FLEXURE OF RING SECTOR PLATES UNDER UNIFORM AND CONCENTRATED NORMAL LOADS

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(*Received* 12 *November* 1975)

The paper presents a method of determining the deflection function when a ring sector plate is subject to arbitrary deflections and slopes along its boundary. Numerical results have been provided when the ring sector plate is subject to uniform or concentrated normal loadings.

Introduction

The problem of determining the stress distributions in a sector plate whose radial edges are stress free and upon whose circular edge self equilibrating tractions are prescribed and the analogous problem in the small deflection theory of flexure of plates have been the subject of several investigations. Eigenfunctions of the biharmonic equation developed in the angular coordinate and various techniques for determining the complex constants involved in the eigenfunction expansions have been used by many authors to provide solutions (see Horvay and Hanson 1957, Silverman 1958, Morley 1963, Gopalacharyulu 1969, Rao *et al.* 1973). However, problems which require the eigenfunctions in the radial coordinate have not been attempted in the literature. In a recent paper (Sarma *et al.* 1974) these eigenfunctions have been developed and a method of obtaining solutions of problems connected with the flexure of long curved plates have been indicated. The aim of the present paper is to bring out the applications of these functions for the flexure of clamped ring sector plates under various normal loadings.

We shall, therefore, consider the general problem of determining the deflection function for a ring sector plate which is subject to prescribed deflections and slopes along its boundary. We shall first indicate the determination of a function $W_h(r, \theta)$, which will give rise to the prescribed deflections and slopes at the four corners of the plate. Secondly, a Fourier series solution $W_f(r, \theta)$ will be developed in the angular coordinate, so that the residual deflections and slopes are reproduced on the curved edges. Finally, we are required to determine a deflection function $W_R(r, \theta)$ so that $W_h + W_f + W_R$ will give the deflection function for the prescribed boundary functions. An eigenvalue problem may be composed to account for the residual deflections and slopes prevailing on the radial edges. The analysis presented by Sarma *et al.* (1974) has been used to determine the functions $W_R(r, \theta)$.

Formulation and Solution of the Problem

Let the ring sector plate occupy the region $a \leqslant r \leqslant b, -\ . \leqslant \theta \leqslant \alpha$. Let $W_a(\theta)$, $W_b(\theta)$, $W_\alpha(r)$ and $W_{-\alpha}(r)$ denote the deflections on the boundaries $r = a$, $r = b$, $\theta = \alpha$ and $\theta = -\alpha$ respectively. Similarly let $\varepsilon_a(\theta)$, $\varepsilon_b(\theta)$, $\varepsilon_\alpha(r)$ and $\varepsilon_{-\alpha}(r)$ denote the slopes on these boundaries. It is assumed that these deflections and the slopes at the corners of the plate are continuous.

The differential equation governing the deflection of the plate is given by

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 W = 0; \; a \leqslant r \leqslant b, -\alpha \leqslant \theta \leqslant \alpha \quad \ldots(1)$$

A deflection function $W_h(r, \theta)$ is determined such that the prescribed deflections W_a, W_b, W_α, $W_{-\alpha}$ and the prescribed slopes ε_a, ε_b, ε_α, $\varepsilon_{-\alpha}$ are reproduced at the four corner points (a, α), (b, α), $(a, -\alpha)$ and $(b, -\alpha)$. We shall split the deflection function $W_h(r, \theta)$ into two parts, one even and the other odd in the angular coordinate θ. We assume

$$\overset{e}{W_h} = C_6 r^4 \cos 2\theta + C_5 r^2 \cos 2\theta + C_4 r^2 \ln r + C_3 \ln r + C_2 r^2 + C_1 \quad \ldots(2a)$$

or

$$\overset{o}{W_h} = D_6\, r^4 \sin 2\theta + D_5 r^2 \sin 4\theta + (D_4 r^2 \ln r + D_3 \ln r + D_2 r^2 + D_4)\theta \quad \ldots(2b)$$

depending upon whether W_h is even or odd in θ.

The constants C_i $(i = 1, 6)$ in the even case are determined using the conditions

$$\overset{e}{W_h}(a, \alpha) = \tfrac{1}{2}\left\{ W_a(\alpha) + W_a(-\alpha) \right\}; \quad \overset{e}{W_h}(b, \alpha) = \tfrac{1}{2}\left\{ W_b(\alpha) + W_b(-\alpha) \right\}$$

$$\frac{\partial \overset{e}{W_h}}{\partial r}(a, \alpha) = \tfrac{1}{2}\left\{ \varepsilon_a(\alpha) + \varepsilon_a(-\alpha) \right\}; \quad \frac{\partial \overset{e}{W_h}}{\partial r}(b, \alpha) = \tfrac{1}{2}\left\{ \varepsilon_b(\alpha) + \varepsilon_b(-\alpha) \right\}$$

$$\frac{1}{a}\frac{\partial \overset{e}{W_h}}{\partial \theta}(a, \alpha) = \tfrac{1}{2}\left\{ \varepsilon_\alpha(a) - \varepsilon_{-\alpha}(a) \right\};$$

$$\frac{1}{b}\frac{\partial \overset{e}{W_h}}{\partial \theta}(b, \alpha) = \tfrac{1}{2}\left\{ \varepsilon_\alpha(b) - \varepsilon_{-\alpha}(b) \right\}$$

The above equations will give rise to a system of six linear algebraic equations for the unknowns C_i $(i=1, 6)$. The explicit determination of these constants though straightforward, however, involves laborious algebraic calculations. Further, since the expressions for the constants are very unwieldy, we have not presented them here.

