UNSTEADY FLOW OF A CONDUCTING DUSTY FLUID THROUGH A RECTANGULAR CHANNEL WITH TIME DEPENDENT PRESSURE GRADIENT

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The unsteady flow of a conducting dusty viscous incompressible fluid through a long rectangular channel under the influence of a uniform magnetic field and a time dependent pressure gradient has been studied. The particular cases when the pressure gradient is (i) periodic function of time, (ii) an absolute constant, (iii) an exponentially decreasing function of time and (iv) $Cte^{-\lambda t}$, have been discussed in detail.

1. INTRODUCTION

Interest in problems of mechanics of systems with more than one phase has developed rapidly in recent years. The study of fluids having uniform distribution of solid spherical particles is of interest in a wide range of areas of technical importance. These areas include fluidization (flow through packed beds), flow in rocket tubes, where small carbon or metallic fuel particles are present, environmental pollution, the process by which rain drops are formed by the coalescence of small droplets, which might be considered as solid particles for the purpose of examining their movement prior to coalescence, combustion, and, more recently, blood flow in capillaries.

Considerable work has already been done on such models of dusty fluid flow. Saffman (1962) has discussed the stability of laminar flow of a dusty gas. The basic theory of multiphase flow is given by Soo (1967). Michael and Miller (1966) and Liu (1966, 1967) studied the flow produced by the motion of an infinite plate in a dusty gas occupying the semi-infinite space above it. Michael (1968) studied the steady motion of a sphere in a dusty gas. Healy (1970) proposed a different set of perturbed equations and studied the flow past a cylinder and a flat plate with position normal to the approach flow. Healy and Yang (1972) obtained an exact solution for the problem using the technique of Laplace transform. Vimala (1972) has discussed the flow of a dusty gas between two oscillating plates. Gupta and Gupta (1976) studied the flow of a dusty gas through a channel with an arbitrary time varying pressure gradient.

The present paper considers the flow of a conducting viscous incompressible fluid with embedded non-conducting identical spherical particles through a long rectangular channel under the influence of a uniform magnetic field and a time varying pressure gradient, taking the fluid and dust particles to be initially at rest. The particular cases when (i) the pressure gradient is a periodic function of time, (ii) the pressure gradient is an absolute constant, (iii) the pressure gradient is an exponentially decreasing function of time, and (iv) the pressure gradient is $Ct\, e^{-\lambda t}$, have also been discussed in detail.

EQUATIONS OF THE PROBLEM

Using the rectangular cartesian coordinate system, the walls of the channel are taken to be the planes $x = \pm a$ and $y = \pm b$. The fluid and dust particle velocities $u(x, y, t)$ and $v(x, y, t)$ respectively, are in z-direction. A uniform magnetic field is applied perpendicular to the planes $y = \pm b$ of rectangular channel. Taking the number density of non-conducting dust particles to be constant throughout the motion, the appropriate momentum equations, after introducing the electromagnetic force obtained by Soo (1968), are:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{KN_0}{\rho}(v - u) - \frac{\sigma B_0^2}{\rho}u \qquad \text{...(1)}$$

$$\frac{\partial v}{\partial t} = \frac{K}{M}(u - v) - \frac{k'}{\rho}\frac{\partial p}{\partial z} \qquad \text{...(2)}$$

where u and v denote the velocities of fluid and dust particle respectively; p is the fluid pressure; M, the mass of a particle; K, the Stokes resistance coefficient, which for spherical particle of radius r is $6\pi\mu r$, μ being the viscosity of the fluid; N_0, the number density of the particle; t, the time; ρ, ρ_p and $\bar{\rho}_p$ are density of the fluid, mass density of the particle and material density of the particle respectively; $\nu = \mu/\rho$, the kinematic viscosity of the fluid; B_0, the magnetic inductivity; σ, the electric conductivity; and $k' = \rho/\rho_p$.

It is assumed that the effect of the induced magnetic field and the electric field produced by the motion of the electrically conducting fluid is negligible and no external field is applied. The dust particles are non-conducting.

