ON THE EFFECT OF NON-UNIFORM ELECTRICAL CONDUCTIVITY ON THE HYDROMAGNETIC STABILITY OF A HOT LAYER

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It has been assumed that in the static state, σ, the electrical conductivity of an infinite horizontal layer of hot fluid with a uniform adverse temperature gradient and a magnetic field acting in the vertical direction, is given by $\sigma = \sigma_0 (1 + \gamma z)$, where z is the distance from the lower boundary and γ is a constant, being small in value. The convective instabilities in two cases, namely, (i) when both the boundaries are free, and (ii) when both the boundaries are rigid, have been investigated on the basis of linear theory, and R_c, the critical value of the Rayleigh number, has been determined to the first order in γ. It has been found that in case (i) R_c is greater or less than that obtained under the assumption that σ is constant according as $\gamma > 0$ or $\gamma < 0$. In case (ii) R_c is found to remain unchanged.

1. Introduction

The stability of an electrically conducting fluid in the presence of a magnetic fluid has drawn the attention of several workers in connection with studies on a variety of phenomena in the fields of astrophysics, geophysics and engineering. One of the models studied by Chandrasekhar (1961, pp. 160) was that of a horizontal layer of electrically conducting incompressible viscous fluid with an adverse temperature gradient in the vertically upward direction in the presence of a magnetic field, assuming electrical conductivity, thermal conductivity, coefficient of viscosity and temperature gradient to be constant. Chandrasekhar (1961, pp. 198) also considered the case when such a system is subjected to rotation about a vertical axis, since its study is useful for understanding such phenomena in the celestial and terrestial fields where Coriolis force becomes important.

In the geothermal region, the temperature variation is non-uniform due to high thermal radiative heat transfer. Stability crtieria for such a layer under radiative heat transfer were studied by Murgai and Khosla (1962) for two extreme cases, namely an optically thin layer and an optically thick layer. Bhattacharyya and Jain (1976) determined the stability criterion of a hot layer of fluid in the presence of a magnetic field, assuming the temperature gradient in the static state to be such as can be accounted for by a uniform distribution of heat source.

In this study, we have investigated the effect of the non-uniform electrical conductivity, σ, as given by $\sigma = \sigma_0(1 + \gamma z)$ in the static state, where z is the distance measured from the lower boundary and γ, a constant being small in value, on the gravitational instability of a hot horizontal layer of conducting fluid in the presence of a uniform adverse temperature gradient and a magnetic field in the vertical direction. Two cases have been studied separately: (i) when both the boundaries are free, and (ii) when both the boundaries are rigid.

2. Governing Equations in the Convective Marginal State

We take a set of cartesian axes with origin at the lower boundary and z-axis vertically upwards. Assuming the marginal state to be convective, we have the following set of linearized equations obtained from the basic hydromagnetic equations (Chandrasekhar, 1961, pp. 146) under Boussinesq approximation and the well-known equation of heat conduction:

$$\left.\begin{aligned}
&-\nabla(\delta\widetilde{\omega}) + \frac{\mu}{4\pi\rho_0} H_3 \frac{\partial \vec{h}}{\partial z} + g\alpha\,\theta\,\vec{l} + \nu\nabla^2 \vec{q} = 0 \\
&\delta\widetilde{\omega} = \frac{1}{\rho_0}\left\{\delta p + \mu\frac{\vec{H}\,.\,\vec{h}}{4\pi}\right\}, \qquad \nu = \frac{\mu_0}{\rho_0} \\
&\nabla\times\vec{e} = 0, \qquad \nabla\times\vec{h} = 4\pi\vec{j} \\
&\vec{j} = \sigma\,[\vec{e} + \mu\vec{q}\times\vec{H}] \\
&\sigma = \sigma_0(1 + \gamma z) \\
&\nabla\,.\,\vec{q} = 0, \qquad \nabla\,.\,\vec{h} = 0 \\
&-w\beta = k\nabla^2\theta
\end{aligned}\right\} \qquad \ldots(1)$$

