ON REGULAR GRAPHS

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Assuming $\mu_1, \mu_2, \ldots, \mu_k$ to be arbitrary non-negative integers and $M(\mu_1, \ldots \mu_k)$, the minimum number of vertices on which there exist k noncomplete regular graphs of degrees $\mu_1, \mu_2, \ldots, \mu_k$ it is proved that

$$M(\mu_1, \ldots, \mu_k) = \begin{cases} 2 \text{ if } \mu = 0 \\ 4 \text{ if } \mu = 1 \\ \mu + 2 \text{ if } \mu \geqslant 2 \text{ is even} \\ \mu + 3 \text{ if } \mu \geqslant 3 \text{ is odd.} \end{cases}$$

where

$$\mu = \max_{1 \leqslant i \leqslant k} \{\mu_i\}.$$

All graphs considered in this paper are finite, undirected, loop-free graphs without multiple edges. For the terms in graph theory used in the paper without specifically defining them, the reader may refer to Harary (1969).

A graph $G(X, E)$ with vertex set X and edge set E is said to be regular of degree n, if every vertex $x \in X$ is joined by single edges to n other vertices of G. A complete graph of order n is a regular graph of degree $n - 1$ on n vertices; it is denoted by K_n. Given any natural number n, the maximum degree of regularity of a graph with n vertices is, therefore, $n - 1$. If $n = 2k, k \geqslant 1$, then it is known (Fiorini and Wilson 1976*a*, *b*; Harary 1969) that the edges of K_n can be coloured with $n - 1 = 2k - 1$ colours, such that no two incident edges receive the same colour. In such an assignment of colours to the edges of K_n each spanning subgraph of K_n whose lines are all of the same colour constitutes what is called a 1-factor (or a matching) of K_n. Thus, there are $n - 1$ 1-factors of K_n corresponding to each of the $n - 1$ colours assigned to the edges of K_n. Clearly, any two edges in a 1-factor are non-incident, while at the same time, the totality of the set of all edges of the 1-factors of K_n exhausts the $\binom{n}{2}$ edges of K_n. Therefore, by removing the edges of a 1-factor of K_n degree of every vertex of K_n is reduced by 1 and hence the resulting graph is a regular graph of degree $n - 2$. It is also noncomplete, as a set of k edges is deleted from K_n. Its order is still n, as we do not delete any vertex in the process. Thus, the maximum degree of a noncomplete regular graph of order n is $n - 2$.

Consider K_n with n even. The way in which the colours are assigned to the edges of K_n should now tell us that no two 1-factors of K_n have a common edge (we then say that the two subgraphs are edge-disjoint; and such a collection of subgraphs is called a decomposition of the graph, if their union reproduces the whole graph). This fact enables us to remove the edges of j 1-factors simultaneously from K_n to obtain a noncomplete regular graph of degree $n-1-j$, $1 \leqslant j \leqslant n-1$. Implicitly, this construction also proves that there exists a noncomplete regular graph of degree n for every given natural number $n \geqslant 2$. We have now enough necessary facts to establish the following lemma.

Lemma 1—Let M_k denote the minimum number of vertices on which there exist k noncomplete regular graphs having different degrees of regularity. Then

$$M_k = \begin{cases} 4 & \text{if } k = 2 \\ k+1 & \text{if } k \geqslant 3 \text{ is odd} \\ k+2 & \text{if } k \geqslant 4 \text{ is even.} \end{cases}$$

PROOF: First, note that $k \geqslant 2$, because we have to have at least two graphs so as to be in a position to speak of different degrees of regularity. We shall now prove that M_k is an even natural number for all $k \geqslant 2$. For this, suppose M_k is odd. Let $G_1, \ldots, G_k$ be noncomplete regular graphs of degrees $\mu_1, \ldots, \mu_k$ respectively and $\mu_1 \neq \mu_2 \neq \ldots \neq \mu_k$ each of whose orders is M_k. We claim that μ_i is even for each i, $1 \leqslant i \leqslant k$. If this were not so, we must have at least one i, $1 \leqslant i \leqslant k$, such that μ_i is odd. But then since the sum of the degrees of the vertices of a graph is always even, it follows that G_i has even number of vertices. This is in conflict with our supposition that $|V(G_i)| = M_k$ is odd. Thus, all μ_i are even. Therefore, the least possible choices for μ_i, without violating any of the conditions laid on them, are $\mu_i = 2(i-1)$, $1 \leqslant i \leqslant k$ (we may choose this order without loss of generality). Now, consider $K_{2(k-1)}$ from which k-noncomplete regular graphs of order $2(k-1)$ having degrees of regularity

