THE SOLUTION OF QUADRUPLE TRIGONOMETRICAL INTEGRAL EQUATIONS AND THEIR APPLICATION TO A CRACK PROBLEM

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This paper deals with quadruple integral equations involving cosine kernels. A closed form solution of quadruple integral equations has been worked out. At the end of this paper, the use of the solution of quadruple integral equations to a two dimensional crack problem has also been investigated.

1. Introduction

Closed form solutions of the triple trigonometric integral equations are given by Tranter (1960) and Srivastava and Lowengrub (1970). A comprehensive survey of the work in the field of dual and triple integral equations has been given by Sneddon (1966). In this paper we shall obtain the closed form solution of the quadruple cosine integral equations which arise in four-part mixed boundary value problems in two-dimensional infinite plane. We here reduce the quadruple integral equations to a singular integral equation which is solvable in closed form. Throughout the paper the analysis is purely formal and no attempt is made to justify the change of order of integrations.

The problem of determining the stress distribution in the neighbourhood of two coplanar Griffith cracks which are opened by constant pressure along the length of the cracks has been considered by Willmore (1949), Tranter (1961) and Lowengrub and Srivastava (1968). In this paper we shall discuss a more general problem than the above. We shall consider the problem of determining the stress field caused by the presence of three Griffith cracks of equal lengths when two of them are subjected to the same pressure. The middle crack lying between the two cracks is stress-free.

The solution of quadruple integral equations can be used in solving other mixed boundary value problems of mathematical physics.

2. Quadruple Integral Equations

Let us consider the quadruple integral equations

$$\int^{\infty} u\, A(u) \cos xu\, du = f_1(x), \quad 0 < x < a \qquad \dots(1)$$

$$\int_0^\infty A(u) \cos xu \cdot du = f_2(x), \; a < x < b \quad \text{... (2)}$$

$$\int_0^\infty u \; A(u) \cos xu \; du = f_3(x), \; b < x < 1 \quad \text{... (3)}$$

$$\int_0^\infty A(u) \cos xu \; du = f_4(x), \; 1 < x \quad \text{... (4)}$$

Assume for a while $f_3 = 0$, $f_4 = 0$. Equation (4) is identically satisfied if we take (Sneddon 1966, pp. 103–104)

$$A(u) = \int_0^1 g(t) \; J_0(ut)dt \quad \text{... (5)}$$

where the function $g(t)$ is to be determined.

If we substitute the value of $A(u)$ from (5) in (1), (2) and (3) and interchange the order of integrations and make use of the relations (2. 1. 13) and (2. 1. 14) of Sneddon (1966, pp. 27–28) we find the triple integral equations

$$\frac{d}{dx} \int_0^x \frac{g(t)dt}{\sqrt{x^2-t^2}} = f_1(x), \qquad 0 < x < a \quad \text{...(6)}$$

$$\int_x^1 \frac{g(t)dt}{\sqrt{t^2-x^2}} = f_2(x), \qquad a < x < b \quad \text{... (7)}$$

$$\frac{d}{dx} \int_0^x \frac{g(t)dt}{\sqrt{x^2-t^2}} = 0, \qquad b < x < 1 \quad \text{... (8)}$$

We now assume that

$$\frac{d}{dx} \int_0^x \frac{g(t)dt}{\sqrt{x^2-t^2}} = R(x^2), \qquad a < x < 1 \quad \text{.. (9)}$$

where $R(x^2)$ is to be determined. Solving equations (6) and (9) with the help of the solution of Abel's type integral equation we get

$$g(t)=\frac{2}{\pi}\,t\left[\int_0^a \frac{f_1(x)dx}{\sqrt{t^2-x^2}} \qquad \int_a^t \frac{R(x^2)dx}{t^2-x^2}\right], \qquad a<t<1 \qquad \text{...(10)}$$

