ON SOME INTEGRAL INEQUALITIES AND THEIR APPLICATIONS TO INTEGRODIFFERENTIAL EQUATIONS

by B. G. Pachpatte, *Department of Mathematics, Deogiri College, Aurangabad (Maharashtra)*

(*Received* 12 *November* 1975)

In this paper we shall present some new integral inequalities which can be used in applications as handy tools. To illustrate the application of some of the inequalities we shall present some interesting results on the behaviour of solutions of integrodifferential equations of the more general type.

1. Introduction

Gronwall (1919) proved a very useful integral inequality what is now referred to as Gronwall's inequality also known as Bellman's Lemma (Coddington and Levinson, 1955, p. 37). This inequality has many uses in the theory of ordinary differential and integral equations in proving uniqueness, comparison, continuous dependence, perturbation and stability results. On the basis of various motivations Gronwall-Bellman inequality has been extended and used considerably in various contexts. Recently, in a series of papers the author has established some new integral inequalities of the Gronwall-Bellman type that have a wide range of applications in the theory of differential and integral equations (Pachpatte 1973; 1975a, b). The aim of the present paper is to establish some useful integral inequalities which claim the following as their origin.

Lemma 1 (Pachpatte 1973) — Let $x(t)$, $f(t)$ and $g(t)$ be real-valued nonnegative continuous functions defined on $I = [0, \infty)$; for which the inequality

$$x(t) \leq x_0 + \int_0^t f(s)\, x(s)\, ds + \int_0^t f(s) \left(\int_0^s g(\tau)\, x(\tau)\, d\tau \right) ds, \quad t \in I$$

holds, where x_0 is a nonnegative constant. Then

$$x(t) \leq x_0 \left[1 + \int_0^t f(s) \exp\left(\int_0^s [\, f(\tau) + g(\tau)\,]\, d\tau \right) ds \right], \quad t \in I.$$

The main results of the present paper are given in section 2. Section 3 deals with the applications of some of these inequalities to the boundedness, asymptotic behaviour, and the rate of growth of solutions of integrodifferential equations of the more general type.

2. Integral Inequalities

In this section we establish some new integral inequalities which can be used as a tool in applications. A useful general version of Lemma 1 may be stated as follows.

Theorem 1—Let $x(t)$, $f(t)$, $g(t)$ and $h(t)$ be real-valued nonnegative continuous functions defined on I, for which the inequality

$$x(t) \leq x_0 + \int_0^t f(s)\, x(s)\, ds + \int_0^t f(s) \left(\int_0^s g(\tau)\, x(\tau)\, d\tau \right) ds$$

$$+ \int_0^t f(s) \left(\int_0^s g(\tau) \left(\int_0^\tau h(k)\, x(k)\, dk \right) d\tau \right) ds \qquad \ldots(1)$$

holds for all $t \in I$, where x_0 is a nonnegative constant. Then

$$x(t) \leq x_0 \left[1 + \int_0^t f(s) \exp\left(\int_0^s f(\tau)\, d\tau \right) \right.$$

$$\left. \times \left[1 + \int_0^s g(\tau) \exp\left(\int_0^\tau [g(k) + h(k)]\, dk \right) d\tau \right] ds \right] \qquad \ldots(2)$$

for all $t \in I$.

Proof: Define a function $m(t)$ by the right member of (1). Then

$$m'(t) = f(t)\, x(t) + f(t) \int_0^t g(\tau)\, x(\tau)\, d\tau + f(t) \int_0^t g(\tau) \left(\int_0^\tau h(k)\, x(k)\, dk \right) d\tau,$$

$m(0) = x_0$, which in view of (1) implies

$$m'(t) \leq f(t)\left[m(t) + \int_0^t g(\tau)m\tau \, d\tau + \int_0^t g(\tau)\left(\int_0^\tau h(k)m(k)\,dk\right) d\tau\right] \qquad \dots(3)$$

Define

$$v(t) = m(t) + \int_0^t g(\tau)m(\tau)\,d\tau + \int_0^t g(\tau)\left(\int_0^\tau h(k)m(k)\,dk\right)d\tau, v(0) = m(0) = x_0 \qquad \dots(4)$$

Then it follows from (3), (4) and the fact that $m(t) \leq v(t)$, the inequality

$$v'(t) \leq f(t)v(t) + g(t)\left[v(t) + \int_0^t h(k)v(k)\,dk\right] \qquad \dots(5)$$

is satisfied. If we define

$$r(t) = v(t) + \int_0^t h(k)\,v(k)\,dk,\ r(0) = v(0) = x_0 \qquad \dots(6)$$

then it follows from (5), (6) and the fact that $v(t) \leq r(t)$, the inequality

$$r'(t) \leq [f(t) + g(t) + h(t)]\,r(t) \qquad \dots(7)$$

is satisfied, which implies the estimation for $r(t)$ such that

$$r(t) \leq x_0 \exp\left[\int_0^t [f(s) + g(s) + h(s)]\,ds\right] \qquad \dots(8)$$