where μ_0, μ, g, α, ρ_0, σ, β and k denote the coefficient of viscosity, magnetic permeability, accleration due to gravity, coefficient of volume expansion, density of the fluid at the temperature T_0 of the lower boundary (in the static state), the electrical conductivity, the uniform adverse temperature gradient in the vertically upward direction and the coefficient of thermometric conductivity respectively. In the above equations, we have taken δp, θ, $\vec{q} = (u, v, w)$, $\vec{h}$, $\vec{e}$ and $\vec{j}$ to denote the variations of pressure, temperature, velocity, magnetic field, electric field and current density respectively from the corresponding quantities of the static state, with $\vec{H} = (0, 0, H_3)$ and $\vec{l}$ as the magnetic field in the static state and the unit vector in the direction of z-axis respectively.

From (1), we have

$$0 = \nabla \times \vec{e} = \nabla \times \left(\frac{\vec{j}}{\sigma} \right) - \mu \nabla \times (\vec{q} \times \vec{H})$$

$$= -\frac{1}{\sigma^2} \nabla \sigma \times \vec{j} + \frac{1}{\sigma} \nabla \times \vec{j} - \mu \nabla \times (\vec{q} \times \vec{H})$$

$$= -\frac{1}{4\pi\sigma^2} \left[\sigma_0 \gamma \vec{l} \times (\nabla \times \vec{h}) + \sigma \nabla^2 \vec{h} + 4\pi\mu H_3 \sigma^2 \frac{\partial \vec{q}}{\partial z} \right]$$

since

$$\nabla \sigma = \sigma_0 \gamma \vec{l}.$$

Hence,

$$\gamma \vec{l} \times (\nabla \times \vec{h}) + (1 + \gamma z) \nabla^2 \vec{h} + \frac{H^3}{\eta} (1 + \gamma z)^2 \frac{\partial \vec{q}}{\partial z} = 0 \qquad \ldots(2)$$

where

$$\eta = \frac{1}{4\pi\mu\sigma_0}.$$

To non-dimensionalise the first and the last of eqn. (1) and eqn. (2), we introduce new variables defined by

$$(x, y, z) \equiv (x'd, y'd, z'd)$$

$$\vec{q} = q_0 \vec{q}', \qquad H_3 = H_0 H_3', \qquad \vec{h} = H_0 \vec{h}'$$

$$\delta\varpi = \frac{\delta p}{\rho_0} + \frac{\mu H_3 h_3}{4\pi} = \frac{\mu H_0^2}{4\pi\rho_0} \delta\varpi'$$

where d is the thickness of the layer,

$$q_0 = \left[\frac{g\alpha k}{\nu\beta} \right]^{1/2}, \qquad H_0 = \left[\frac{4\pi\rho_0 \nu q_0}{\mu d} \right]^{1/2}$$

and

$$\delta\varpi' = \frac{4\pi}{\mu H_0^2} \delta p + H_3' h_3'.$$

After simplifying and dropping the dashes, we get

$$\nabla^2 \vec{q} + H_3 \frac{\partial \vec{h}}{\partial z} + S\theta \vec{l} - \nabla (\delta\varpi) = 0 \qquad \ldots(3)$$

$$\bar{\gamma} \vec{l} \times (\nabla \times \vec{h}) + (1 + \bar{\gamma} z) \nabla^2 \vec{h} + \frac{H_3}{\bar{\eta}} (1 + \bar{\gamma} z)^2 \frac{\partial \vec{q}}{\partial z} = 0 \qquad \ldots(4)$$

$$\nabla^2 \theta = -Sw \qquad \ldots(5)$$

$$\nabla \cdot \vec{q} = 0, \qquad \nabla \cdot \vec{h} = 0 \qquad \ldots(6)$$

where

$$S^2 = \frac{g\alpha\beta d^4}{\nu k} = R,$$

the Rayleigh number, $\bar{\gamma} = \gamma d$ and

$$\bar{\eta} = \frac{\eta}{q_0 d},$$

S being the positive square root of R.

