_{t=s+1}^{n-1} [1 - f(t)] \,.$$

$$\times \left[\frac{x_0 \prod_{t=n_0}^{s-1} [1 + f(t) + g(t)\, Q(t)]}{1 + x_0 \sum_{t=n_0}^{s-1} g(t) \prod_{\tau=n_0}^{t} [1 + f(\tau) + g(\tau)\, Q(\tau)]} \right], \qquad \ldots(45)$$

for all $n \in N$, where

$$Q(n) = \frac{x_0 \prod_{s=n_0}^{n-1} [1 + f(s) + h(s)]}{1 - x_0 \sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^{s} [1 + f(t) + h(t)]}, \; n \in N, \qquad \ldots(46)$$

in which $\sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^{s} [1 + f(t) + h(t)] < x_0^{-1}$, for all $n \in N$.

The proof of this theorem follows by an argument similar to that used in the proof of Theorem 9 in view of the proof of Theorem 5 given in Section 2. We invite the reader to accomplish the proof.

Finally, we note that the discrete inequalities established in this section can be used to study the stability, boundedness and asymptotic behaviour of the solutions of a class of more general finite difference equations and stochastic discrete time systems similar to those obtained by this author earlier (Pachpatte 1973*c*, 1976). Since the details of these results are similar to those given in these papers, we do not discuss it here. Other applications of the inequalities established in this paper will appear elsewhere.

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