Then from (5), we have

$$v'(t) \leq f(t)\,v(t) + g(t)x_0 \exp\left[\int_0^t [f(s) + g(s) + h(s)]\,ds\right],$$

which implies the estimate for $v(t)$ such that

$$v(t) \leq x_0 \exp\left(\int_0^t f(s)\,ds\right)\left[1 + \int_0^t g(s)\exp\left[\int_0^s (g(\tau) + h(\tau))\,d\tau\right]ds\right]$$

Substituting this value of $v(t)$ in (3) we have

$$m'(t) \leq x_0 f(t) \exp\left(\int_0^t f(s)ds\right)\left[1+\int_0^t g(s)\exp\left[\int_0^s (g(\tau)+h(\tau))\,d\tau\right]ds\right] \quad \ldots(9)$$

Now, integrating both sides of (9) from 0 to t and substituting the value of $m(t)$ in (1) we obtain the desired bound in (2).

We state the following generalization of Lemma 1 which may be convenient in some applications.

Theorem 2—Let $x(t)$, $k(t)$, $p(t)$, $f(t)$, $g(t)$ and $h(t)$ be real-valued nonnegative continuous functions defined on I, for which the inequality

$$x(t) \leq k(t) + p(t)\Bigg[\int_0^t f(s)x(s)\,ds + \int_0^t f(s)p(s)\left(\int_0^s g(\tau)x(\tau)\,d\tau\right)ds$$

$$+\int_0^t f(s)p(s)\left(\int_0^s g(\tau)p(\tau)\left(\int_0^\tau h(n)x(n)\,dn\right)d\tau\right)ds\Bigg] \quad \ldots(10)$$

holds for all $t \in I$. Then

$$x(t) \leq k(t) + p(t)\Bigg[\int_0^t f(s)\Bigg[k(s) + p(s)\Bigg\{\int_0^s \exp\left(\int_\tau^s f(n)\,p(n)\,dn\right).\Big(k(\tau)[f(\tau)+g(\tau)\Big]$$

$$+\,g(\tau)\,p(\tau)\int_0^\tau k(n)[f(n)+g(n)+h(n)]$$

$$\times\exp\left(\int_n^\tau p(\zeta)\;[f(\zeta)+g(\zeta)+h(\zeta)]\,d\zeta\right)dn\Big)d\tau\Bigg\}\Bigg]ds\Bigg] \quad \ldots(11)$$

for all $t \in I$.

By setting $m(t)$ equal to the expression in the brackets [ ] given in (10) and following the similar argument as in the proof of Theorem 1 we obtain the desired bound in (11).

Another interesting and useful generalization of Lemma 1 is embodied in the following theorem.

Theorem 3—Let $x(t), f(t), g(t)$ and $h(t)$ be real-valued nonnegative continuous functions defined on I, for which the inequality

$$x(t) \leq x_0 + \int_0^t f(s)x(s)\,ds + \int_0^t f(s)\left(\int_0^s g(\tau)x(\tau)\,d\tau\right)ds$$

$$+ \int_0^t f(s)\left(\int_0^s g(\tau)\left(\int_0^\tau h(k)x^\alpha(k)\,dk\right)d\tau\right)ds$$

holds for all $t \in I$, where x_0 is a nonnegative constant and $0 \leq \alpha < 1$. Then

$$x(t) \leq x_0 + \int_0^t f(s)\exp\left(\int_0^s f(\tau)\,d\tau\right)\left[x_0 + \int_0^s g(\tau)\exp\left(\int_0^\tau g(k)\,dk\right)\right.$$

$$\left.\left[x_0^{1-\alpha} + (1-\alpha)\int_0^\tau h(k)\exp\left((-1+\alpha)\int_0^k [f(n)+g(n)]dn\right)dk\right]^{\frac{1}{1-\alpha}} d\tau\right]ds$$

for all $t \in I$.

The proof of this theorem follows by the similar argument as in the proof of Theorem 1 with suitable modifications. We omit the details.

The next result presents another useful generalization of Lemma 1. This form is found to be convenient in some applications.

Theorem 4—Let $x(t), f(t), g(t)$ and $h(t)$ be real-valued nonnegative continuous functions defined on I, for which the inequality

$$x(t) \quad x_0 + \int_0^t f(s)x(s)\,ds + \int_0^t f(s)\left(\int_0^s g(\tau)x(\tau)\,d\tau\right)ds$$

$$+ \int_0^t h(s)x(s)\left[x(s) + \int_0^s g(\tau)x(\tau)\,d\tau\right]ds \qquad \ldots(12)$$

holds for all $t \in I$, where x_0 is a positive constant. If

$$1 - x_0 \int_0^t h(s) \exp\left(\int_0^s [f(\tau) + g(\tau)]\, d\tau\right) ds > 0 \text{ for all } t \in I, \text{ then}$$

$$x(t) \leq x_0 \exp\left(\int_0^t h(s)Q(s)\, ds\right) + \int_0^t f(s)Q(s) \exp\left(\int_s^t h(\tau)Q(\tau)\, d\tau\right) ds \qquad \ldots(13)$$

for all $t \in I$, where

$$Q(t) = \frac{x_0 \exp\left(\int_0^t [f(s) + g(s)]ds\right)}{1 - x_0 \int_0^t h(s) \exp\left(\int_0^s [f(\tau) + g(\tau)]d\tau\right) ds}, \; t \in I \qquad (14)$$