$$2(k-1)-2 = 2k-4,\quad 2(k-1)-3 = 2k-5,\quad 2k-6, \ldots, 2, 1, 0$$

can be obtained, as mentioned above. Since the maximum degree of regularity of a graph with M_k vertices is $M_k - 1$, it follows that

$$\max_{1 \leqslant i \leqslant k} \{\mu_i\} = \mu_k \leqslant M_k - 1 < M_k.$$

Since $2(k-1) = \mu_k < M_k$ and there exist k noncomplete regular graphs of order $2(k-1)$ as shown above, we are led to violate the minimality of M_k. Therefore, we must have M_k to be even for all integers $k \geqslant 2$. We are now ready to establish the formula for M_k. Obviously, $M_2 = 4$ (indeed, $G_1 = \bar{K}_4$ and $G_2 = 2K_2$ are the two required regular graphs).

Next, as already observed, the maximum degree of regularity of a noncomplete graph on M_k vertices is $M_k - 2$ and there are $M_k - 1$ noncomplete regular graphs of order M_k having degrees of regularity $M_k - 2$, $M_k - 3, \ldots, 2, 1, 0$. So, if we are to select k such graphs, we must have $k \leqslant M_k - 1$, so that $M_k \geqslant k + 1$. Now, two cases arise for k; it could be either odd or even. We treat them separately.

Case 1: k *is odd*—In this case, $k+1$ is even. We can then obtain from K_{k+1} k noncomplete regular graphs of order $k+1$ having degrees of regularity $0, 1, 2, \ldots, k-1$, as mentioned earlier. Since M_k is the minimum order, so that this could be done, we must have $M_k \leqslant k+1$. Also since $M_k \geqslant k+1$, we get $M_k = k+1$ in this case.

Case 2: k *is even*—In this case, $M_k \neq k+1$, as M_k is proved to be even for all $k \geqslant 2$. Therefore, $M_k \geqslant k+2$. Now, $k+2$ is also an even integer. Therefore, we can construct from K_{k+2} $k+1$ noncomplete regular graphs of order $k+2$ having degrees of regularity $0, 1, 2, \ldots, k$ (we can select any k from these in order to show the existence of k such graphs), as described earlier. Since M_k is the minimum such number, we must have $M_k \leqslant k+2$, proving $M_k = k+2$ in this case.

Q.E.D.

Theorem 1—Let $\mu_1, \ldots, \mu_k$ be arbitrary non-negative integers and let $M(\mu_1, \ldots, \mu_k)$ denote the minimum number of vertices on which there exist k noncomplete regular graphs of degrees $\mu_1, \ldots, \mu_k$ respectively. Then

$$M(\mu_1, \ldots, \mu_k) = \begin{cases} 2 \text{ if } \mu = 0 \\ 4 \text{ if } \mu = 1 \\ \mu + 2 \quad \text{if } \mu \geqslant 2 \text{ is even} \\ \mu + 3 \quad \text{if } \mu \geqslant 3 \text{ is odd,} \end{cases}$$