Substituting the value of $g(t)$ from (10) in (7) and interchanging the order of integrations, we get

$$\frac{1}{\pi}\int_a^1 R(u^2)\log\left|\frac{\sqrt{1-x^2}+\sqrt{1-u^2}}{\sqrt{1-x^2}-\sqrt{1-u^2}}\right|du$$

$$=f_2(x)+\frac{1}{\pi}\int_0^a f_1(u)\log\left|\frac{\sqrt{1-x^2}+\sqrt{1-u^2}}{\sqrt{1-x^2}-\sqrt{1-u^2}}\right|du, \quad a<x<b \qquad \text{...(11)}$$

We now have with the help of (8) and (9)

$$\left.\begin{aligned} R(u^2) &= h(u^2), \; a<u<b \\ &= 0, \; b<u<1 \end{aligned}\right\} \qquad \text{...(12)}$$

where $h(u^2)$ is unknown function. Now we can write (11) in the following form by making use of (12) :

$$\frac{1}{\pi}\int_a^b h(u^2)\log\left|\frac{\dfrac{1}{\sqrt{1-x^2}}+\dfrac{1}{\sqrt{1-u^2}}}{\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{\sqrt{1-u^2}}}\right|du$$

$$=f_2(x)+\frac{1}{\pi}\int_0^a f_1(u)\log\left|\frac{\dfrac{1}{\sqrt{1-x^2}}+\dfrac{1}{\sqrt{1-u^2}}}{\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{\sqrt{1-u^2}}}\right|du=M(x)\,(\text{say}),\; a<x<b \qquad \text{...(13)}$$

Using the results obtained by Parihar (1971, Lemma 2, p. 257) we get the solution of the integral eqn. (13) in the following form :

$$h(u^2)=\frac{2u\;\overline{1-u^2}}{\pi\sqrt{(u^2-a^2)(b^2-u^2)}}\int_a^b \frac{\sqrt{(x^2-a^2)(b^2-x^2)}\;M'(x)\,dx}{(u^2-x^2)\;\sqrt{1-x^2}}$$

$$+\frac{B\sqrt{1-a^2}}{2F\left[\frac{\pi}{2},\sqrt{\frac{b^2-a^2}{1-a^2}}\right]}\frac{u}{\sqrt{(1-u^2)(u^2-a^2)(b^2-u^2)}}, \quad a<u<b \qquad \dots(14)$$

where

$$B=\frac{2\sqrt{1-a^2}}{F\left[\frac{\pi}{2},\sqrt{\frac{1-b^2}{1-a^2}}\right]}\int_a^b \frac{xM(x)\,dx}{\sqrt{(1-x^2)(x^2-a^2)(b^2-x^2)}}$$

$$-\frac{4}{\pi}\int_a^b \frac{y\sqrt{1-y^2}\;dy}{\sqrt{(y^2-a^2)(b^2-y^2)}}\int_a^b \frac{\sqrt{(x^2-a^2)(b^2-x^2)}\;M'(x)\,dx}{(y^2-x^2)(1-x^2)^{1/2}} \qquad \dots(15)$$

F denotes elliptic integral of the first kind and $M'(x)$ denotes differentiation with respect to x of the function $M(x)$.

Equations (5), (10), (12), (14) and (15) enable us to complcte the solution of the system of equations (1), (2), (3) and (4) for $f_3=0$, $f_4=0$. To complete the solution, assume that

$$f_1=0,\ f_2=0 \text{ and } f_3\neq 0,\ f_4\neq 0$$

$$\text{Let } A(u)=\int_x^\infty g(t)\,J_0(ut)\,dt \qquad \dots(16)$$

The function $A(u)$ given by (16) satisfies (1) identically and on substitution in (2), (3) and (4) leads to triple integral equations

$$\int_a^\infty \frac{g(t)dt}{\sqrt{t^2-x^2}}=0, \quad a<x<b \qquad \dots(17)$$

$$\frac{d}{dx}\int_a^x \frac{g(t)dt}{\sqrt{x^2-t^2}}=f_3(x), \quad b<x<1 \qquad \dots(18)$$

$$\int_x^\infty \frac{g(t)dt}{\sqrt{t^2-x^2}}=f_4(x), \quad 1<x \qquad \dots(19)$$

We suppose that

$$\frac{d}{dx}\int_x \frac{g(t)dt}{\sqrt{t^2-x^2}} = -R_2(x^2),\ a < x < 1 \qquad ...(20)$$