PROOF : Define a function $m(t)$ by the right member of (12). Then

$$m'(t) = f(t)x(t) + f(t) \int_0^t g(\tau)x(\tau)\, d\tau + h(t)x(t)\left[x(t) + \int_0^t g(\tau)\, x(\tau)\, d\tau\right], \; m(0) = x_0$$

which in view of (12) implies

$$m'(t) \leq f(t)\left[m(t) + \int_0^t g(\tau)m(\tau)\, d\tau\right] + h(t)m(t)\left[m(t) + \int_0^t g(\tau)m(\tau)\, d\tau\right] \qquad \ldots(15)$$

If we put

$$v(t) = m(t) + \int_0^t g(\tau)m(\tau)\, d\tau, \; v(0) = m(0) = x_0 \qquad \ldots(16)$$

it follows from (15), (16) and the fact that $m(t) \leq v(t)$, the inequality

$$v'(t) \leq h(t)v^2(t) + [f(t) + g(t)]\, v(t) \qquad \ldots(17)$$

is satisfied. The inequality (17) can be written as

$$v^{-2}(t)v'(t) - [f(t) + g(t)]\, v^{-1}(t) \leq h(t) \qquad \ldots(18)$$

Put $v^{-1}(t) = n(t)$, so that $v^{-2}(t)v'(t) = -n'(t)$, and $n(0) = x_0^{-1}$, then we obtain

$$n'(t) + [f(t) + g(t)]\, n(t) \geqslant -h(t) \qquad \text{...(19)}$$

which implies the estimation for $n(t)$ such that

$$n(t) \exp\left(\int_0^t [f(s) + g(s)]ds\right) \geqslant n(0) - \int_0^t h(s) \exp\left(\int_0^s [f(\tau) + g(\tau)]d\tau\right) ds$$

Now, substituting $n(t) = v^{-1}(t)$ in the above inequality, we have

$$v(t) \leqslant \frac{x_0 \exp\left(\int_0^t [f(s) + g(s)]\, ds\right)}{1 - x_0 \int_0^t h(s) \exp\left(\int_0^s [f(\tau) + g(\tau)]\, d\tau\right) ds} = Q(t)$$

since $n(0) = \dfrac{1}{x_0}$. Then from (15) we have

$$m'(t) \leqslant h(t)Q(t)m(t) + f(t)Q(t)$$

which implies the estimate for $m(t)$ such that

$$m(t) \leq x_0 \exp\left(\int_0^t h(s)Q(s)\, ds\right) + \int_0^t f(s)Q(s) \exp\left(\int_s^t h(\tau)Q(\tau)\, d\tau\right) ds$$

Now, substituting the value of $m(t)$ in (12) we obtain the desired bound in (13).

We next establish the following integral inequality which can be used in obtaining the lower bounds on an unknown function.

Theorem 5—Let $u(t)$, $v(t)$, $p(t)$, $f(t)$, $g(t)$ and $h(t)$ be real-valued nonnegative continuous functions defined on $J = [a,b]$; and

$$u(t) \geqslant v(s) - p(t) \left[\int_s^t f(k)v(k)dk + \int_s^t f(k)\left(\int_k^t g(\tau)v(\tau)d\tau\right)dk\right.$$

$$\left. + \int_s^t f(k)\left(\int_k^t g(\tau)\left(\int_\tau^t h(\zeta)\, v(\zeta)\, d\zeta\, d\tau\right)dk\right] \qquad \text{...(20)}$$

for $a \leq s \leq t \leq b$, then

$$u(t) \geqslant v(s)\Big[1+p(t)\int_s^t f(\tau)\,\exp\Big(\int_\tau^t p(k)f(k)\,dk\Big)$$

$$\times\Big[1+\int_\tau^t g(k)\exp\Big(\int_k^t [h(\zeta)+g(\zeta)]d\zeta\Big)dk\Big]\,d\tau\,\Big]^{-1} \qquad \ldots(21)$$

for $a \leq s \leq t \leq b$.