Boundary conditions—We have considered two cases in this paper: (i) when both the boundaries are free, and (ii) when both the boundaries are rigid. Moreover, in both cases, the boundaries are assumed to be perfectly conducting thermally. Thus, on the boundaries

$$\left.\begin{aligned} w &= \frac{\partial^2 w}{\partial z^2} = \theta = 0 \qquad \text{in case (i)} \\ w &= \frac{\partial w}{\partial z} = \theta = 0 \qquad \text{in case (ii)} \end{aligned}\right\} \qquad \ldots(7)$$

3. Normal Mode Analysis

We take the solutions of eqns. (3)–(6) to be of the form

$$\chi(x, y, z) = \bar{\chi}(z) \exp(ia_x x + ia_y y) \qquad \ldots(8)$$

where $\chi(x, y, z)$ is a variable such as u, v, w, h_1, h_2, h_3, θ and $\delta\widetilde{\omega}$ (h_1, h_2, h_3 being the components of $\vec{h}$). We furhter assume that $\bar{\chi}(z)$ and the parameter S are expandable in the form

$$\left.\begin{aligned} \bar{\chi}(z) &= \chi^{(0)} + \bar{\gamma}\chi^{(1)} + \ldots \\ \text{and} \qquad S &= S_0 + \bar{\gamma} S_1 + \ldots \end{aligned}\right\} \qquad \ldots(9)$$

when $\bar{\gamma}$ is small in value. Substituting (8) in (3)–(6) and separating out the terms with like powers of $\bar{\gamma}$ using (9), we get

$$\left.\begin{aligned}
&(D^2 - a^2) u^{(0)} + H_3 D h_1^{(0)} - ia_x \delta\widetilde{\omega}^{(0)} = 0 \\
&(D^2 - a^2) v^{(0)} + H_3 D h_2^{(0)} - ia_y \delta\widetilde{\omega}^{(0)} = 0 \\
&(D^2 - a^2) w^{(0)} + H_3 D h_3^{(0)} + S_0 \theta^{(0)} - D\delta\widetilde{\omega}^{(0)} = 0 \\
&(D^2 - a^2) h_1^{(0)} + \frac{H_3}{\bar{\eta}} D u^{(0)} = 0 \\
&(D^2 - a^2) h_2^{(0)} + \frac{H_3}{\bar{\eta}} D v^{(0)} = 0 \\
&(D^2 - a^2) h_3^{(0)} + \frac{H_3}{\bar{\eta}} D w^{(0)} = 0 \\
&(D^2 - a^2) \theta^{(0)} = - S_0 w^{(0)} \\
&ia_x u^{(0)} + ia_y v^{(0)} + D w^{(0)} = 0 \\
&ia_x h_1^{(0)} + ia_y h_2^{(0)} + D h_3^{(0)} = 0
\end{aligned}\right\} \qquad \ldots(10)$$