By making use of (17) we find that

$$\left.\begin{aligned} R_2(x^2) &= 0, & a < x < b, \\ &= h_2(x^2), & b < x < 1 \end{aligned}\right\} \qquad ...(21)$$

Here $R_2(x^2)$ and $h_2(x^2)$ are unknown functions. Differentiating equation (19) with respect to x and using the solution of Abel's type integral equation, we get from (19) and (20)

$$g(t) = \frac{2t}{\pi}\left[\int_t^1 \frac{R_2(u^2)du}{\sqrt{u^2-t^2}} - \int_1 \frac{f_4'(u)du}{\sqrt{u^2-t^2}}\right],\ a < t < 1 \qquad ...(22)$$

We have, for $b < x < 1$,

$$\int_a^x \frac{t\ dt}{\sqrt{x^2-t^2}}\int_t^1 \frac{R_2(u^2)du}{\sqrt{u^2-t^2}} = \int_a^x\left[\int_t^x+\int_x^1\right]\frac{tR_2(u^2)\,du\,dt}{\sqrt{x^2-t^2}\sqrt{u^2-t^2}}$$

$$=\left[\int_a^x\int_a^u+\int_x^1\int_a^x\right]\frac{t\ R_2(u^2)\ dt\ du}{\sqrt{x^2-t^2}\sqrt{u^2-t^2}}$$

$$=\int_a^1 R_2(u^2)\ du\int_a^{\min(u,x)}\frac{t\,dt}{\sqrt{x^2-t^2}\sqrt{u^2-t^2}}$$

$$=\frac{1}{2}\int_a^1 R_2(u^2)\log\left|\frac{\sqrt{u^2-a^2}+\sqrt{x^2-a^2}}{\sqrt{u^2-a^2}-\sqrt{x^2-a^2}}\right|du$$

Substituting the expression for $g(t)$ from (22) into (18) and interchanging the order of integrations and using the above result we can easily show that

$$\frac{1}{\pi}\frac{d}{dx}\int_a^1 R_2(u^2)\log\left|\frac{\sqrt{u^2-a^2}+\sqrt{x^2-a^2}}{\sqrt{u^2-a^2}-\sqrt{x^2-a^2}}\right|du$$

$$=f_3(x)+\frac{1}{\pi}\frac{d}{dx}\int_1^\infty f_4'(u)\log\left|\frac{\sqrt{u^2-a^2}+\sqrt{x^2-a^2}}{\sqrt{u^2-a^2}-\sqrt{x^2-a^2}}\right|du,\quad b<x<1 \quad \text{...(23)}$$

The above equation can be written in the following form by making use of (21).

$$\frac{1}{\pi}\frac{d}{dx}\int_b^1 h_2(u^2)\log\left|\frac{\sqrt{u^2-a^2}+\sqrt{x^2-a^2}}{\sqrt{u^2-a^2}-\sqrt{x^2-a^2}}\right|du$$

$$=f_3(x)+\frac{1}{\pi}\frac{d}{dx}\int_1^\infty f_4'(u)\log\left|\frac{\sqrt{u^2-a^2}+\sqrt{x^2-a^2}}{\sqrt{u^2-a^2}-\sqrt{x^2-a^2}}\right|du,\quad b<x<1 \quad \text{...(24)}$$

We find, after interchanging the order of differentiation and integration in equation (24), $h_2(u^2)$ must satisfy the integral equation

$$\frac{2}{\pi}\int_b^1 \frac{h_2(u^2)\sqrt{u^2-a^2}\,du}{u^2-x^2} = \frac{f_3(x)\sqrt{x^2-a^2}}{x}+\frac{2}{\pi}\int_1^\infty \frac{\sqrt{u^2-a^2}\,f_4'(u)\,du}{u^2-x^2}$$

$$= N(x)\ \text{(say)},\ b<x<1 \quad \text{...(25)}$$

Now using the finite Hilbert transform theorem given by Srivastava and Lowengrub (1970), we obtain the solution of equation (25) in the form