PROOF : For fixed t in the interval I, we define for $a \leq s \leq t$

$$m(s) = u(t) + p(t)\Big[\int_s^t f(k)v(k)dk + \int_s^t f(k)\Big(\int_k^t g(\tau)v(\tau)\,d\tau\Big)dk$$

$$+\int_s^t f(k)\Big(\int_k^t g(\tau)\Big(\int_\tau^t h(\zeta)\;v(\zeta))\,d\zeta\Big)d\tau\Big)dk\Big],\; m(t) = u(t) \qquad \ldots(22)$$

From (22) we have

$$m'(s) = -p(t)\Big[f(s)v(s) + f(s)\int_s^t g(\tau)v(\tau)d\tau + f(s)\int_s^t g(\tau)\Big(\int_\tau^t h(\zeta)\;v(\zeta)\;d\zeta\Big)d\tau\Big]$$

which in view of $v(s) \leq m(s)$ implies

$$m'(s) \geqslant -p(t)f(s)\Big[m(s) + \int_s^t g(\tau)m(\tau)d\tau + \int_s^t g(\tau)\Big(\int_\tau^t h(\zeta)\;m(\zeta)\,d\zeta\Big)d\tau\Big] \qquad \ldots(23)$$

Define

$$n(s) = m(s) + \int_s^t g(\tau)m(\tau)\,d\tau + \int_s^t g(\tau)\Big(\int_\tau^t h(\zeta)\;m(\zeta)\,d\zeta\Big)d\tau,\; n(t) = m(t) = u(t). \quad \ldots(24)$$

Then it follows from (23), (24) and the fact that $m(s) \leq n(s)$, the inequality

$$n'(s) + g(s)\left[n(s) + \int_s^t h(\tau)n(\tau)\,d\tau\right] + p(t)f(s)n(s) \geqslant 0 \qquad \ldots(25)$$

is satisfied. If we define

$$r(s) = n(s) + \int_s^t h(\tau)n(\tau)\,d\tau,\ r(t) = n(t) = u(t) \qquad ..(26)$$

then it follows from (25), (26) and the fact that $n(s) \leq r(s)$, the inequality

$$r'(s) + [h(s) + g(s) + p(t)f(s)]\, r(s) \geqslant 0$$

is satisfied, which implies the estimation for $r(s)$ such that

$$r(s) \leq u(t)\ \exp\left[\int_s^t (h(\tau) + g(\tau) + p(t)f(\tau))d\tau\right]$$

Then, from (25), we have

$$n'(s) + p(t)f(s)n(s) \geqslant g-(s)u(t)\exp\left[\int_s^t \left(h(\tau) + g(\tau) + p(t)f(\tau)\right)d\tau\right]$$

which implies the estimation for $n(s)$ such that

$$n(s) \leqslant u(t)\exp\left(\int_s^t p(t)f(\tau)\,d\tau\right)\left[1 + \int_s^t g(\tau)\ \exp\left(\int_\tau^t (h(k)+g(k))\,dk\right)d\tau\right]$$

Substituting this value of $n(s)$ in (23) we have

$$m'(s) \geq -p(t)f(s)u(t)\exp\left(\int_s^t p(t)f(\tau)\,d\tau\right)$$

$$\times\left[1 + \int_s^t g(\tau)\exp\left(\int_\tau^t (h(k) + g(k))\,dk\right)d\tau\right] \qquad \ldots(27)$$

Now, integrating both sides of (27) from s to t and substituting the value of $m(s)$ in (20) we obtain the desired bound in (21).

As an application of Theorem 5 we establish the following interesting and useful integral inequality.

Theorem 6—Let $u(t)$, $v(t)$, $p(t)$, $f(t)$, $g(t)$ and $h(t)$ be real-valued nonnegative continuous functions defined on J; $G(r)$ be a continuous, strictly increasing, convex and submultiplicative function for $r \geqslant 0$; $G(0) = v$, $\lim_{r \to \infty} G(r) = \infty$; $\alpha(t)$, $\beta(t)$ be continuous functions on J; $\alpha(t)$, $\beta(t) > 0$, $\alpha(t) + \beta(t) = 1$; and

$$u(t) \geqslant v(s) - p(t)\, G^{-1}\left[\int_s^t f(k)G(v(k))\,dk + \int_s^t f(k)\left(\int_k^t g(\tau)G(v(\tau))\,d\tau\right)dk + \int_s^t f(k)\left(\int_k^t g(\tau)\left(\int_\tau^t h(\zeta)\,G(v(\zeta))\,d\zeta\right)d\tau\right)dk\right] \quad \ldots(28)$$

for $a \leqslant s \leqslant t \leqslant b$. Then

$$u(t) \geqslant \alpha(t)G^{-1}\Bigg(\alpha^{-1}(t)G(v(s))\Bigg[1 + \beta(t)G(p(t)\,\beta^{-1}(t))\int_s^t f(\tau) \times \exp\left(\int_\tau^t \beta(t)G(p(t)\beta^{-1}(t))\,f(k)\,dk\right)\cdot\left[1 + \int_\tau^t g(k)\exp\left(\int_k^t [g(\zeta) + h(\zeta)]\,d\zeta\right)dk\right]d\tau\Bigg]^{-1}\Bigg) \quad \ldots(29)$$

for $a \leqslant s \leqslant t \leqslant b$.