and also

$$\left.\begin{aligned}
&(D^2 - a^2)\, u^{(1)} + H_3 D h_1^{(1)} - i a_x \delta \widetilde{\omega}^{(1)} = 0 \\
&(D^2 - a^2)\, v^{(1)} + H_3 D h_2^{(1)} - i a_y \delta \widetilde{\omega}^{(1)} = 0 \\
&(D^2 - a^2)\, w^{(1)} + H_3 D h_3^{(1)} + S_0 \theta^{(1)} + S_1 \theta^{(0)} - D \delta \widetilde{\omega}^{(1)} = 0 \\
&(D^2 - a^2)\, h_1^{(1)} + \frac{H_3}{\bar{\eta}} D u^{(1)} + z (D^2 - a^2)\, h_1^{(0)} \\
&\qquad + \frac{2H_3}{\bar{\eta}} z D u^{(0)} + \{ i a_x h_3^{(0)} - D h_1^{(0)} \} = 0 \\
&(D^2 - a^2)\, h_2^{(1)} + \frac{H_3}{\bar{\eta}} D v^{(1)} + z (D^2 - a^2)\, h_2^{(0)} \\
&\qquad + \frac{2H_3}{\bar{\eta}} z\, D v^{(0)} + \{ i a_y h_3^{(0)} - D h_2^{(0)} \} = 0 \\
&(D^2 - a^2)\, h_3^{(1)} + \frac{H_3}{\bar{\eta}} D w^{(1)} + z (D^2 - a^2)\, h_3^{(0)} \\
&\qquad + \frac{2H_3}{\bar{\eta}} z D w^{(0)} = 0 \\
&(D^2 - a^2)\, \theta^{(1)} = - S_0 w^{(1)} - S_1 w^{(0)} \\
&i a_x u^{(1)} + i a_y v^{(1)} + D w^{(1)} = 0 \\
&i a_x h_1^{(1)} + i a_y h_2^{(1)} + D h_3^{(1)} = 0
\end{aligned}\right\} \quad \ldots(11)$$

where

$$D \equiv \frac{d}{dz} \quad \text{and} \quad a^2 = a_x{}^2 + a_y{}^2.$$

From (10), we have, after elimination of all variables, except $w^{(0)}$ and $\theta^{(0)}$,

$$\left.\begin{aligned}
&\{(D^2 - a^2)^2 - Q D^2\}\, w^{(0)} = S_0 a^2\, \theta^{(0)} \\
\text{and} \quad &(D^2 - a^2)\, \theta^{(0)} = - S_0 w^{(0)}
\end{aligned}\right\} \quad \ldots(12)$$

Similarly, from (11), we have

$$\left.\begin{aligned}
&\{(D^2 - a^2)^2 - Q D^2\}\, w^{(1)} = S_0 a^2 \theta^{(1)} + Q\, \{ z D^2\, w^{(0)} \\
&\qquad\qquad + D w^{(0)} \} + S_1 a^2\, \theta^{(0)} \\
\text{and} \quad &(D^2 - a^2)\, \theta^{(1)} = - S_0 w^{(1)} - S_1 w^{(0)} \\
&Q \text{ being equal to } \frac{H_3{}^2}{\bar{\eta}}.
\end{aligned}\right\} \quad \ldots(13)$$

Substituting (8) in (7) and equating the terms with like powers of $\bar{\gamma}$ using (9), we get the following set of boundary conditions to be satisfied at $z = 0$ and $z = 1$:

$$\left.\begin{aligned} w^{(0)} &= D^2 w^{(0)} = \theta^{(0)} = 0 \\ w^{(1)} &= D^2 w^{(1)} = \theta^{(1)} = 0 \end{aligned}\right\} \qquad \text{...(14)}$$

in case (i)

and

$$\left.\begin{aligned} w^{(0)} &= Dw^{(0)} = \theta^{(0)} = 0 \\ w^{(1)} &= Dw^{(1)} = \theta^{(1)} = 0 \end{aligned}\right\} \qquad \text{...(15)}$$

in case (ii).

4. Determination of R_c

To determine the minimum value of R and hence the minimum value of S given by $S = S_0 + \bar{\gamma} S_1 + \ldots$, where $\bar{\gamma}$ is small in value, we first obtain the minimum value of S_0 and the corresponding eigenfunction $w^{(0)}$ and a_c, the critical value of a, from (12) and the first set of boundary conditions of (14) or (15) (depending upon the nature of the boundary). This problem is the same as that considered by Chandrasekhar (1961, pp. 170, 172) in the case when the electrical conductivity is constant and S_0^2 is found to be the same as the critical Rayleigh number obtained by him.

