SOME SOLUTIONS OF EINSTEIN-MAXWELL EQUATIONS FOR CYLINDRICALLY SYMMETRIC SPACE-TIME WITH TWO DEGREES OF FREEDOM IN GENERAL RELATIVITY

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Einstein-Maxwell equations have been solved for cylindrically symmetric space-time with two degrees of freedom. The electromagnetic field has been assumed to be purely electric.

1. Introduction

Solution of Einstein-Maxwell equations for cylindrical symmetry has been studied extensively (Witten 1962, Singh, Radhakrishna and Sharan 1965, Roy and Tripathi 1972). In this paper, we obtain some solutions of Einstein-Maxwell equations for cylindrically symmetric space-time with two degrees of freedom. The electromagnetic field is taken to be an electric one in the coordinates adopted, the electric intensity being in the radial direction. The metric of the space-time is taken in the form

$$ds^2 = e^{2\gamma-2\psi}(dt^2 - d\rho^2) - e^{2\psi}\, dz^2 - 2\chi e^{2\psi}\, d\phi dz - (\chi^2 e^{2\psi} + \rho^2 e^{2\xi-2\psi})\, d\phi^2 \qquad \ldots(1)$$

where $x^1 = \rho$, $x^2 = z$, $x^3 = \phi$ and $x^4 = t$ and γ, ψ, ξ and χ are functions of ρ and t. The field equations to be satisfied are

$$R_{ij} = -\kappa E_{ij} \qquad \ldots(2)$$

where

$$E_{ij} = -F_{i\Omega} F_j{}^{\Omega} + \tfrac{1}{4} F_{\alpha\beta}F^{\alpha\beta} g_{ij}, \qquad \ldots(3)$$

$$F_{[ij,k]} = 0 \qquad \ldots(4)$$

and

$$F^{ij}_{;j} = J^i. \qquad \ldots(5)$$

In the above, a semicolon indicates covariant differentiation. For the metric (1), the components R_{12}, R_{13}, R_{24} and R_{34} vanish identically. This leads to

$$F_{13} F_{23} \frac{1}{\rho^2} e^{2\psi-2\xi} = F_{12} F_{23} \frac{\chi}{\rho^2} e^{2\psi-2\xi} + F_{14} F_{24} e^{2\psi-2\gamma}$$

$$F_{12}F_{32}\left[e^{-2\psi}+\frac{\chi^2}{\rho^2}e^{2\psi-2\xi}\right]=F_{13}F_{32}\frac{\chi}{\rho^2}e^{2\psi-2\xi}+F_{14}F_{34}e^{2\psi-2\gamma}$$

$$F_{31}F_{41}e^{2\psi-2\gamma}=F_{32}F_{43}\frac{\chi}{\rho^2}e^{2\psi-2\xi}-F_{32}F_{42}\left[e^{-2\psi}+\frac{\chi^2}{\rho^2}e^{2\psi-2\xi}\right]$$

$$F_{21}F_{41}e^{2\psi-2\gamma}=F_{23}F_{42}\frac{\chi}{\rho^2}e^{2\psi-2\xi}-F_{23}F_{43}\frac{1}{\rho^2}e^{2\psi-2\xi}. \qquad ...(6)$$

This set is obviously satisfied, when $F_{12}=F_{13}=F_{23}=F_{24}=F_{34}=0$ and $F_{14}\neq 0$.