PROOF: Rewrite (28) as

$$v(s) \leqslant \alpha(t)(u(t)\,\alpha^{-1}(t)) + \beta(t)(p(t)\beta^{-1}(t))G^{-1}\left[\int_s^t f(k)G(v(k))\,dk + \right.$$

(equation continued next page)

$$+\int_s^t f(k)\left(\int_k^t g(\tau)G(v(\tau))d\tau\right)dk+\int_s^t f(k)\left(\int_k^t g(\tau)\left(\int_\tau^t h(\zeta)\,G(v(\zeta))\,d\zeta\right)d\tau\right)dk\Bigg]$$

for $a\leqslant s\leqslant t\leqslant b$. Since G is convex, submultiplicative and monotonic, we have

$$\alpha(t)G(u(t)\,\alpha^{-1}(t))\geq G(v(s))-\beta(t)G(p(t)\beta^{-1}(t))\Bigg[\int_s^t f(k)G(v(k))dk$$

$$+\int_s^t f(k)\left(\int_k^t g(\tau)G(v(\tau))d\tau\right)dk+\int_s^t f(k)\left(\int_k^t g(\tau)\left(\int_\tau^t h(\zeta)\,G(v(\zeta))\,d\zeta\right)d\tau\right)dk\Bigg]$$

Now, an application of Theorem 5 yields the desired bound in (29).

Gollwitzer (1969) and Langenhop (1960) have obtained lower bounds on unknown functions. However, the bounds obtained here are different from those obtained by the above authors.

We note that the integral inequalities similar to those given by Pachpatte (1975b, §2) can be established by following the similar arguments as given by Pachpatte (1975b). Since this translation is quite straightforward in view of our Theorem 1, we leave it for the reader to fill in where needed.

3. Applications to Integrodifferential equations

Recently, Imanaliev and Ved′ (1972) and Karpievic (1972) have studied the stability of the solutions of a certain class of integrodifferential equations. The problem considered in this section is in the general spirit of the investigations in Imanaliev and Ved′ (1972) and Karpievic (1972). We are here concerned with the boundedness, asymptotic behaviour, and the rate of growth of the solutions of integrodifferential system of the form

$$x'(t)=F(t)+A(t)\,x(t)+\int_{t_0}^t B(t,\,s)x(s)ds+H\left(t,\,x(t),\int_{t_0}^t K(t,s,x(s))ds\right)$$

$$+\,x(t)W\left(t,\,x(t),\int_{t_0}^t K(t,s,x(s))ds\right),\quad x(t_0)=y_0 \qquad \ldots(30)$$

as a perturbation of the nonlinear integrodifferential system

$$y'(t) = F(t) + A(t)\, y(t) + \int_{t_0}^{t} B(t,s)\, y(s)\, ds, \;\; y(t_0) = y_0 \qquad \ldots(31)$$

The main tools in our analysis are the variation of constants formula developed by Grossman and Miller (1970) and the integral inequalities established in section 2. Here x, y, F, H, K and W are the elements of R^n, a real n-dimensional Euclidean space, and $y_0 \neq 0$. We assume that $F \in C[R^+, R^n]$, $K \in C[R^+ \times R^+ \times R^n, R^n]$, and $H, W \in C[R^+ \times R^n \times R^n, R^n]$. The symbol $|\,..\,|$ will denote some convenient norm R^n as well as a corresponding consistent matrix norm. We denote by $x(t) = x(t, t_0, y_0)$ the solution of (30) through the initial point (t_0, y_0) and $y(t) = y(t, t_0, y_0)$ the solution of (31) through the initial point (t_0, y_0) for $t_0 \geqslant 0$.

For p in the interval $1 \leq p < \infty$, L^p is the usual Lebesgue space of measurable function g such that $\|g\|_p = \left(\int_0^\infty |g(t)|\, dt \right)^{\frac{1}{p}} < \infty$ is the set of all functions which are locally L^p on R^+. Consider the system (31) with $y(f) \in R^n$, $A(f)$ is an n by n matrix belonging to $LL^1(R^t)$, $B(t, s)$ is an n by n matrix that is locally integrable in both variables. It is known (Grossman and Miller 1970) that the unique solution $y(f)$ of (31) is represented by

$$y(f) = R(t, t_0)\, y_0 + \int_{t_0}^{t} R(t, s)\; F(s)\, ds \qquad \ldots(32)$$

where $R(t, s)$ is an n by n matrix that is continuous in (t, s) and satisfies

$\frac{\partial R(t, s)}{\partial s} = -\, R(t, s)\, A(s) - \int_s^t R(t, u) B(u, s)\; du,\; R(t, t) = I_0,$ on the interval $0 \leq s \leq t$, where I_0 denotes the identity matrix. Here $R(t, s)$ is called the resolvent kernel of equation (30).