We next solve (13), together with the boundary conditions (14) or (15) (depending upon the boundary) with known $w^{(0)}$, S_0 and a and determine S_1. We find it useful to eliminate $\theta^{(1)}$ from eqn. (13) and get

$$\begin{aligned} &[(D^2 - a^2)\{D^2 - a^2)^2 - QD^2\} + S_0^2 a^2]\, w^{(1)} \\ &\quad = Q\,[z\,(D^2 - a^2)\, D^2 w^{(0)} + (3D^2 - a^2)\, Dw^{(0)}] - S_0 S_1 a^2 w^{(0)} \end{aligned} \qquad \text{...(16)}$$

(i) Both the boundaries are free:

We have from Chandrasekhar (1961, pp. 170)

$$w^{(0)} = A_0 \sin \pi z \qquad \text{...(17)}$$

$$S_0^2 = \frac{(\pi^2 + a^2)\{\pi^2 + a^2)^2 + Q\pi^2\}}{a^2} \qquad \text{...(18)}$$

and a is such that if $a^2 = \pi^2 x$, then x is given by $2x^3 + 3x^2 = 1 + Q/\pi^2$, so that S_0^2 given by (18) is minimum.

Substituting (17) into (16), we get

$$\begin{aligned} &[(D^2 - a^2)\{(D^2 - a^2)^2 - QD^2\} + S_0^2 a^2]\, w^{(1)} \\ &\quad = QA_0 \pi^2 (\pi^2 + a^2)\, z \sin \pi z - QA_0 \pi (3\pi^2 + a^2) \cos \pi z \\ &\qquad - 2S_0 S_1 a^2 A_0 \sin \pi z. \end{aligned} \qquad \text{...(19)}$$

We multiply both sides of (19) by sin πz, integrate between $z = 0$ and $z = 1$ and take into account (18) and the boundary conditions for $w^{(1)}$ in (14), to obtain

$$\pi \left\{ [D^4 w^{(1)}]_{z=1} + [D^4 w^{(1)}]_{z=0} \right\}$$

$$= \frac{QA_0 \pi^2 (\pi^2 + a^2)}{4} - S_0 S_1 A_0 a^2 . \quad \text{...(20)}$$

But from (13) and (14), it is observed that on the boundary

$$D^4 w^{(1)} = QDw^{(0)} = QA_0 \, \pi \cos \pi z$$

Hence, from (20)

$$S_0 S_1 = \frac{Q \pi^2 (\pi^2 + a^2)}{4a^2}. \quad \text{...(21)}$$

Therefore,

$$R_c = S^2 = (S_0 + \bar{\gamma} S_1 + \ldots)^2 = S_0^2 + 2 S_0 S_1 \bar{\gamma} + \ldots$$

$$= R_1 + \bar{\gamma} R_2 + \ldots \quad \text{...(22)}$$

where $R_1 = S_0^2 =$ The critical value of the Rayleigh number obtained under the assumption that the electrical conductivity is constant (Chandrasekhar, pp. 170) and

$$R_2 = \frac{Q \pi^2 (\pi^2 + a^2)}{2a^2}. \quad \text{...(23)}$$

TABLE I

Q	a_c	R_1	R_2
0	2·233	657·511	0
10	2·590	923·070	121·914
50	3·270	1762·04	474·43
100	3·702	2653·71	848·48
500	4·998	8578·88	3441·25
1,000	5·684	15207·0	6439·4
5,000	7·585	63135·9	28885·5
10,000	8·588	119832·0	55921·0
40,000	10·95	445507·0	213556·0

In Table I are presented the values of R_2 for a set of values of Q taking the corresponding values of a_c and R_1 from Table XIV of Chandrasekhar (1961, pp. 170).