2. Derivation of the Line-elements and Discussions

Section 1

In this section, we take the metric potentials to be functions of ρ alone. The field equations are

$$(a)\quad R_{11}=\frac{\kappa}{2}F_{14}^2e^{2\psi-2\gamma},$$

$$(b)\quad R_{22}=-\frac{\kappa}{2}F_{14}^2e^{6\psi-4\gamma},$$

$$(c)\quad R_{33}=-\frac{\kappa}{2}F_{14}^2[\chi^2e^{6\psi-4\gamma}+\rho^2e^{2\psi+2\xi-4\gamma}],$$

$$(d)\quad R_{44}=-\frac{\kappa}{2}F_{14}^2e^{2\psi-2\gamma},$$

$$(e)\quad R_{23}=-\frac{\kappa}{2}F_{14}^2\chi e^{6\psi-4\gamma}. \qquad ...(7)$$

Eliminating F_{14}, we obtain the following differential equations:

$$(a)\quad R_{11}+R_{44}=0,$$

$$(b)\quad R_{11}+e^{2\gamma-4\psi}R_{22}=0,$$

$$(c)\quad R_{11}+\frac{1}{\rho^2}e^{2\gamma-2\xi}R_{33}-\frac{\chi}{\rho^2}e^{2\gamma-2\xi}R_{23}=0,$$

$$(d)\quad R_{23}-\chi R_{22}=0, \qquad ...(8)$$

which give rise to the following differential equations:

$$\xi_1^2+2\psi_1^2+\xi_{11}-2\gamma_1\xi_1+2\frac{\xi_1}{\rho}-2\frac{\gamma_1}{\rho}+\frac{1}{2\rho^2}\chi_1^2e^{4\psi-2\xi}=0, \qquad ...(9)$$

$$\xi_1^2+2\psi_1^2+\xi_{11}+\gamma_{11}-\xi_1\gamma_1-\frac{\gamma_1}{\rho}+2\frac{\xi_1}{\rho}=0, \qquad ...(10)$$

$$2\xi_1^2 + 2\psi_1^2 - 2\frac{\psi_1}{\rho} - \frac{\gamma_1}{\rho} + 4\frac{\xi_1}{\rho} + 2\xi_{11} + \gamma_{11} - 2\psi_{11} - 2\psi_1\xi_1 - \gamma_1\xi_1$$

$$+ \frac{1}{\rho^2} e^{4\psi - 2\xi}\left[\chi_1^2 + \frac{1}{2}\chi\chi_{11} - \frac{1}{2}\chi\chi_1\xi_1 + 2\chi\chi_1\psi_1 - \frac{1}{2\rho}\chi\chi_1\right] = 0 \quad \text{...(11)}$$

$$2\chi_1\psi_1 + \frac{1}{2}\chi_{11} - \frac{1}{2}\frac{\chi_1}{\rho} - \frac{1}{2}\chi_1\xi_1 = 0. \quad \text{...(12)}$$

From eqns. (10), (11) and (12), we obtain

$$\xi_1^2 - 2\frac{\psi_1}{\rho} + 2\frac{\xi_1}{\rho} + \xi_{11} - 2\psi_{11} - 2\psi_1\xi_1 + \frac{\chi_1^2}{\rho^2} e^{4\psi - 2\xi} = 0. \quad \text{...(13)}$$

Equation (12) on integration gives

$$\frac{\chi_1}{\rho} e^{4\psi - \xi} = a \quad \text{...(14)}$$

a being a constant of integration. From eqns. (9) and (14), we obtain

$$\gamma_1 = \frac{1}{2}\left(\xi_1 + \frac{1}{\rho}\right) + \frac{\xi_{11} - \frac{1}{\rho^2}}{2\left(\xi_1 + \frac{1}{\rho}\right)} + \frac{\psi_1^2}{\left(\xi_1 + \frac{1}{\rho}\right)} + \frac{a^2 e^{-4\psi}}{4\left(\xi_1 + \frac{1}{\rho}\right)}. \quad \text{...(15)}$$

From eqns. (13) and (14), we get

$$\psi_{11} + \psi_1\left(\xi_1 + \frac{1}{\rho}\right) = \frac{1}{2}\left(\xi_1^2 + 2\frac{\xi_1}{\rho} + \xi_{11}\right) + \frac{a^2}{2} e^{-4\psi}. \quad \text{...(16)}$$