Introducing the non-dimensional quantities

$$x^* = \frac{x}{a},\quad y^* = \frac{y}{a},\quad z^* = \frac{z}{a},\quad p^* = \frac{pa^2}{\rho\nu^2},\quad t^* = \frac{t\nu}{a^2},\quad u^* = \frac{ua}{\nu}$$

and

$$v^* = \frac{va}{\nu},$$

eqns. (1) and (2) become (dropping the stars)

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial z} + \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \beta(v - u) - Hu \qquad \text{...(3)}$$

$$\frac{\partial v}{\partial t} = \gamma'(u - v) - k'\,Q\frac{\partial p}{\partial z} \qquad \text{...(4)}$$

where

$$\beta = \frac{K_0}{\gamma} = \frac{N_0 K a^2}{\rho \nu}, \quad K_0 = \frac{N_0 M}{\rho}, \quad \gamma = \frac{M \nu}{K a^2}, \quad \gamma' = \frac{1}{\gamma}, \quad H = \frac{\sigma B_0^2}{\rho}$$

and

$$Q = \frac{\nu}{a}.$$

Initially, the fluid and particles are at rest. The flow takes place under the influence of time dependent pressure gradient with no-slip boundary conditions. From symmetric consideration, the flow in region $x \geqslant 0$, $y \geqslant 0$, is considered. Accordingly, the boundary conditions are:

$$t > 0 \qquad \left.\begin{array}{l} \left.\begin{array}{l} u(1, y, t) = 0 \\ v(1, y, t) = 0 \end{array}\right\} 0 \leqslant y \leqslant h \\ \dfrac{\partial u}{\partial x} = 0, \quad \dfrac{\partial v}{\partial x} = 0 \text{ at } x = 0 \end{array}\right\} \qquad \ldots(5)$$

and

$$\left.\begin{array}{l} \left.\begin{array}{l} u(x, h, t) = 0 \\ v(x, h, t) = 0 \end{array}\right\} 0 \leqslant x \leqslant 1 \\ \dfrac{\partial u}{\partial y} = 0, \quad \dfrac{\partial v}{\partial y} = 0, \text{ at } y = 0 \end{array}\right\} \qquad \ldots(6)$$

where

$$h = b/a.$$

Solution of the Problem

For solving the problem, we choose the finite cosine transform defined as

$$\bar{u}(m, y, t) = \int_0^1 u(x, y, t) \cos q_m x \, dx \qquad \ldots(7)$$

$$\bar{\bar{u}}(x, n, t) = \int_0^h u(x, y, t) \cos q_n y \, dy \qquad \ldots(8)$$

where

$$q_m = \frac{2m+1}{2}\pi, \quad q_n = \frac{2n+1}{2h}\pi.$$

It can be shown that the inversion formulae for the finite cosine transforms defined by (7) and (8) are given by

$$u(x, y, t) = 2 \sum_{m=0}^{\infty} \bar{u}(m, y, t) \cos q_m x \qquad \ldots(9)$$

and

$$u(x, y, t) = \frac{2}{h} \sum_{n=0}^{\infty} \bar{\bar{u}}(x, n, t) \cos q_n y. \quad \ldots(10)$$

Multiplying eqns. (3) and (4) by $\cos q_m x \,.\, \cos q_n y$ and then integrating twice within the limits 0 to 1 and 0 to h and using the boundary conditions (5) and (6), we get

$$\frac{\partial U}{\partial t} = \frac{(-1)^{m+n}}{q_m q_n} f(t) - (q_m^2 + q_n^2) U + \beta (V - U) - HU \quad \ldots(11)$$

$$\frac{\partial V}{\partial t} = \gamma' (U - V) - \frac{(-1)^{m+n} k' Q}{q_m q_n} f(t) \quad \ldots(12)$$

where

$$U = \int_0^1 \int_0^h u(x, y, t) \cos q_m x \,.\, \cos q_n y \, dx \, dy$$

$$V = \int_0^1 \int_0^h v(x, y, t) \cos q_m x \,.\, \cos q_n y \, dx \, dy$$

and

$$-\frac{\partial p}{\partial z} = f(t).$$

Again, applying Laplace transform to eqns. (11) and (12) under the transform initial condition

$$U = 0, \quad V = 0 \quad \text{at} \quad t = 0,$$

we get

$$s\bar{U} = \frac{(-1)^{m+n}}{q_m q_n} \bar{f}(s) - (q_m^2 + q_n^2) \bar{U} + \beta (\bar{V} - \bar{U}) - H\bar{U} \quad \ldots(13)$$

$$s\bar{V} = \gamma' (\bar{U} - \bar{V}) - \frac{(-1)^{m+n} k' Q}{q_m q_n} \bar{f}(s) \quad \ldots(14)$$

where $\bar{U}$, $\bar{V}$ and $\bar{f}(s)$ are the Laplace transforms of the respective quantities.