(ii) Both the boundaries are rigid:

For the sake of simplicity, we shift the origin at the centre of the layer. From (12), we get

$$(D^2 - a^2)\left[\{(D^2 - a^2)^2 - QD^2\} + S_0^2 a^2\right] w^{(0)} = 0 \qquad \ldots(24)$$

which we like to put in the form

$$(D^2 - q_1^2)(D^2 - q_2^2)(D^2 - q_3^2)\, w^{(0)} = 0 \qquad \ldots(25)$$

taking

$$(D^2 - a^2)\left[\{(D^2 - a^2)^2 - QD^2\} + S_0^2 a^2\right] \equiv (D^2 - q_1^2)(D^2 - q_2^2) \times (D^2 - q_3^2) \qquad \ldots(26)$$

where

$q_1^2\ (i = 1, 2, 3)$ are the roots of

$$(x - a_1^2)\left[\{x - a^2)^2 - Qx\} + S_0^2 a^2\right] = 0. \qquad \ldots(27)$$

The boundary conditions for $w^{(0)}$ are put, due to (12) and (14), in the form

$$w^{(0)} = Dw^{(0)} = \{(D^2 - a^2)^2 - QD^2\}\, w^{(0)} = 0. \qquad \ldots(28)$$

for $z = \pm \frac{1}{2}$.

The above problem of determination of the minimum eigenvalue of S_0^2, which is the same as the determination of the critical Rayleigh number in the case when conductivity is assumed to be constant, has been solved by Chandrasekhar (1961, pp. 175) following the variational method. Since the solution obtained by the variational method is not useful for the present discussion, we follow the direct method, though we accept the observation of Chandrasekhar that the minimum value of R, the Rayleigh number, which is the same as S_0^2 in our case, is obtained by considering the solution as an even function of z.

Thus, we get from (25) taking the even solution

$$w^{(0)} = A_1 \cosh q_1 z + A_2 \cosh q_2 z + A_3 \cosh q_3 z. \qquad \ldots(29)$$

Due to (28), we find, after some rearrangement, that

$$\begin{vmatrix} 1 & 1 & 1 \\ q_1 \tanh \frac{q_1}{2} & q_2 \tanh \frac{q_2}{2} & q_3 \tanh \frac{q_3}{2} \\ \frac{1}{(q_1^2 - a^2)} & \frac{1}{(q_2^2 - a^2)} & \frac{1}{(q_3^2 - a^2)} \end{vmatrix} = 0 \qquad \ldots(30)$$

and, also

$$\frac{A_1 \cosh \frac{q_1}{2}}{q_2 \tanh \frac{q_2}{2} - q_3 \tanh \frac{q_3}{2}}$$

$$= \frac{A_2 \cosh \frac{q_2}{2}}{q_3 \tanh \frac{q_3}{2} - q_1 \tanh \frac{q_1}{2}} = \frac{A_3 \cosh \frac{q_3}{2}}{q_1 \tanh \frac{q_1}{2} - q_2 \tanh \frac{q_2}{2}} = A_0 \,(\text{say}) \quad ...(31)$$

Eqn. (30) gives S_0 as a function of a, which, when minimized with respect to a, gives the critical values of S_0 and a. We assume that this minimization has been done; a is given by its critical value and the corresponding eigenfunction $w^{(0)}$ is given by (29), where A_1, A_2, A_3 are given by (31) with q_i's ($i = 1, 2, 3$) having been calculated for the critical values of S_0 and a.

We multiply both sides of (16) by $w^{(0)}$, integrate from $z = -\frac{1}{2}$ to $z = +\frac{1}{2}$ and make use of the boundary conditions (15) and eqn. (24), to obtain

$$\left[D^2 w^{(0)} D^3 w^{(1)} - D^3 w^{(0)} D^2 w^{(1)}\right]_{-1/2}^{+1/2}$$

$$= Q \int_{-1/2}^{+1/2} z w^{(0)} (D^2 - a^2) D^2 w^{(0)} \, dz + Q \int_{-1/2}^{+1/2} w^{(0)} (3D^2 - a^2) D w^{(0)} \, dz$$

$$- 2 S_0 S_1 a^2 \int_{-1/2}^{+1/2} \{w^{(0)}\}^2 \, dz$$

$$= -2 S_0 S_1 a^2 \int_{-1/2}^{+1/2} \{w^{(0)}\}^2 \, dz \quad ...(32)$$

as both $z w^{(0)} (D^2 - a^2) D^2 w^{(0)}$ and $w^{(0)} (3D^2 - a^2) D w^{(0)}$ are odd functions of z.