From eqns. (9), (10) and (14), we obtain

$$\gamma_{11} + \gamma_1\left(\xi_1 + \frac{1}{\rho}\right) = \frac{a^2}{2} e^{-4\psi}. \quad \text{...(17)}$$

Equations (16) and (17) lead to

$$(\psi_{11} - \gamma_{11}) + (\psi_1 - \gamma_1)\left(\xi_1 + \frac{1}{\rho}\right) = \frac{1}{2}\left(\xi_1 + \frac{1}{\rho}\right)^2 + \frac{1}{2}\left(\xi_{11} - \frac{1}{\rho^2}\right) \quad \text{...(18)}$$

which on integration gives

$$\gamma_1 = \psi_1 - \frac{1}{2}\frac{(\rho e^{\xi})_1}{\rho e^{\xi}} - \frac{b}{\rho e^{\xi}} \quad \text{...(19)}$$

where b is a constant of integration. From eqns. (15) and (19), we obtain

$$\left[\psi_1 - \frac{1}{2}\left(\xi_1 + \frac{1}{\rho}\right)\right]^2 + \frac{3}{4}\left(\xi_1 + \frac{1}{\rho}\right)^2 + \frac{b}{\rho e^{\xi}}\left(\xi_1 + \frac{1}{\rho}\right)$$

$$+ \frac{1}{2}\left(\xi_{11} - \frac{1}{\rho^2}\right) + \frac{a^2}{4} e^{-4\psi} = 0. \quad \text{...(20)}$$

By the substitution

$$\psi - \tfrac{1}{2}\xi - \tfrac{1}{2}\log\rho = \alpha \qquad \text{...(21)}$$

eqn. (16) reduces to

$$\alpha_{11} + \alpha_1\left(\xi_1 + \frac{1}{\rho}\right) - \frac{a^2}{2\rho^2}e^{-4\alpha-2\xi} = 0 \qquad \text{...(22)}$$

which on integration gives

$$[\rho e^{\xi}\alpha_1]^2 = C^2 - \tfrac{1}{4}a^2 e^{-4\alpha} \qquad \text{...(23)}$$

C being a constant of integration. From eqns. (20) and (23), we obtain

$$\tfrac{1}{4}[(\rho e^{\xi})_1]^2 + b(\rho e^{\xi})_1 + \tfrac{1}{2}(\rho e^{\xi})(\rho e^{\xi})_{11} + C^2 = 0 \qquad \text{...(24)}$$

Let us now define the variable R by means of the equation $\rho e^{\xi} = d\rho/dR$. Equation (23) then leads to

$$\frac{d\alpha}{\sqrt{C^2 - \tfrac{1}{4}a^2 e^{-4\alpha}}} = dR \qquad \text{...(25)}$$

which on integration gives

$$e^{2\alpha} = \frac{a}{2c}\cosh[2cR - \beta] \qquad \text{...(26)}$$

where β is a constant of integration. Eqns. (21) and (26) lead to

$$e^{4\psi} = \left[\frac{d\rho}{dR}\right]^2 \frac{a^2}{4C^2}\cosh^2[2CR - \beta]. \qquad \text{...(27)}$$

Equation (19) on integration gives

$$\gamma = \alpha - \int \frac{b}{\rho e^{\xi}}\, d\rho + \frac{1}{2}\log S \qquad \text{...(28)}$$

where S is a constant of integration and α is given by eqn. (26). From eqns. (14) and (26), we obtain

$$\frac{d\chi}{dR} = \frac{4C^2}{a}\operatorname{sech}^2[2CR - \beta] \qquad \text{...(29)}$$

which on integration gives

$$\chi = \frac{2C}{a}\tanh[2CR - \beta] + \delta \qquad \text{...(30)}$$

δ being a constant of integration. The substitutions

(*a*) $\rho e^{\xi} = Y$

and

(*b*) $\dfrac{dY}{dR} = p \qquad \text{...(31)}$

reduce eqn. (24) to

$$\frac{dp}{dY}=\frac{p^2-4bpY-4C^2Y^2}{2pY}. \qquad \text{...(32)}$$

From eqn. (32), we have

$$\frac{2\,vdv}{v^2+4bv+4C^2}=-\frac{dY}{Y} \qquad \text{...(33)}$$

where

$$p=vY. \qquad \text{...(34)}$$

Three cases arise.