$$h_2(u^2) = -\frac{2}{\pi}\frac{u}{\sqrt{u^2-a^2}}\left\|\frac{u^2-b^2}{1-u^2}\right\|^{\frac{1}{2}}\int_b^1 x\left(\frac{1-x^2}{x^2-b^2}\right)^{\frac{1}{2}}\frac{N(x)}{x^2}$$

$$+\frac{uC}{\sqrt{(u^2-a^2)(u^2-b^2)(1-u^2)}},\quad b<u<1 \quad \text{...(26)}$$

where C is an arbitrary constant. Making use of (19), (20) and (21), we can find the expression for $g(t)$ in the following form:

$$g(t) = -\frac{2t}{\pi} \int_t^{\infty} \frac{G(u^2)du}{\sqrt{u^2-t^2}}, \quad a < t < \infty \qquad \text{...(27)}$$

where

$$\left. \begin{aligned} G(u^2) &= 0, & a < u < b; \\ &= -h_2'(u^2), & b < u < 1; \\ &= f_4'(u), & 1 < u < \infty \end{aligned} \right\} \qquad \text{...(28)}$$

Substituting the expression for $g(t)$ from (27) into (17) and interchanging the order of integrations and then using the well-known integral

$$\int_x^u \frac{t\,dt}{\sqrt{t^2-x^2}\,\sqrt{u^2-t^2}} = \frac{\pi}{2},$$

we find that

$$\int_x^{\infty} G(u^2)\,du = 0, \quad a < x < b \qquad \text{...(29)}$$

Hence, by making use of (28) we find from (29) that

$$\int_b^1 h_2(u^2)\,du = \int_1^{\infty} f_4'(u)\,du. \qquad \text{...(30)}$$

With the help of (26) and (30), we get

$$C = \frac{\sqrt{1-a^2}}{F} \int_1^{\infty} f_4(u)du + \frac{2\sqrt{1-a^2}}{F} \int_b^1 \frac{u\sqrt{u^2-b^2}\,du}{\sqrt{(u^2-a^2)(1-u^2)}}$$

$$\times \int_b^1 x \left(\frac{1-x^2}{x^2-b^2}\right)^{\frac{1}{2}} \frac{N(x)dx}{x^2-u^2}, \qquad \text{...(31)}$$

where

$$F = F\left[\frac{\pi}{2}, \sqrt{\frac{1-b^2}{1-a^2}}\right]$$

Equations (16), (21), (22), (26) and (31) enable us to calculate $A(u)$. By combining the two solutions, viz., one for $f_3 = 0, f_4 = 0$ and other for $f_1 = 0, f_2 = 0, f_3 \neq 0, f_4 \neq 0$, we obtain the solution of the quadruple integral equations (1), (2), (3) and (4).

3. Two-dimensional Crack Problem

To illustrate the use of the solution of quadruple trigonometric integral equations we consider the problem of determining the stress distribution in the neighbourhood of three collinear Griffith cracks. The cracks are defined by $-1 \leqslant x \leqslant -b,\ b \leqslant x \leqslant 1,\ -a \leqslant x \leqslant a,\ y = 0$. The cracks $-1 \leqslant x \leqslant -b,\ b \leqslant x \leqslant 1, y = 0$, are opened by the application of a prescribed internal pressure $p(x)$ and the faces of the crack $-a \leqslant x \leqslant a,\ y = 0$ are stress free. Denoting the components of stress $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ and the displacement vector by $[u_x, u_y, 0]$, we have the following boundary conditions :

$$\left.\begin{aligned} \sigma_{yy}(x, 0) &= 0, \quad -a \leqslant x < a; \\ \sigma_{yy}(x, 0) &= -p(x), \quad -1 \leqslant x \leqslant -b,\ b \leqslant x \leqslant 1; \\ u_y(x, 0) &= 0, \quad a < |x| < b,\ |x| > 1; \\ \sigma_{xy}(x, 0) &= 0, \quad - \quad < x < \infty. \end{aligned}\right\} \qquad \ldots(32)$$