If we put $w^{(1)} = w_e^{(1)} + w_0^{(1)}$, where $w_e^{(1)}$ and $w_0^{(1)}$ are the even and odd parts respectively of $w^{(1)}$, we get from (32)

$$\left[D^2 w^{(0)} D^3 w_e^{(1)} - D^2 w^{(0)} D^2 w_e^{(1)}\right]_{z=1/2} + S_0 S_1 a^2 \int_{-1/2}^{+1/2} \{w^{(0)}\}^2 \, dz = 0. \quad ...(33)$$

After separating out the even part of (16), and putting the operator $[(D^2 - a^2) \{(D^2 - a^2)^2 - Q D^2\} + S_0^2 a^2]$ in the form given by (26), we have, substituting $w^{(0)}$, as given by (29) and (31)

$$(D^2 - q_1^2)(D^2 - q_2^2)(D^2 - q_3^2) w_e^{(1)}$$

$$= -2 S_0 S_1 a^2 A_0 \left[\left(q_2 \tanh \frac{q_2}{2} - q_3 \tanh \frac{q_3}{2} \right) \frac{\cosh q_1 z}{\cosh \frac{q_1}{2}} \right.$$

$$+ \left(q_3 \tanh\frac{q_3}{2} - q_1 \tanh\frac{q_1}{2}\right) \frac{\cosh q_2 z}{\cosh\frac{q_2}{2}}$$

$$+ \left(q_1 \tanh\frac{q_1}{2} - q_2 \tanh\frac{q_2}{2}\right) \frac{\cosh q_3 z}{\cosh\frac{q_3}{2}}\Bigg] \qquad \ldots(34)$$

From (34), the solution of $w_e^{(1)}$ is found as

$$w_e^{(1)} = B_1 \cosh q_1 z + B_2 \cosh q_2 z + B_3 \cosh q_3 z$$
$$- (S_0 S_1 a^2) A_0 \Bigg[\lambda_1 \frac{z \sin q_1 z}{\cosh\frac{q_1}{2}} + \lambda_2 \frac{z \sinh q_2 z}{\cosh\frac{q_2}{2}}$$
$$+ \lambda_3 \frac{z \sinh q_3 z}{\cosh\frac{q_3}{2}}\Bigg] \qquad \ldots(35)$$

where

$$\lambda_1 = \frac{q_2 \tanh\frac{q_2}{2} - q_3 \tanh\frac{q_3}{2}}{q_1 (q_1^2 - q_2^2)(q_1^2 - q_3^2)}$$

$$\lambda_2 = \frac{q_3 \tanh\frac{q_3}{2} - q_1 \tanh\frac{q_1}{2}}{q_2 (q_2^2 - q_3^2)(q_2^2 - q_1^2)}$$

$$\lambda_3 = \frac{q_1 \tanh\frac{q_1}{2} - q_2 \tanh\frac{q_2}{2}}{q_3 (q_3^2 - q_1^2)(q_3^2 - q_2^2)} \qquad \ldots(36)$$