Then solutions of (30) and (31) with the same initial values are related by

$$x(t) = y(t) + \int_{t_0}^{t} R(t, s)\, H\!\left(s, x(s), \int_{t_0}^{s} K(s, \tau, x(\tau))\, d\tau \right) ds \; +$$

(equation continued next page)

$$+\int_{t_0}^{t} R(t, s)\, x(s)\, W\left(s,\ x(s),\ \int_{t_0}^{s} K(s, \tau, x(\tau))\, d\tau \right) ds \qquad ...(33)$$

(see Grossman and Miller (1970) for details).

In this section we are interested in the following stability definitions in terms of the behaviour of solutions of (31) and its resolvent kernel which are the natural extensions of the concepts recently introduced by this author (Pachpatte 1975d) for nonlinear differential equations.

Definition 1—The solution $y(t)$ of (31) is said to be globally uniformly stable relative to its resolvent kernel if there exists a constant $M > 0$ such that

$$|y(t, t_0, y_0)| \leqslant M |y_0|$$

and

$$|R(t, s)| \leqslant M$$

for all $0 \leqslant t_0 \leqslant s \leqslant t < \infty$ and $|y_0| < \infty$.

Definition 2 — The solution $y(t)$ of (31) is said to be exponentially asymptotically stable relative to its resolvent kernel if there exist constants $M > 0$, $\alpha > 0$ such that

$$|y(t, t_0, y_0)| \leqslant M |y_0| \exp (-\alpha(t-t_0)$$

and

$$|R(t, s)|\ M \exp \leqslant (-\alpha(t-s)$$

for all $0 \leqslant t_0 \leqslant s \leqslant t < \infty$ and $|y_0|$ sufficiently small.

Definition 3— The solution $y(t)$ of (31) is said to be uniformly slowly growing relative to its resolvent kernel if, and only if, for every $\alpha > 0$ there exists a constant $M > 0$, possibly depending on α, such that

$$|y(t, t_0, y_0)| \leqslant M |y_0| \ \exp \alpha(-(t-t_0)$$

and

$$|R(t, s)| \leqslant M \exp \alpha(t-s)$$

for all $0 \leqslant t_0 \leqslant s \leqslant t < \infty$ and $|y_0| < \infty$.

We note (Pachpatte 1975d) that a continuous function $z(t)$ is slowly growing if and only if for every $\alpha > 0$ there exists a constant M, which may depend on α such that

$$| z(t) | \leqslant M e^{\alpha t}_{\alpha}, \quad t \leqslant 0.$$

Our first theorem in this section deals with the boundedness of the solutions of (30) under some suitable conditions on the perturbation terms and on the solutions of (32).

Theorem 7—Let the solution $y(t)$ of (31) be globally uniformly stable relative to its resolvent kernel. Suppose that the functions H, W and K in (30) satisfy

$$| H(t, x, z) | \leqslant f(t)[| x | + | z |], t \in R^+ \qquad \text{...(34)}$$

$$| W(t, x, z) | \leqslant h(t)[| x | + | z |], t \in R^+ \qquad \text{...(35)}$$

$$| K(t, s, x) | \leqslant g(s) | x |, \; 0 \leqslant s \leqslant t < \infty \qquad \text{...(36)}$$

Here $f, g, h \in C[R^+, R^+]$ and

$$1 - M | y_0 | \int_{t_0}^{t} Mh(s) \exp \left(\int_{t_0}^{s} [Mf(\tau) + g(\tau)] \, d\tau \right) ds > 0$$

for all $t \in R^+$, such that

$$\int_{t_0}^{\infty} f(s) Q_0(s) \, ds < \infty, \int_{t_0}^{\infty} h(s) Q_0(s) \, ds < \infty \qquad \text{...(37)}$$

where

$$Q_0(t) = \frac{M | y_0 | \exp \left(\int_{t_0}^{t} [Mf(s) + g(s)] \, ds \right)}{1 - M | y_0 | \int_{t_0}^{t} M h(s) \exp \left(\int_{t_0}^{s} [Mf(\tau) + g(\tau)] \, d\tau \right) ds} \qquad \text{...(38)}$$

and $M > 0$, $y_0 \neq 0$ are constants. Then all solutions of (30) are bounded on R^+.