Solving eqns. (13) and (14), we get

$$\bar{U} = \frac{(-1)^{m+n}}{q_m q_n} \frac{(s + \gamma' - k' Q\beta) \bar{f}(s)}{(s - \alpha_1)(s - \alpha_2)} \quad \ldots(15)$$

$$\bar{V} = \frac{(-1)^{m+n}}{q_m q_n} \left[\frac{\gamma' (s + \gamma' - k' Q\beta)}{(s + \gamma')(s - \alpha_1)(s - \alpha_2)} - \frac{k' Q}{(s + \gamma')} \right] \bar{f}(s) \quad \ldots(16)$$

where

$$\alpha_1 = -\tfrac{1}{2}\left[(\gamma' + \beta + H + q_m^2 + q_n^2) + \{(\gamma' + \beta + H + q_m^2 + q_n^2)^2 - 4\gamma'(q_m^2 + q_n^2 + H)\}^{1/2}\right]$$

$$\alpha_2 = -\tfrac{1}{2}\left[(\gamma' + \beta + H + q_m^2 + q_n^2) - \{(\gamma' + \beta + H + q_m^2 + q_n^2)^2 - 4\gamma'(q_m^2 + q_n^2 + H)\}^{1/2}\right].$$

Now to obtain u and v, we may invert the Laplace transform by convolution and then applying the inversion formulae for the finite cosine transforms, we get

$$u = \frac{4}{h}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}}{q_m q_n}\left[\int_0^t f(t-\eta)\{A_1 e^{\alpha_1\eta} + A_2 e^{\alpha_2\eta}\}\,d\eta\right] \times \cos q_m x \,.\, \cos q_n y \qquad \ldots(17)$$

and

$$v = \frac{4}{h}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}}{q_m q_n}\left[\int_0^t f(t-\eta)\{\gamma'(B_1 e^{\alpha_1\eta} + B_2 e^{\alpha_2\eta}) + B_3 e^{-\gamma'\eta}\}\,d\eta\right]\cos q_m x \,.\, \cos q_n y \qquad \ldots(18)$$

where

$$A_1 = \frac{(\alpha_1 + \gamma' - k'Q\beta)}{(\alpha_1 - \alpha_2)}, \quad A_2 = -\frac{(\alpha_2 + \gamma' - k'Q\beta)}{(\alpha_1 - \alpha_2)},$$

$$B_1 = \frac{A_1}{(\alpha_1 + \gamma')}, \quad B_2 = \frac{A_2}{(\alpha_2 + \gamma')}, \quad B_3 = -\left\{1 + \frac{\gamma'\beta}{(\alpha_1 + \gamma')(\alpha_2 + \gamma')}\right\}k'Q.$$

Particular Cases

(i) *When the Pressure Gradient is Periodic Function of Time*

Substituting $f(t) = C\sin\omega t$ (where C and ω are constants) in the above equations and on simplifying, we get velocities of the fluid and the dust particles

$$u = \frac{4C}{h}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}}{q_m q_n}\left[\frac{\omega A_1}{(\alpha_1^2 + \omega^2)}e^{\alpha_1 t} + \frac{\omega A_2}{(\alpha_2^2 + \omega^2)}e^{\alpha_2 t} + \left\{\frac{\omega^2 + (\gamma' - k'Q\beta)^2}{(\alpha_1^2 + \omega^2)(\alpha_2^2 + \omega^2)}\right\}^{1/2}\sin(\omega t - \psi_1)\right]\cos q_m x \,.\, \cos q_n y \quad \ldots(19)$$

and

$$v = \frac{4C}{h} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{q_m q_n} \left[\frac{\gamma' \omega B_1}{(\alpha_1^2 + \omega^2)} e^{\alpha_1 t} + \frac{\gamma' \omega B_2}{(\alpha_2^2 + \omega^2)} e^{\alpha_2 t} \right.$$

$$+ \frac{\omega B_3}{(\gamma'^2 + \omega^2)} e^{-\gamma' t} + \gamma' \left\{ \frac{\omega^2 + (\gamma' - k' Q\beta)^2}{(\alpha_1^2 + \omega^2)(\alpha_2^2 + \omega^2)(\gamma'^2 + \omega^2)} \right\}^{1/2}$$