Case 1 : $b>c$—Equation (33) on integration gives

$$Y=K\,[v+L]^{-[2L/(L-M)]}\,[v+M]^{[2M/(L-M)]} \qquad \text{...(35)}$$

where $L=2\,[b+\sqrt{b^2-c^2}]$ and $M=2\,[b-\sqrt{b^2-c^2}]$, k being a constant of integration. Eqns. (31) and (33) lead to

$$v=\frac{1}{2}\left[(L-M)\coth\left\{\frac{(L-M)(n+R)}{8}\right\}-(L+M)\right] \qquad \text{...(36)}$$

n being a constant of integration. From eqns. (35) and (36), we get

$$\rho e^{\xi}=\frac{4K}{(L-M)^2}\,e^{-[(L+M)\,(n+R)]/4}\sinh^2\left[\frac{(L-M)\,(n+R)}{8}\right]. \qquad \text{...(37)}$$

After suitable coordinate transformations, the metric of the space-time is given by

$$\begin{aligned} ds^2 &= \exp\left[-\frac{(L+M)}{4}\,R\right]\sinh^2\frac{(L-M)(n+R)}{8} \\ &\times\left[\operatorname{cosech}^4\frac{(L-M)(n+R)}{8}\,dT^2-\frac{4ks}{(L-M)^2}\right. \\ &\times\exp\left[-\frac{(L+M)(2R+n)}{4}\right]dR^2-\cosh\,[2CR-\beta] \\ &\times\left\{dz^2+2\left[\frac{2c}{a}\tanh\,(2cR-\beta)+\delta\right]d\phi\,dz+\left[\left\{\frac{2c}{a}\tanh\right.\right.\right. \\ &\left.\left.\left.\times\,(2cR-\beta)+\delta\right\}^2+\frac{4c^2}{a^2}\operatorname{sech}^2(2CR-\beta)\right]d\phi^2\right\}\Bigg]. \end{aligned} \qquad \text{...(38)}$$

Case 2 : $b=c$—Equation (33) on integration gives

$$Y=\frac{l}{(v+2c)^2}\,\exp\left[-\frac{4c}{v+2c}\right] \qquad \text{...(39)}$$

l being a constant of integration. From eqns. (31) and (33), we obtain

$$v = \frac{2}{k + R} - 2c \qquad \ldots(40)$$

where k is a constant of integration. Equations (39) and (40) lead to

$$\rho e^{\xi} = \frac{1}{4} l (k + R)^2 \exp[-2c(k + R)]. \qquad \ldots(41)$$

After suitable transformations, the metric of the space-time takes the form

$$ds^2 = \left[\frac{1}{(k + R)^2} dT^2 - \frac{1}{4} ls (k+R)^2 e^{-2C(k+2R)} dR^2\right]$$
$$- e^{-2cR} (k + R)^2 \cosh(2cR - \beta) \left[dz^2 + 2\left\{\frac{2c}{a}\tanh(2cR - \beta) + \delta\right\}\right.$$
$$\times d\phi dz + \left[\left\{\frac{2c}{a}\tanh(2CR - \beta) + \delta\right\}^2\right.$$
$$\left.\left. + \frac{4c^2}{a^2}\operatorname{sech}^2(2cR - \beta)\right] d\phi^2\right] \qquad \ldots(42)$$

Case 3: $b < c$—Equation (33) on integration gives

$$Y = \frac{Q}{(v + 2b)^2 + 4(c^2 - b^2)}\left[\exp\left\{\frac{2b}{\sqrt{c^2 - b^2}}\tan^{-1}\frac{v + 2b}{2\sqrt{c^2 - b^2}}\right\}\right] \qquad \ldots(43)$$