PROOF: Using the variation of constants formula developed by Grossman and Miller (1970) the solutions of (30) and (31) with the same initial value are related by

$$x(t) = y(t) + \int_{t_0}^{t} R(t, s) H\left(s, x(s), \int_{t_0}^{s} K(s, \tau, x(\tau))\, d\tau\right) ds$$

$$+ \int_{t_0}^{t} R(t, s)\, x(s) W\left(s, x(s), \int_{t_0}^{s} K(s, \tau, x(\tau))\, d\tau\right) ds \qquad \dots (39)$$

Using (39), (34), (35), (36) together with the global uniform stability of the solution $y(t)$ of (31) relative to its resolvent kernel we obtain

$$| x(t) \leqslant | M | y_0 | + \int_{t_0}^{t} M f(s) | x(s) | ds + \int_{t_0}^{t} M f(s) \left(\int_{t_0}^{s} g(\tau) | x(\tau) | d\tau \right) ds$$

$$+ \int_{t_0}^{t} M h(s) | x(s) | \left[| x(s) | + \int_{t_0}^{s} g(\tau) | x(\tau) | d\tau \right] ds$$

Now an application of Theorem 4 yields

$$| x(t) | \leqslant M | y_0 | \exp\left(\int_{t_0}^{t} M h(s) Q_0(s)\, ds \right)$$

$$+ \int_{t_0}^{t} M f(s) Q_0(s) \exp\left(\int_{s}^{t} M\, h(\tau) Q_0(\tau)\, d\tau \right) ds$$

where $Q_0(t)$ is as defined in (38). The above estimation in view of the assumption (37) implies the boundedness of all solutions of (30) on R^+, and the theorem is proved.

Our next theorem shows that under some suitable conditions on the perturbation terms in (30), the exponential asymptotic stability of (31) relative to the resolvent kernel implies that all the solutions of (30) approach zero as $t \to \infty$.

Theorem 8—Let the solution $y(t)$ of (31) be exponentially asymptotically stable relative to its resolvent kernel. Suppose that the functions H and W in (30) satisfy the hypotheses (34) and (35) of Theorem 7 and let the function K satisfy

$$| K(t, s, x) | \leqslant e^{-\alpha t}\, g(s) | x |, \quad 0 \leqslant s \leqslant t < \infty \qquad \dots (40)$$

Here $\alpha > 0$ is a constant, and $f, g, h \in C[R^+, R^+]$ and

$$1 - M \mid y_0 \mid \exp(\alpha t_0) \int_{t_0}^{t} Mh(s) \exp(-\alpha s) \exp\left(\int_{t_0}^{s} [Mf(\tau) + g(\tau) \exp(-\alpha\tau)]\, d\tau \right) ds > 0, \text{ for all } t \in R^+ \text{ such that}$$

$$\int_{t_0}^{\infty} f(s) Q_1(s)\, ds < \infty, \int_{t_0}^{\infty} h(s) \exp(-\alpha s) Q_1(s)\, ds < \infty \qquad \ldots(41)$$

where $Q_1(t) =$

$$\frac{M \mid y_0 \mid \exp(\alpha t_0) \exp\left(\int_{t_0}^{t} [Mf(s) + g(s) e^{-\alpha s}]\, ds \right)}{1 - M \mid y_0 \mid \exp(-\alpha t_0) \int_{t_0}^{t} Mh(s) \exp(-\alpha s) \exp\left(\int_{t_0}^{s} [Mf(\tau) + g(\tau) \exp(-\alpha\tau)]\, d\tau \right) ds} \qquad \ldots(42)$$

and $M > 0$, $y_0 \neq 0$ are constants. Then all solutions of (30) approach zero as $t \to \infty$.

Proof : It is known that the solutions of (30) and (31) with the same initial values are related by the integral equation (39). Using (39), (34), (35), (40) together with exponential asymptotic stability of the solution $y(t)$ of (31) relative to its resolvent kernel, we obtain

$$|x(t)| \leq M \mid y_0 \mid \exp(-\alpha(t - t_0)) + \int_{t_0}^{t} M \exp(-\alpha(t-s)) f(s) \Big[\mid x(s) \mid + \exp(-\alpha s) \int_{t_0}^{s} g(\tau) \mid x(\tau) \mid d\tau \Big] ds$$

$$+ \int_{t_0}^{t} M \exp(-\alpha(t-s)) h(s) \mid x(s) \mid \Big[\mid x(s) \mid + \exp(-\alpha s) \int_{t_0}^{s} g(\tau) \mid x(\tau) \mid d\tau \Big] ds$$

The above inequality can be written as

$$| x(t) | \exp(\alpha t) \leqslant M | y_0 | \exp(\alpha t_0) + \int_{t_0}^{t} M f(s) | x(s) | \exp(\alpha s)\, ds$$

$$+ \int_{t_0}^{t} M f(s) \left(\int_{t_0}^{s} g(\tau) \exp(-\alpha\tau) | x(\tau) | \exp(\alpha\tau)\, d\tau \right) ds$$

$$+ \int_{t_0}^{t} M h(s) \exp(-\alpha s) | x(s) | \exp(\alpha s) \Big[|x(s)| \exp(\alpha s)$$

$$+ \int_{t_0}^{s} g(\tau) \exp(-\alpha\tau) | x(\tau) | \exp(\alpha\tau)\, d\tau \Big] ds$$

Now applying Theorem 4 with $x(t) = | x(t) | \exp(\alpha t)$ then multiplying by $\exp(-\alpha t)$ we obtain

$$| x(t) | \leqslant My_0 \exp(-\alpha(t-t_0)) \exp\left(\int_{t_0}^{t} M \exp(-\alpha s) Q_1(s) ds \right)$$

$$+ \exp(-\alpha t) \int_{t_0}^{t} M f(s) Q_1(s) \exp\left(\int_{s}^{t} M h(\tau) \exp(-\alpha\tau) Q_1(\tau)\, d\tau \right) ds$$

The above estimation in view of assumption (41) yields the desired result if we choose M and $| x_0 |$ small enough, and the proof of the theorem is complete.