$$\left. \times \sin(\omega t - \psi_2) - \frac{k' Q}{(\gamma'^2 + \omega^2)^{1/2}} \sin(\omega t - \psi_3) \right] \cos q_m x . \cos q_n y \quad \ldots(20)$$

respectively,

where

$$\psi_1 = \tan^{-1}\left(-\frac{\omega}{\alpha_1}\right) + \tan^{-1}\left(-\frac{\omega}{\alpha_2}\right) - \tan^{-1}\frac{\omega}{(\gamma' - k' Q\beta)},$$

$$\psi_2 = \tan^{-1}\left(-\frac{\omega}{\alpha_1}\right) + \tan^{-1}\left(-\frac{\omega}{\alpha_2}\right) + \tan^{-1}\left(\frac{\omega}{\gamma'}\right) - \tan^{-1}\frac{\omega}{(\gamma' - k' Q\beta)},$$

$$\psi_3 = \tan^{-1}\left(\frac{\omega}{\gamma'}\right).$$

(ii) *When the Pressure Gradient is Constant*

Substituting $f(t) = C$ (where C is an absolute constant) in the above equations and on simplifying, we get velocities of the fluid and the dust particles

$$u = \frac{4C}{h} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{q_m q_n} \left[\frac{\gamma' - k' Q\beta}{\alpha_1 \alpha_2} + \frac{A_1}{\alpha_1} e^{\alpha_1 t} + \frac{A_2}{\alpha_2} e^{\alpha_2 t} \right]$$

$$\times \cos q_m x . \cos q_n y \quad \ldots(21)$$

$$v = \frac{4C}{h} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{q_m q_n} \left[\left(\frac{\gamma' - k' Q\beta}{\alpha_1 \alpha_2} - \frac{k' Q}{\gamma'} \right) + \frac{\gamma' B_1}{\alpha_1} e^{\alpha_1 t} \right.$$

$$\left. + \frac{\gamma' B_2}{\alpha_2} e^{\alpha_2 t} - \frac{k' Q B_3}{\gamma'} e^{-\gamma' t} \right] \cos q_m x . \cos q_n y \quad \ldots(22)$$

respectively.

(iii) *When the Pressure Gradient is Exponentially Decreasing Function of Time*

Substituting $f(t) = C e^{-\lambda t}$ (where C and λ are constants) in the above equations and on simplifying, we get velocities of fluid and dust particles

$$u = \frac{4C}{h} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{q_m q_n} \left[\frac{\gamma' - k' Q\beta - \lambda}{(\alpha_1 + \lambda)(\alpha_2 + \lambda)} e^{-\lambda t} + \frac{A_1}{(\alpha_1 + \lambda)} e^{\alpha_1 t} \right.$$

$$\left. + \frac{A_2}{(\alpha_2 + \lambda)} e^{\alpha_2 t} \right] \cos q_m x . q_n y \quad \ldots(23)$$

and

$$v = \frac{4C}{h} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{q_m q_n} \left[\left\{ \frac{\gamma'(\gamma' - k'Q\beta - \lambda)}{(\gamma' - \lambda)(\alpha_1 + \lambda)(\alpha_2 + \lambda)} - \frac{k'Q}{(\gamma' - \lambda)} \right\} e^{-\lambda t} + \frac{\gamma' B_1}{(\alpha_1 + \lambda)} e^{\alpha_1 t} + \frac{\gamma' B_2}{(\alpha_2 + \lambda)} e^{\alpha_2 t} - \frac{k'QB_3}{(\gamma' - \lambda)} e^{-\gamma' t} \right] \cos q_m x \,.\, \cos q_n y \qquad \ldots(24)$$

respectively.

(iv) *When* $f(t) = Cte^{-\lambda t}$

The velocities of the fluid and the dust particles are

$$u = \frac{4C}{h} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{q_m q_n} \left[\frac{A_1}{(\alpha_1 + \lambda)^2} e^{\alpha_1 t} + \frac{A_2}{(\alpha_2 + \lambda)^2} e^{\alpha_2 t} + \left\{ \frac{\gamma' - k'Q\beta - \lambda}{(\alpha_1 + \lambda)(\alpha_2 + \lambda)} t + \frac{(2\lambda + \alpha_1 + \alpha_2)(\gamma' - k'Q\beta) + (\alpha_1\alpha_2 - \lambda^2)}{(\alpha_1 + \lambda)^2 (\alpha_2 + \lambda)^2} \right\} e^{-\lambda t} \right] \times \cos q_m x \,.\, \cos q_n y \qquad \ldots(25)$$