where

$$\mu = \max_{1 \leqslant i \leqslant k} \{\mu_i\}.$$

PROOF: Obviously, if $\mu = 0, 1$, then one has $M(\mu_1 \ldots, \mu_k) = 2, 4$ respectively. So, we may assume $\mu \geqslant 2$ henceforth. We can prove that $M(\mu_1 \ldots, \mu_k)$ is an even integer for all natural numbers $k \geqslant 2$ exactly in the same manner as in the case of M_k in Lemma 1. However, if $k = 1$, then we have to treat the problem separately as follows. In this case, $\mu = \mu_1$ so that $M(\mu)$ is the minimum number of vertices on which there exists a noncomplete regular graph of degree μ. Since there exists a noncomplete regular graph of degree μ having $M(\mu)$ vertices, we must have $\mu \leqslant M(\mu) - 1$, so that $M(\mu) \geqslant \mu + 1$. If possible, suppose $M(\mu)$ is odd. Again, if μ is odd, then any regular graph of degree μ must have even number of vertices and so we must have μ to be even. Since we can obtain from $K_{\mu+2}$ a noncomplete regular graph of degree μ having $\mu + 2$ vertices and since $M(\mu)$ is the minimum of such orders, we get $M(\mu) \leqslant \mu + 2$. But since $M(\mu)$ is odd, this implies that $M(\mu) \leqslant \mu + 1$, so that we now obtain $M(\mu) = \mu + 1$. This is a contradiction to the definition of $M(\mu)$, since any regular graph of degree μ having $\mu + 1$ vertices is complete. Therefore, $M(\mu)$ must be even. Thus, we have proved now that $M(\mu_1 \ldots, \mu_k)$ is an even integer for all natural numbers k.

We are now ready to proceed to prove the formula for $M(\mu_1, \ldots, \mu_k)$. Since there exists a noncomplete regular graph of order $M(\mu_1, \ldots, \mu_k)$ having degree μ, we must have $\mu \leqslant M(\mu_1, \ldots, \mu_k) - 1$, so that $M(\mu_1, \ldots, \mu_k) \geqslant \mu + 1$. Two cases arise for μ.

Case 1: *μ is even*—In this case, $\mu + 1$ is odd, so that $M(\mu_1, \ldots, \mu_k) \neq \mu + 1$, as $M(\mu_1, \ldots, \mu_k)$ is even. Therefore, $M(\mu_1 \ldots, \mu_k) \geqslant \mu + 2$. But then since $\mu + 2$ is even, we obtain $\mu + 1$ noncomplete regular graphs of order $\mu + 2$ having degrees of regularity $0, 1, 2, \ldots, \mu$ from $K_{\mu+2}$ [since $\mu_i \geqslant (i-1)$ for all i, we may select any k of these $\mu + 1$ regular graphs], so that $M(\mu_1, \ldots, \mu_k) \leqslant \mu + 2$. This proves $M(\mu_1, \ldots, \mu_k) = \mu + 2$.

Case 2: *μ is odd*—In this case, $\mu + 1$ is even. Now $M(\mu_1, \ldots, \mu_k) \neq \mu + 1$, because if the equality holds, then it would come out that there are k noncomplete regular graphs of order $\mu + 1$ with degrees $\mu_1, \ldots, \mu_k$, one of which is of degree μ. But this is impossible, because a regular graph of order $\mu + 1$ having degree μ must be complete. Thus, $M(\mu_1, \ldots, \mu_k) \geqslant \mu + 2$. Since μ is odd, $\mu + 2$ is so and, therefore, it follows that $M(\mu_1, \ldots, \mu_k) \geqslant \mu + 3$. Also, on $\mu + 3$ vertices there are $\mu + 1$ noncomplete regular graphs of degrees $0, 1, 2, \ldots, \mu$ (we can choose any k of these for the purpose) it follows that $M(\mu_1, \ldots, \mu_k) \leqslant \mu + 3$, so that $M(\mu_1, \ldots, \mu_k) = \mu + 3$ in this case.

Q.E.D.

Remark 1: Note that the expressions for $M(\mu_1, \ldots, \mu_k)$ do not depend upon whether all μ_i are distinct or not, since we are allowed to choose as many copies of the regular graphs as required (so k can tend to infinity).

This motivates the problem of finding the minimum number of vertices on which k nonisomorphic regular graphs exist.

Remark 2: One can prove Theorem 1 independently and derive Lemma 1. But our approach gives scope for enumerating the isomorphism classes of noncomplete regular graphs indicated in Remark 1.

REFERENCES

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