Q being a constant of integration. From eqns. (31) and (33), we obtain

$$v = 2\sqrt{c^2 - b^2}\left[\tan\{\sqrt{c^2 - b^2}(\lambda - R)\}\right] - 2b \qquad \ldots(44)$$

where λ is a constant of integration. From eqns. (43) and (44), we get

$$\rho e^{\xi} = \frac{Q}{4(c^2 - b^2)} \exp[-2b(R - \lambda)] \cos^2[\sqrt{c^2 - b^2}(R - \lambda)] \qquad \ldots(45)$$

After suitable transformations of coordinates, the metric of the space-time can be put into the form

$$ds^2 = \sec^2\{\sqrt{c^2 - b^2}(R - \lambda)\}\left[dT^2 - \frac{Qs}{4(c^2 - b^2)} e^{-2b(2R - \lambda)}\right.$$
$$\left.\times \cos^4\{\sqrt{c^2 - b^2}(R - \lambda)\} dR^2\right] - e^{-2bR}\cosh(2cR - \beta)\cos^2$$
$$\times \{\sqrt{c^2 - b^2}(R - \lambda)\}\left[dz^2 + 2\left\{\frac{2c}{a}\tanh(2cR - \beta) + \delta\right\} d\phi dz\right.$$
$$\left. + \left[\left\{\frac{2c}{a}\tanh(2cR - \beta) + \delta\right\}^2 + \frac{4c^2}{a^2}\operatorname{sech}^2(2cR - \beta)\right] d\phi^2\right] \ldots(46)$$

For the space-times given by metrics (38), (42) and (46), the values of F_{14} are respectively given by

$$F_{14}^2 = \frac{(L+M)^2}{32x} e^{[(L+M)(2-R)]/4} \operatorname{cosech}^4 \frac{(L-M)(n+R)}{8} \qquad \ldots(47)$$

$$F_{14}^2 = \frac{2}{\kappa} \frac{1}{(k+R)^4} \qquad \ldots(48)$$

and

$$F_{14}^2 = \frac{2}{\kappa}(c^2 - b^2) \sec^4 [\sqrt{c^2 - b^2}(R - \lambda)]. \qquad \ldots(49)$$

Equation (4) is obviously satisfied and in all the three cases we find that

$$J^1 = J^2 = J^3 = J^4 = 0.$$

Section 2

In this section, we consider the case when the metric potentials are functions of both ρ and t. The field equations in this case are

$$(a) \quad R_{11} = \frac{\kappa}{2} F_{14}^2 e^{2\psi - 2\gamma}$$

$$(b) \quad R_{22} = -\frac{\kappa}{2} F_{14}^2 e^{6\psi - 4\gamma},$$

$$(c) \quad R_{33} = -\frac{\kappa}{2} F_{14}^2 [\chi^2 e^{6\psi - 4\gamma} + \rho^2 e^{2\psi + 2\xi - 4\gamma}],$$

$$(d) \quad R_{44} = -\frac{\kappa}{2} F_{14}^2 e^{2\psi - 2\gamma},$$

$$(e) \quad R_{23} = -\frac{\kappa}{2} F_{14}^2 \chi e^{6\psi - 4\gamma},$$

$$(f) \quad R_{14} = 0. \qquad \ldots(50)$$

Eliminating F_{14}, we obtain the following differential equations:

$$(a) \quad R_{11} + R_{44} = 0,$$

$$(b) \quad R_{11} + e^{2\gamma - 4\psi} R_{22} = 0,$$

$$(c) \quad R_{11} + \frac{1}{\rho^2} e^{2\gamma - 2\xi} R_{33} - \frac{\chi}{\rho^2} e^{2\gamma - 2\xi} R_{23} = 0,$$