In addition, we require that all the stress and displacement components vanish at infinity and that

$$p(-x) = p(x).$$

The solution of the equations of elastic equilibrium appropriate to half plane $y \geqslant 0$ is given by

$$u_y(x,y) = \frac{2(1+\eta)}{\pi E} \int_0^\infty A(u)\,(2-2\eta + uy)e^{-uy} \cos xu\, du \qquad \ldots(33)$$

$$u_x(x,y) = \frac{-2(1+\eta)}{\pi E} \int_0^\infty A(u)\,(1-2\eta-uy)e^{-uy} \sin ux\cdot du \qquad \ldots(34)$$

where E is the Young's modulus and η the Poisson's ratio of the material. In this case we have

$$\sigma_{yy}(x, y) = \frac{-2}{\pi} \int_0^\infty u\, A(u)\,(1+uy)e^{-uy} \cos ux\, du \qquad \ldots(35)$$

and

$$\sigma_{xy}(x,y) = -y\int_0^\infty u^2 A(u)\, e^{-uy} \sin ux\, du \qquad ...(36)$$

The boundary conditions (32) are satisfied provided that $A(u)$ is determined by the quadruple integral equations

$$\int_0^\infty uA(u) \cos xu\, du = 0, \quad 0<x<a \qquad ...(37)$$

$$\int_0^\infty A(u) \cos xu\, du = 0, \quad a<x<b \qquad ...(38)$$

$$\int_0^\infty u\,A(u) \cos xu\, du = \frac{\pi}{2}p(x), \quad b<x<1 \qquad ...(39)$$

$$\int_0^\infty A(u) \cos xu\, du = 0, \quad x>1. \qquad ...(40)$$

Making use of (16), (21), (22), (26) and (31) we get the solution of equations (37), (38), (39) and (40) in the following form:

$$A(u) = \int_a^\infty g_2(t)\, J_0(ut)\, dt$$

$$g_2(t) = t\int_b^1 \frac{h_3(u^2)\, du}{\sqrt{u^2-t^2}}, \quad a<t<b$$

$$h_3(u^2) = -\frac{2}{\pi}\frac{u}{\sqrt{u^2-a^2}}\left(\frac{u^2-b^2}{1-u^2}\right)^{\frac{1}{2}}\int_b^1\left(\frac{1-x^2}{x^2-b^2}\right)^{\frac{1}{2}}\sqrt{\frac{x^2-a^2}{x^2-u^2}}\; p(x)\, dx$$

$$+\frac{u\,C_1}{\sqrt{(u^2-a^2)(u^2-b^2)(1-u^2)}}, \quad b<u<1 \qquad ...(41)$$

where

$$C_1 = \frac{2}{\pi}\frac{\sqrt{1-a^2}}{F}\int_b^1 \frac{u\sqrt{u^2-b^2}\,du}{\sqrt{(u^2-a^2)(1-u^2)}}\int_b^1 \left(\frac{1-x^2}{x^2-b^2}\right)^{\frac{1}{2}} \frac{(x^2-a^2)^{\frac{1}{2}}}{x^2-u^2}\, p(x)\, dx \qquad ...(42)$$

As $a \to 0$, we obtain the solution of triple integral equations, which is in agreement with the result of Srivastava and Lowengrub (1970).

Making use of the well-known identity

$$\left[\frac{(u^2-b^2)(1-x^2)}{(1-u^2)(x^2-b^2)}\right]^{\frac{1}{2}} \cdot \left[1+\frac{x^2-u^2}{u^2-b^2}\right] = \left[\frac{(1-u^2)(x^2-b^2)}{(u^2-b^2)(1-x^2)}\right]^{\frac{1}{2}} \left[1-\frac{x^2-u^2}{1-u^2}\right]$$

we can write (41) and (42) in the alternative form

$$h_3(u^2) = -\frac{2}{\pi}\frac{u}{\sqrt{u^2-a^2}}\left(\frac{1-u^2}{u^2-b^2}\right)^{\frac{1}{2}}\int_b^1\left(\frac{x^2-b^2}{1-x^2}\right)^{\frac{1}{2}}\frac{\sqrt{x^2-a^2}\,p(x)dx}{x^2-u^2}$$

$$+ \frac{uC_2}{\sqrt{(u^2-a^2)\,(u^2-b^2)(1-u^2)}} \qquad ...(43)$$

where

$$C_2 = \frac{2\sqrt{1-a^2}}{\pi F}\int_b^1 \frac{u}{\sqrt{u^2-a^2}}\left(\frac{1-u^2}{u^2-b^2}\right)^{\frac{1}{2}} du \int_b^1\left(\frac{x^2-b^2}{1-x^2}\right)^{\frac{1}{2}}\frac{(x^2-a^2)^{\frac{1}{2}}\, p(x)\, dx}{x^2-u^2} \qquad ...(44)$$