and

$$v = \frac{4C}{h} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{q_m q_n} \left[\frac{\gamma' B_1}{(\alpha_1 + \lambda)^2} e^{\alpha_1 t} + \frac{\gamma' B_2}{(\alpha_2 + \lambda)^2} e^{\alpha_2 t} + \frac{B_3}{(\gamma' - \lambda)^2} e^{-\gamma' t} + \left\{ \frac{(\gamma' - k'Q\beta - \lambda)\gamma' t}{(\alpha_1 + \lambda)(\alpha_2 + \lambda)(\gamma' - \lambda)} + \frac{(\gamma' - k'Q\beta) D}{\alpha_1 \alpha_2 \lambda} - \frac{k'Q\{(\gamma' - \lambda)t - 1\}}{(\gamma' - \lambda)} \right\} e^{-\lambda t} \right] \cos q_m x . \cos q_n y \qquad \ldots(26)$$

where

$$\begin{aligned} D = [\{&(\alpha_1 - \alpha_2)(\alpha_1 + \gamma')(\alpha_2 + \gamma')(\alpha_1 + \lambda)^2 (\alpha_2 + \lambda)^2 (\gamma' - \lambda)^2 (\gamma' - k'Q\beta) \\ &+ \alpha_2 \gamma' \lambda^2 (\gamma' - k'Q\beta + \alpha_1)(\alpha_2 + \gamma')(\alpha_2 + \lambda)^2 (\gamma' - \lambda)^2 - \alpha_1 \gamma' \lambda^2 \\ &\times (\gamma' - k'QB + \alpha_2)(\alpha_1 + \gamma')(\alpha_1 + \lambda)^2 (\gamma' - \lambda)^2 + \alpha_1 \alpha_2 \lambda^2 k'Q\beta \\ &\times (\alpha_1 - \alpha_2)(\alpha_1 + \lambda)^2 (\alpha_2 + \lambda)^2 + \alpha_1 \alpha_2 \gamma' (k'Q\beta - \gamma' + \lambda)(\alpha_1 - \alpha_2) \\ &\times (\alpha_1 + \gamma')(\alpha_2 + \gamma')(\alpha_1 + \lambda)(\alpha_2 + \lambda)(\gamma' - \lambda)\} : \{(\gamma' - k'Q\beta) \\ &\times (\alpha_1 - \alpha_2)(\alpha_1 + \gamma')(\alpha_2 + \gamma')(\alpha_1 + \lambda)^2 (\alpha_2 + \lambda)^2 (\gamma' - \lambda)^2\}]. \end{aligned}$$

DISCUSSION

In the particular case (iv), if we put $\lambda = 0$, the velocities of the fluid and the dust particles can be obtained for a pressure gradient, which is linearly dependent on time.

If $H = 0$, all the velocity expressions for fluid and dust particles can be obtained in the absence of magnetic field under the influence of various pressure gradients and also if $K' = 0$, the results are in agreement with those of Gupta and Gupta (1976).

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REFERENCES

Gupta, R. K., and Gupta, S. C. (1976). Flow of a dusty gas through a channel with arbitrary time varying pressure gradient. *ZAMP*, 27 (1), 119.

Healy, J. Val. (1970). Perturbed two-phase cylindrical-type flow. *Phys. Fluids* **73** (3), 551.

Healy, J. Val., and Yang, H. T. (1972). The Stokes problems for a suspension of particles. *Astronautica Acta*, 17, 851.

Liu, J. T. (1966). Flow induced by an oscillating infinite plate in a dusty gas. *Phys. Fluid*, **9**, 1716.

——— (1967). Flow induced by the impulsive motion of an infinite plate in a dusty gas. *Astronautica Acta*, **13**, 369.

Michael, D. H. (1968). The steady motion of a sphere in a dusty gas. *J. Fluid Mech.*, **31**, 175.

Michael, D. H., and Miller, D. A. (1966). Plane parallel flow of a dusty gas. *Mathematika*, **13**, 97.

Saffman, P. G. (1962). On the stability of laminar flow of a dusty gas. *J. Fluid Mech.*, **13**, 120.

Soo, S. L. (1967). Fluid Dynamics of Multiphase Systems, Ist Ed. Blaisdell Pub. Co., Waltham, Mass., U.S.A.

Vimala, C. S. (1972). Flow of a dusty gas between two oscillating plates. *Def. Sci. J.*, p. 231.