$$(d) \quad R_{23} - \chi R_{22} = 0,$$

$$(e) \quad R_{14} = 0. \qquad \ldots(51)$$

Equations (51) in this case give rise to the following equations:

$$\xi_1^2 + \xi_4^2 + 2\psi_1^2 + 2\psi_4^2 + \xi_{11} + \xi_{44} - 2\gamma_1\xi_1 - 2\gamma_4\xi_4 + 2\frac{\xi_1}{\rho} - 2\frac{\gamma_1}{\rho}$$

$$+ \frac{1}{2\rho^2}[\chi_1^2 + \chi_4^2]\, e^{4\psi-2\xi} = 0, \qquad \ldots(52)$$

$$\xi_1^2 + 2\psi_1^2 + \xi_{11} + \gamma_{11} - \gamma_{44} - \xi_4\gamma_4 - \xi_1\gamma_1 - \frac{\gamma_1}{\rho} + 2\frac{\xi_1}{\rho}$$

$$+ \frac{1}{2\rho^2}\chi_4^2\, e^{4\psi-2\xi} = 0, \qquad \ldots(53)$$

$$2\xi_1^2 + 2\psi_1^2 - 2\frac{\psi_1}{\rho} - \frac{\gamma_1}{\rho} + 4\frac{\xi_1}{\rho} + 2\xi_{11} + \gamma_{11} - 2\psi_{11} + 2\psi_{44} - \gamma_{44}$$

$$- \xi_4\gamma_4 + 2\xi_4\psi_4 - 2\xi_1\psi_1 - \gamma_1\xi_1 - \xi_{44} - \xi_4^2 + \frac{1}{2\rho^2}\chi_1^2\, e^{4\psi-2\xi}$$

$$+ \frac{1}{\rho^2}e^{4\psi-2\xi}\left[\frac{1}{2}\chi_1^2 - \frac{1}{2}\chi_4^2 + \frac{1}{2}\chi\chi_{11} - \frac{1}{2}\chi\chi_{44} - \frac{1}{2}\chi\chi_1\xi_1 + \frac{1}{2}\chi\chi_4\xi_4\right.$$

$$\left. + 2\chi\chi_1\psi_1 - 2\chi\chi_4\psi_4 - \frac{1}{2}\frac{\chi\chi_1}{\rho}\right] = 0, \qquad \ldots(54)$$

$$2\chi_1\psi_1 - 2\chi_4\psi_4 + \frac{1}{2}\chi_{11} - \frac{1}{2}\chi_{44} - \frac{1}{2}\frac{\chi_1}{\rho} + \frac{1}{2}\chi_4\xi_4 - \frac{1}{2}\chi_1\xi_1 = 0 \qquad \ldots(55)$$

$$2\psi_1\psi_4 + \xi_1\xi_4 + \xi_{14} - \xi_1\gamma_4 - \gamma_1\xi_4 + \frac{\xi_4}{\rho} - \frac{\gamma_4}{\rho} + \frac{1}{2}\frac{\chi_1\chi_4}{\rho^2}e^{4\psi-2\xi} = 0\,. \quad \ldots(56)$$

From eqns. (53), (54) and (55), we obtain

$$\xi_1^2 - 2\frac{\psi_1}{\rho} + 2\frac{\xi_1}{\rho} + \xi_{11} - 2\psi_{11} + 2\psi_{44} + 2\psi_4\xi_4 - 2\psi_1\xi_1$$

$$- \xi_{44} - \xi_4^2 + \frac{1}{\rho^2}e^{4\psi-2\xi}[\chi_1^2 - \chi_4^2] = 0. \qquad \ldots(57)$$