We can easily deduce that

$$\left[\sigma_{yy}(x,0)\right]_{a<x<b} = -\frac{2}{\pi}\frac{x}{\sqrt{(x^2-a^2)}}\int_b^1\frac{(t^2-a^2)^{\frac{1}{2}}\, h_3(t^2)\,dt}{t^2-x^2} \qquad ...(45)$$

and

$$\left[\sigma_{yy}(x,0)\right]_{x>1} = \frac{2}{\pi}\frac{x}{\sqrt{x^2-a^2}}\int_b^1\frac{(t^2-a^2)^{\frac{1}{2}}\, h_3(t^2)\,dt}{x^2-t^2} \qquad ...(46)$$

Substituting (43) and (41) respectively into the equations (45) and (46), we obtain the expressions

$$[\sigma_{yy}(x,0)]_{a>x>b} = \frac{2}{\pi}\frac{x}{\sqrt{x^2-a^2}}\left(\frac{1-x^2}{b^2-x^2}\right)^{\frac{1}{2}}\int_b^1\left(\frac{y^2-b^2}{1-y^2}\right)^{\frac{1}{2}}\frac{\sqrt{y^2-a^2}\,p(y)\,dy}{y^2-x^2}$$

$$-\frac{xC_2}{\sqrt{(x^2-a^2)(b^2-x^2)(1-x^2)}} \quad \dots (47)$$

and

$$[\sigma_{yy}(x,0)]_{x>1} = \frac{2}{\pi}\frac{x\sqrt{x^2-b^2}}{\sqrt{(x^2-a^2)(x^2-1)}}\int_b^1\frac{(1-y^2)^{1/2}}{(y^2-b^2)^{1/2}}\frac{\sqrt{y^2-a^2}\,p(y)\,dy}{x^2-y^2}$$

$$+\frac{x\,C_1}{\sqrt{(x^2-a^2)(x^2-b^2)(x^2-1)}} \quad \dots (48)$$

Hence, we now find the stress intensity factors at the tips of the cracks in the form

$$N_a = \lim_{x\to a+}\sqrt{x-a}\,[\sigma_{yy}(x,0)]$$

$$=\frac{\sqrt{2a}}{\pi\sqrt{(b^2-a^2)(1-a^2)}}\left[(1-a^2)\int_b^1\frac{\sqrt{y^2-b^2}\,p(y)\,dy}{\sqrt{(1-y^2)(y^2-a^2)}}-\frac{\pi}{2}C_2\right] \quad \dots (49)$$

$$N_b = \lim_{x\to b-}\sqrt{b-x}\,[\sigma_{yy}(x,0)]$$

$$=\frac{\sqrt{2b}}{\pi\sqrt{(b^2-a^2)(1-b^2)}}\left[(1-b^2)\int_b^1\frac{\sqrt{y^2-a^2}\,p(y)\,dy}{\sqrt{(1-y^2)(y^2-b^2)}}-\frac{\pi}{2}C_2\right] \quad \dots (50)$$

and

$$N_c = \lim_{x\to 1+}\sqrt{x-1}\,[\sigma_{yy}(x,0)]$$

$$= \frac{\sqrt{2}}{\pi \sqrt{(1-a^2)(1-b^2)}} \left[(1-b^2) \int_b^1 \frac{\sqrt{y^2-a^2}\, p(y)\, dy}{\sqrt{(y^2-b^2)(1-y^2)}} + \frac{\pi}{2} C_1 \right] \quad \dots (51)$$

where C_1 and C_2 are given by (42) and (44) respectively.

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