To this end, we establish the following theorem which demonstrates that all the solutions of (30) grow more slowly than any positive exponential.

Theorem 9—Let the solution $y(t)$ of (31) be uniformly slowly growing relative to its resolvent kernel. Suppose that the functions H and W in (30) satisfy the hypotheses (34) and (35) of Theorem 7 and let the function K satisfy

$$| K(t,s,x) | \leqslant \exp(\alpha t)\, g(s)\, | x |, \quad 0 \leqslant s \leqslant t > \infty$$

Here $\alpha > 0$ is a constant, and $f, g, h \in C[R^+, R^+]$ and

$$1-M|y_0| \exp(-\alpha t_0)\int_{t_0}^{t} Mh(s)\exp(\alpha s)\exp\left(\int_{t_0}^{s}[Mf(\tau)+g(\tau)\exp(\alpha\tau)]\,d\tau\right)ds>0$$

for all $t \in R^+$ such that

$$\int_{t_0}^{\infty} f(s)Q_2(s)ds < \infty, \int_{t_0}^{\infty} h(s)\exp(\alpha s)Q_2(s)ds < \infty,$$

where

$$Q_2(t)=\frac{M|y_0|\exp(-\alpha t_0)\exp\left(\int_{t_0}^{t}[Mf(s)+g(s)\exp(\alpha s)]ds\right)}{1-M|y_0|\exp(-\alpha s)\int_{t_0}^{s} Mh(s)\exp(\alpha s)\exp\left(\int_{t_0}^{s}[Mf(\tau)+g(\tau)\exp(\alpha\tau)]d\tau\right)ds}$$

and $M>0$, $y_0 \neq 0$ are constants. Then all solutions of (30) are slowly growing.

The proof of this theorem follows by the similar argument as in the proof of Theorem 8 with suitable modifications, and hence we omit the details.

Finally, we note that the results obtained in this section can be modified very easily to cover the case when the perturbation terms in (30) are of the forms

$$H\left((t,x(t)),\int_{t_0}^{t} K\left((t,s,x(s)),\int_{t_0}^{s} a(s,\tau,x(\tau))d\tau\right)ds\right), W \equiv 0$$

and

$$H\left((t,x(t)),\int_{t_0}^{t} K\left(t,s,x(s),\int_{t_0}^{s} a(s,\tau,x^{\alpha}(\tau))d\tau\right)ds\right), W \equiv 0$$

by using the inequalities established in Theorems 1 and 3 respectively, under some suitable conditions on the functions involved. These theorems will not be given here since there are no new essential ideas to explain.

References

Grossman, S. I., and Mill, R. K. (1970). Perturbation theory for Volterra integrodifferential systems, *J. Diff. eqns.*, **8**, 457–74.

Coddington, E. A. and Levinson, N. (1955). Theory of Ordinary Differential Equations. McGraw-Hill Book Co., Inc., New York.

Gollwitzer, H. E. (1969). A note on a functional inequality. *Proc. Am. math. S. oc.*, **43**, 642-47.

Gronwall, T. H. (1919). Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. *Ann. Math.*, **20**, 292-96.

Imanaliev, M. I. and Ved, Ju. A. (1972). Integral perturbations in the theory of the stability of systems of differential equations (Russian). *Izv. Akad. Nauk. Kirgiz. SSR.*, **5**, 23-30.

Karpievic, N. A. (1972). The stability of a certain class of integrodifferential equations (Russian). *Vestnik Beloruss. Gos. Univ. Ser.* **I**, **94**, 12-15.

Langenhop, C. E. (1960). Bounds on the norm of a solution of a general differential equation. *Proc. Am. math. Soc.*, **11**, 795-99.

Pachpatte, B. G. (1973). A note on Gronwall-Bellman inequality. *J. Math. Analysis Applic.*, **44**, 758-62.

———(1975a). A note on integral inequalities of the Bellman-Bihari type. *J. Math. Analysis Applic.*, **49**, 295-301.

———(1975b). On some integral inequalities similar to Bellman-Bihari inequalities. *J. Math. Analysis Applic.*, **49**, 794-802.

———(1975c). A note on an integral inequality. *J. Math. Physical. Sci.*, **9**, 11-14.

———(1975d). Perturbations of nonlinear systems of differential equations. *J. Math. Analysis Applic.*, **51**.