If we suppose ξ to be constant, the integrability condition for eqns. (52) and (56) leads to $R_{22} = 0$ and thus F_{14} vanishes contrary to our assumption. If, however, we take ξ to be independent of time, eqns. (52) and (56) immediately reduce to the form

$$\gamma_1\left(\xi_1 + \frac{1}{\rho}\right) = \frac{1}{2}\xi_1^2 + \psi_1^2 + \psi_4^2 + \frac{1}{2}\xi_{11} + \frac{\xi_1}{\rho} + \frac{1}{4\rho^2}(\chi_1^2 + \chi_4^2)\, e^{4\psi-2\xi} \qquad \ldots(58)$$

and

$$\gamma_4\left(\xi_1 + \frac{1}{\rho}\right) = 2\psi_1\psi_4 + \frac{1}{2\rho^2}\chi_1\chi_4\, e^{4\psi-2\xi}\,. \qquad \ldots(59)$$

Integrability condition leads to

$$2\psi_4\psi_{11} - \gamma_4\left(\xi_{11} - \frac{1}{\rho^2}\right) + \frac{1}{2\rho^2}e^{4\psi-2\xi}\left[(4\psi_1 - 2\xi_1)\chi_1\chi_4 - \frac{2\chi_1\chi_4}{\rho} + \chi_{11}\chi_4\right]$$
$$= 2\psi_4\psi_{44} + \frac{\psi_4}{\rho^2}(\chi_1^2 + \chi_4^2)e^{4\psi-2\xi} + \frac{1}{2\rho^2}\chi_4\chi_{44}e^{4\psi-2\xi}. \qquad \text{...(60)}$$

From eqns. (55), (57) and (60), we get

$$\psi_4\left(\xi_{11} - \frac{1}{\rho^2}\right) + \psi_4\left(\xi_1 + \frac{1}{\rho}\right)^2 - 2\psi_1\psi_4\left(\xi_1 + \frac{1}{\rho}\right)$$
$$- \frac{1}{2\rho^2}\chi_1\chi_4\left(\xi_1 + \frac{1}{\rho}\right)e^{4\psi-2\xi} = \gamma_4\left(\xi_{11} - \frac{1}{\rho^2}\right). \qquad \text{...(61)}$$

From eqns. (59) and (61), we obtain

$$[\psi_4 - \gamma_4]\left[\xi_{11} + \frac{2\xi_1}{\rho} + \xi_1^2\right] = 0. \qquad \text{...(62)}$$

The condition, $\xi_{11} + 2\xi_1/\rho + \xi_1^2 = 0$, leads to $R_{22} = 0$, which is a contradiction. Therefore, eqn. (62) leads to

$$\psi_4 = \gamma_4. \qquad \text{...(63)}$$

From eqns. (55), (57), (59), (60) and (63), we get

$$2\psi_1 = \left(\xi_1 + \frac{1}{\rho}\right). \qquad \text{...(64)}$$

Equations (59), (63) and (64) lead to either $\chi_4 = 0$ or $\chi_1 = 0$. Since $\chi_4 = 0$ leads to inconsistency, we have

$$\chi_1 = 0. \qquad \text{...(65)}$$

From eqns. (57), (64) and (65), we get

$$\psi_{44} - \frac{1}{2\rho^2}\chi_4^2 e^{4\psi-2\xi} = 0. \qquad \text{...(66)}$$

From eqns. (55) and (65), we obtain

$$\chi_{44} + 4\chi_4\chi_4 = 0 \qquad \text{...(67)}$$

which implies that ψ_4 is a function of t alone. Equations (64) implies that ψ_1 is a function of ρ alone. We may, therefore, assume that

$$\psi = F(\rho) + G(t) \qquad \text{...(68)}$$

where F and G are respectively functions of ρ and t alone. From eqns. (53), (64), (65) and (66), we get

$$6\psi_1^2 + 2\psi_{11} + \gamma_{11} - 2\psi_1\gamma_1 = 0. \qquad \text{...(69)}$$

From eqns. (52), (64), (65) and (66), we obtain

$$6\psi_1^2 + 2\psi_4^2 + 2\psi_{11} - 4\psi_1\gamma_1 + \psi_{44} = 0. \qquad \text{...(70)}$$