The expressions for the constants D_i $(i = 1, 6)$ in the odd case are determined using the conditions

$$\overset{o}{W}_h(a,\alpha) = \tfrac{1}{2}\Big\{ W_a(\alpha) - W_a(-\alpha) \Big\}; \quad \overset{o}{W}_h(b,\alpha) = \tfrac{1}{2}\Big\{ W_b(\alpha) - W_b(-\alpha) \Big\}$$

$$\frac{\partial \overset{o}{W}_h}{\partial r}(a,\alpha) = \tfrac{1}{2}\{ \varepsilon_a(\alpha) - \varepsilon_a(-\alpha)\}; \quad \frac{\partial \overset{o}{W}_h}{\partial r}(b,\alpha) = \tfrac{1}{2}\{ \varepsilon_b(\alpha) - \varepsilon_b(-\alpha)\}$$

$$\frac{1}{a}\frac{\partial \overset{o}{W}_h}{\partial \theta}(a,\alpha) = \tfrac{1}{2}\Big\{ \varepsilon_\alpha(a) + \varepsilon_{-\alpha}(a) \Big\}; \quad \frac{1}{b}\frac{\partial \overset{o}{W}_h}{\partial \theta}(b,\alpha) = \tfrac{1}{2}\Big\{ \varepsilon_\alpha(b) + \varepsilon_{-\alpha}(b) \Big\}$$

The function $W_h = \overset{e}{W}_h + \overset{o}{W}_h$ is thus determined.

The deflections and slopes determined using $W_h(r,\theta)$ are subtracted from those prescribed on the four boundaries. The resulting deflections and slopes will now be zero at the corners. The residual deflections and slopes are now calculated on the four boundaries.

A Fourier series solution $W_f(r,\theta)$ is now developed in the coordinate θ so as to satisfy the residual deflections and slopes on the curved boundaries. For this we take

$$W_f = \Sigma f_m(r) \begin{matrix}\cos\\ \sin\end{matrix} (\alpha_m\, \theta) \qquad \ldots(3)$$

where $\alpha_m = \overset{e}{\alpha}_m = (2m-1)\dfrac{\pi}{2\alpha}$ and $\alpha_m = \overset{o}{\alpha}_m = m\pi/\alpha$ depending upon whether W_f is even or odd in θ.

Upon substituting (3) in (1) we get

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\alpha_m{}^2}{r^2} \right)^2 f_m = 0 \qquad \ldots(4)$$

The general solution of (4) is given by

$$f_m(r) = A_m r^{\alpha_m} + B_m r^{-\alpha_m} + C_m r^{\alpha_m+2} + D_m r^{-\alpha_m+2} \qquad \ldots(5)$$

If the residual deflections and slopes on the curved boundaries are expanded in a Fourier series (cosine or sine series depending on the problem on hand) and are equated to the corresponding deflections and slopes determined using (3), we get four linear algebraic equations for the unknowns A_m, B_m, C_m and D_m. The analysis being straightforward, the details are not presented here. With the determination of these constants the function $W_f = \overset{e}{W}_f + \overset{o}{W}_f$ is determined.

In order to satisfy the prescribed deflections and slopes on the four boundaries 'exactly', it remains to determine a function W_R so that it satisfies the clamped conditions on the curved boundaries and will give rise to the residual deflections and slopes on the radial boundaries. This is done by formulating an eigenvalue problem in the radial coordinate. A linear combination of the eigenfunctions is sought to satisfy the residual deflections and slopes on the radial boundaries. In a recent paper Sarma *et al.* (1974) solutions for the flexure of long flat curved plates under end loadings have been presented. The analysis for finite curved plates verbatim follows as in Sarma *et al.* (1974). We have therefore preferred to present only the solutions of the title problem for some boundary loadings.

Numerical Solutions

(i) *Ring Sector Plates Under Uniform Load* q

A particular function $W_p(r, \theta)$ which corresponds to the uniform load q and for which the deflections and slopes vanish along the curved edges $r = a$ and $r = b$ is taken in the form

$$W_p = \frac{q}{D}\left[C_1 + C_2 r^2 + C_3 \ln r + C_4 r^2 \ln r + \frac{r^4}{64}\right] \quad \ldots(6)$$

The constants C_i $(i = 1, 4)$ are determined using the conditions

$$W_p = \frac{\partial W_p}{\partial r} = 0 \text{ on } r = a \text{ and } r = b$$

For numerical calculations the aspect ratio b/a is set equal to 2, the semi-sector angle α to 30° and the Poisson's ratio σ to 0.3. The eigenfunction analysis is carried out using the first five eigenvalues. The deflections are slopes on $\theta = \pm\alpha$ calculated via the truncated eigenfunction series, differ from zero by an error which is less than 10^{-4} times the deflection at the point $\left(\frac{b+a}{2}, 0\right)$.