From (28), we have $w_e^{(1)} = Dw_e^{(1)} = \{(D^2 - a^2)^2 - QD^2\} w_e^{(1)} = 0$ for $z = \frac{1}{2}$, which gives

$$B_1 \cosh\frac{q_1}{2} + B_2 \cosh\frac{q_2}{2} + B_3 \cosh\frac{q_3}{2} - \left(\frac{S_0 S_1 a^2}{2}\right) A_0 t_1 = 0 \qquad \ldots(37)$$

$$q_1 B_1 \sinh\frac{q_1}{2} + q_2 B_2 \sinh\frac{q_2}{2} + q_3 B_3 \sinh\frac{q_3}{2} - \left(\frac{S_0 S_1 a^2}{2}\right) A_0 t_2 = 0 \qquad \ldots(38)$$

$$f(q_1) B_1 \cosh\frac{q_1}{2} + f(q_2) B_2 \cosh\frac{q_2}{2} + f(q_3) B_3 \cosh\frac{q_3}{2}$$
$$- \left(\frac{S_0 S_1 a^2}{2}\right) A_0 t_3 = 0 \qquad \ldots(39)$$

where

$$f(q_1) = (q_i^2 - a^2)^2 - Q q_i^2, \qquad (i = 1, 2, 3) \qquad \ldots(40)$$

t_1, t_2, t_3 being some lengthy expressions involving q_i's which have not been presented here to avoid unnecessary complications.

We obtain another equation in B_1, B_2, B_3 and A_0 by substituting $w^{(0)}$ and $w_e^{(1)}$ given by (29) and (35) respectively in (33) and integrating the last term. We express the equation in the form

$$B_1 r_1 + B_2 r_2 + B_3 r_3 + A_0 r_4 = 0. \qquad \ldots(41)$$

Again, for the sake of simplicity, we have refrained from presenting the expressions represented by r_1, r_2, r_3 and r_4 in the above equation.

Eliminating B_1, B_2, B_3 and A_0 from (38)–(41), we get

$$S_0 S_1 a^2 \Delta = 0 \qquad \ldots(42)$$

where

$$\Delta = \begin{vmatrix} \cosh\frac{q_1}{2} & \cosh\frac{q_2}{2} & \cosh\frac{q_3}{2} & t_1 \\ q_1 \sinh\frac{q_1}{2} & q_2 \sinh\frac{q_2}{2} & q_3 \sinh\frac{q_3}{2} & t_2 \\ f(q_1)\cosh\frac{q_1}{2} & f(q_2)\cosh\frac{q_2}{2} & f(q_3)\cosh\frac{q_3}{2} & t_3 \\ r_1 & r_2 & r_3 & r_4 \end{vmatrix} \qquad \ldots(43)$$

We conclude from (43) that if $\Delta \neq 0$, then S_1 must be equal to zero. Determination of Δ involves considerable complications. But we observe that if $\Delta = 0$, then S_1 becomes indeterminate, a fact which is physically unrealistic. We, therefore, conclude that $\Delta \neq 0$, and hence $S_1 = 0$. The value of the critical Rayleigh number R_c is given by

$$R_c = (S_0 + \bar{\gamma} S_1 + \ldots)^2 = S_0^2 = R_1 \qquad \ldots(44)$$

to the first order in $\bar{\gamma}$, where $R_1 = S_0^2$ is the same as the critical Rayleigh number obtained by Chandrasekhar (1961, pp. 175).

5. Discussion

Assuming that the electrical conductivity, σ, in the static state is given by $\sigma = \sigma_0(1 + \gamma z)$, where γ is small, we see from (22) that in case both the boundaries are free, the critical Rayleigh number is greater or less than that obtained under the assumption that the conductivity is constant according as $\gamma > 0$ or $\gamma < 0$. Moreover, if R_c, the critical Rayleigh number, is given by $R_1 + \bar{\gamma} R_2$, then the ratio R_2/R_1 increases with increase in Q (Table I). It can be shown from (18) and (23) that

$$\frac{R_2}{R_1} = \frac{1}{2}\left[1 + \frac{\pi^2}{Q}(1 + x)^2\right]^{-1}$$

As $Q \to \infty$, $x \to \dfrac{Q^{1/3}}{2^{1/3}\,\pi^{2/3}}$ and hence $\dfrac{R_2}{R_1} \to \dfrac{1}{2}$.

From (44), we find that in the case when both the boundaries are rigid, the value of the critical Rayleigh number remains unchanged to the first order in γ.

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