Eqns. (67) and (68) give

$$\chi_4 = L(\rho)\, e^{-4F-4G} \qquad ...(71)$$

which leads to

$$e^{4F} = mL(\rho) \qquad ...(72)$$

m being an arbitrary constant and $L(\rho)$ a function of integration. From eqns. (66), (68) and (72), we obtain

$$G_{44}\, e^{4G} = \frac{1}{2\rho^2} e^{-2\xi} \frac{e^{4F}}{m^2} = n \qquad ...(73)$$

n being a constant. Equation (73) on integration gives

$$G_4^2 = -\frac{n}{2} e^{-4G} + \frac{1}{2} h \qquad ...(74)$$

where h is a constant of integration From eqn. (74), we get

$$e^{2G} = \sqrt{\frac{n}{h}} \cosh\left[\sqrt{2h}\,(l+t)\right] \qquad ...(75)$$

l being a constant of integration. From eqns. (68), (70), (73) and (74), we obtain

$$\gamma_1 = \frac{3}{2} F_1 + \frac{1}{2}\frac{F_{11}}{F_1} + \frac{1}{4}\frac{h}{F_1}\,. \qquad ...(76)$$

From eqns. (68) and (69), we get

$$6F_1^2 + 2F_{11} + \gamma_{11} - 2F_1\gamma_1 = 0. \qquad ...(77)$$

Eqns. (76) and (77) lead to

$$\gamma_{11} = -3F_1^2 - F_{11} + \frac{h}{2}\,. \qquad ...(78)$$

From eqns. (76) and (78), we obtain

$$6\mu^4 - h\mu^2 + 5\mu^2\mu_1 - \mu_1^2 + \mu\mu_{11} - \tfrac{1}{2} h\mu_1 = 0 \qquad ...(79)$$

where

$$\mu = F_1 = \psi_1 = \frac{1}{2}\left(\xi_1 + \frac{1}{\rho}\right) \qquad ...(80)$$

From eqn. (79), we get

$$e^{\int [3\mu - (h/2\mu)]\, d\rho}\left[\frac{\mu_1}{\mu} + 2\mu\right] = \text{constant} = M\,(\text{say}). \qquad ...(81)$$

To obtain a solution for μ, we assume that

$$\mu_1 = \frac{1}{f(\mu)}\,. \qquad ...(82)$$

The function then satisfies Abel's equation of the first kind

$$f' = -\frac{f}{\mu} + \left(5\mu - \frac{1}{2}\frac{h}{\mu}\right) f^2 + (6\mu^3 - h\mu) f^3. \qquad \ldots(83)$$

When f is known, μ can be obtained as a function of ρ. Thus, in view of eqns. (64), (68), (71), (72), (75) and (76), the metric potentials of the space-time are given by

$$\psi = \int \mu d\rho + \frac{1}{2}\log\left[\sqrt{\frac{n}{h}}\cosh\{\sqrt{2h}\,(l+t)\}\right]$$

$$\xi = 2\int \mu d\rho - \log\rho + k,$$

$$\chi = \frac{\sqrt{h}}{\sqrt{2}\,mn}\tanh[\sqrt{2h}\,(l+t)] + b,$$

$$\gamma = \int \left(\frac{3}{2}\mu + \frac{1}{2}\frac{\mu_1}{\mu} + \frac{1}{4}\frac{h}{\mu}\right) d\rho + \frac{1}{2}\log\left[\sqrt{\frac{n}{h}}\cosh\{\sqrt{2h}\,(l+t)\}\right] \qquad \ldots(84)$$

where b and k are constants of integration. For this space-time, the value of F_{14} is given by

$$F_{14}^2 = -\frac{2}{k} e^{\int [\mu + (h/2\mu)]\, d\rho} [\mu\mu_1 + 2\mu^3]. \qquad \ldots(85)$$

Maxwell's equations are seen to be satisfied with

$$J^1 = J^2 = J^3 = J^4 = 0.$$

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