However, using ten eigenvalues this accuracy can be increased to an order of 10^{-5}. The moment M_θ on $\theta = \alpha$ and the moments M_r on $r = a$ and $r = b$ are presented in Table I. The deflections and moments on the line $\theta = 0$ and $r = (b+a)/2$ are presented in Table II.

(ii) *Ring Sector Plates Under Concentrated Load* q

We assume that a concentrated load q acts at the point $\left(\frac{b+a}{2}, 0\right)$.

A particular solution which corresponds to the concentrated load q is given by $W_p = \rho^2 \ln \rho$, where $\rho^2 = r^2 - (b + a)\, r \cos\theta + (b + a)^2/4$. Suitable biharmonic functions are added to W_p so that the deflections and slopes at the corner points $(a, \pm\alpha)$, $(b, \pm\alpha)$ vanish. Accordingly the particular solution is taken in the form

$$W_p = \frac{q}{16\pi D}\left[C_1 + C_2 r^2 + C_3 \ln r + C_4 r^2 \ln r + C_5 r^2 \cos 2\theta + C_6 r^4 \cos 2\theta + \rho^2 \ln \rho\right]$$

The calculations are performed as per the analysis outlined earlier. For numerical work we set $b/a = 2.0$, $\alpha = 30°$ and $\sigma = 0.3$. The Fourier series solution W_f is determined using the first twenty terms of the Fourier expansions. The eigenfunction analysis is carried out using the first five eigenvalues. The deflections and slopes calculated along the boundary of the plate differ from zero by an error equal to 10^{-2} times the maximum deflection at the point $\left(\frac{b+a}{2}, 0\right)$. However, using ten eigenvalues this accuracy in the error can be increased to an order of 10^{-3}. The moment M_θ on $\theta = \alpha$ and the moment M_r on $r = a$ and $r = b$ are presented in Table III. The deflections and moments on the lines $\theta = 0$ and $r = (b+a)/2$ are presented in Table IV.

TABLE I

r	$\alpha_1 qa^2 (M_\theta)_{\theta \doteq \alpha}$	θ	$\alpha_2 qa^2 (M_r)_{r \doteq a}$	$\alpha_3 qa^2 (M_r)_{r \doteq b}$
	α_1		α_2	α_3
1.0	0.0000	0°	−0.0894	−0.0700
1.2	−0.0346	6°	−0.0849	−0.0674
1.4	−0.0571	12°	−0.0709	−0.0595
1.6	−0.0486	18°	−0.0463	−0.0439
1.8	−0.0204	24°	−0.0148	−0.0195
2.0	0.0000	30°	0.0000	0.0000

TABLE II

r	$\frac{\beta_1 qa^4}{D}(W)_{\theta \doteq 0}$	$\beta_2 qa^2 (M)_{\theta = 0}$	0	$\frac{\beta_3 qa^4}{D}(W)_{r=(b+a)/2}$	$\beta_4 qa^2 (M_r)_{r=(b+a)/2}$
	β_1	β_2		β_3	β_4
1.0	0.00000	−0.02682	0°	0.00226	0.03754
1.2	0.00101	0.00151	6°	0.00214	0.03583
1.4	0.00212	0.01774	12°	0.00179	0.03026
1.6	0.00205	0.01814	18°	0.00119	0.01980
1.8	0.00090	0.00477	24°	0.00045	0.00360
2.0	0.00000	−0.02097	30°	0 00000	−0.01791

TABLE III

r	$r_1qa^2(M_\theta)_{\theta=\alpha}$	θ	$r_2qa^2(M_r)_{r=a}$	$r_3qa^2(M_r)_{r=b}$
	r_1		r_2	r_3
1.0	0.0000	0°	−0.1812	−0.1714
1.2	−0.0261	6°	−0.1600	−0.1589
1.4	−0.0572	12°	−0.1011	−0.0853
1.6	−0.0431	18°	−0.0514	−0.0394
1.8	−0.0215	24°	−0.0141	−0.0101
2.0	0.0000	20°	0.0000	0.0000

TABLE IV

r	$\delta_1 \frac{qa^4}{D}(W)_{\theta=0}$	$\delta_2 qa^2(M_\theta)_{\theta=0}$	θ	$\delta_3 \frac{qa^4}{D}(W)_{r=(b+a)/2}$	$\delta_4 qa^2(M_r)_{r=(b+a)/2}$
	δ_1	δ_2		δ_3	δ_4
1.0	0.00000	−0.04760	0°	0.00715	*
1.2	0.00241	0.00293	6°	0.00532	0.11467
1.4	0.00620	0.12985	12°	0.00318	0.05736
1.6	0.00610	0.20386	18°	0.00147	0.02372
1.8	0.00227	0.09945	24°	0.00042	0.00305
2.0	0.00000	−0.02644	30°	0.00000	−0.01361

* Load point

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