SOME THEOREMS ON AFFINE MOTION IN A RECURRENT FINSLER SPACE

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In the present paper an infinitesimal affine motion in a recurrent Finsler space is considered. In such space many important results have been obtained.

1. INTRODUCTION

Let us consider an n-dimensional Finsler space Fn (Rund 1959) having positively homogeneous metric function $F(x, \dot{x})$ of degree one in $\dot{x}^i$, s. The fundamental metric tensor of the space is defined by

$$g_{ij}(x, \dot{x}) = \tfrac{1}{2}\dot{\partial}_j \dot{\partial}_i F^2(x, \dot{x}), \quad (\dot{\partial}_i \equiv \partial / \partial \dot{x}^i) \qquad \text{...(1.1)}$$

which is symmetric in its lower indices.

Let $X^i(x, \dot{x})$ be any contravariant vector field depending both upon positional and directional arguments. Cartan's covariant derivative of $X^i(x, \dot{x})$ with respect to x^k is given by (Rund 1959)

$$X^i_{|k} = \partial_k X^i - (\dot{\partial}_m X^i)\, G^m_k + X^h \Gamma^{*i}_{hk} \qquad \text{...(1.2)}$$

where $\Gamma^{*i}_{hk}(x, \dot{x})$ are connection coefficients.

The curvature tensor $K^i_{hjk}(x, \dot{x})$ arising from this covariant is given by

$$K^i_{hjk}(x, \dot{x}) = 2\left\{ \partial_{[k}\Gamma^{*i}_{j]h} - \dot{\partial}_r \Gamma^{*i}_{h[j} G^r_{k]} + \Gamma^{*r}_{h[j} \Gamma^{*i}_{k]r} \right\} \qquad \text{...(1.3)}$$

Let us consider an infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)\, dt \qquad \text{...(1.4)}$$

where $v^i(x)$ is any vector field and dt an infinitesimal constant. In view of the covariant derivative and (1.4) the Lie-derivatives of any tensor field

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$2A_{[hk]} = A_{hk} - A_{kh}$ and $2A_{(hk)} = A_{hk} + A_{kh}$.

$\underset{j}{\dot{T}}(x, \dot{x})$ and connection coefficient $\Gamma^{*i}_{jk}(x, \dot{x})$ are given by (Yano 1957)

$$Lv\, T^i_j(x, \dot{x}) = T^i_{j|h} v^h + (\dot{\partial}_s T^i_j) v^s_{|r} \dot{x}^r + T^i_h v^h_{|j} - T^h_j v^i_{|h} \quad \text{and} \qquad ...(1.5)$$

$$Lv\, \Gamma^{*i}_{jk}(x, \dot{x}) = v^i_{|jk} + K^i_{jkh} v^h + (\dot{\partial}_s \Gamma^{*i}_{jk}) v^s_{|r} \dot{x}^r \qquad ...(1.6)$$

We have the following commutation formulae:

$$\dot{\partial}_l (Lv\, T^i_j) - Lv\, \dot{\partial}_l T^i_j = 0 \quad \text{and} \qquad ...(1.7)$$

$$(Lv\, \Gamma^{*i}_{jh})_{|k} - (Lv\, \Gamma^{*i}_{kh})_{|j} = Lv\, K^i_{hjk} + 2\dot{x}^s \dot{\partial}_r \Gamma^{*i}_{h[j} Lv\, \Gamma^{*r}_{k]s} \qquad ...(1.8)$$

If in a Finsler space Fn the curvature tensor $K^i_{hjk}(x, \dot{x})$ satisfies the relation

$$K^i_{hjk|m} = \beta_m K^i_{hjk} \qquad ...(1.9)$$

where $\beta_m(x)$ is any non-zero vector then such type of Fn is called recurrent Finsler space. We shall denote it by Fn^* throughout this paper.

2. Affine Motion in Fn^*

With the help of the infinitesimal point transformation (1.4) we can obtain a deformed space with affine connection $\Gamma^{*i}_{jk} + (Lv\, \Gamma^{*i}_{jk})\, dt$.

If the original space and deformed space have the same affine connection, the transformation is called affine motion of the space Fn. In our Fn^* space we shall consider such type of affine motion. In order that it be the case, it is necessary and sufficient that we have

$$Lv\, \Gamma^{*i}_{jk} = v^i_{|jk} + K^i_{jkh} v^h + (\dot{\partial}_s \Gamma^{*i}_{jk}) v^s_{|r} \dot{x}^r = 0 \qquad ...(2.1)$$

In view of (2.1), eqn. (1.8) reduces to

$$Lv\, K^i_{jkh} = 0. \qquad ...(2.2)$$

With respect to the Lie-derivative, for any tensor T^i_{jk}, we have

$$(Lv T^i_{jk|l}) - (Lv T^i_{jk})_{|l} = T^i_{sk} Lv\Gamma^{*s}_{jl} + T^i_{js} Lv\, \Gamma^{*s}_{kl} - T^s_{jk} Lv\Gamma^{*i}_{sl}$$

$$+ (\dot{\partial}_s T^i_{jk})\, Lv\Gamma^{*s}_{rl} \dot{x}^r \qquad ...(2.3)$$

Apply this commutation formula to the curvature tensor $K^i_{hjk}(x, \dot{x})$ and noting the equations (2.1) and (2.2), we get

$$Lv(K^i_{hjk|l}) = 0 \qquad \text{...(2.4)}$$

Continuing this process, we can see that the Lie-derivatives of curvature tensors and their successive derivatives must be all zero. This fact suggest us the

Lemma 2.1—In order that an affinely connected space admit a group of motion G_1 of affine motions, it is necessary and sufficient that there exists a positive number K such that the first K sets of the equations (2.2), (2.4)...... be compatible invariables v^i and $v^i_{\backslash j}$ and all such solutions satisfy the $(K+1)$ set of the equations.

The above Lemma is due to Knebelman (1945) which gives us a certification of the existence of affine motion. If we consider an affine motion in Fn^* then we should have (2.1) and (2.2). In view of the equations (1.9) and (2.2), we get the following set of relations one after another:

$$\begin{cases} Lv\ K^i_{hjk} = 0 \\ Lv\ (K^i_{hjk\backslash l1}) = K^i_{hjk}\ Lv\ \beta_{l1} \\ Lv\ (K^i_{hjk\,|l1|\,l2}) = K^i_{hjk}[\beta_{l2} Lv \beta_{l1} + (Lv\beta_{l1})_{\backslash l2}] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \end{cases} \qquad (2.5)$$

where we have used the commutation formula (1.7).

3. Discussions

If an Fn^* admits an infinitesimal affine motion, we get (Takano 1961)

$$\beta_h v^h = 0 \qquad \text{...(3.1)}$$

Consequently, from (1.9) and this fact, we can see

$$K^i_{hjk\,|m}\ v^m = 0 \qquad \text{...(3.2)}$$

In view of eqns. (1.9), (2.2) and (3.2), the Lie-derivative of the curvature tensor $K^i_{hjk}(x, \dot{x})$ reduces to

$$(\dot{\partial}_s\ K^i_{hjk})\ v^s{}_{|r}\dot{x}^r + K^i_{sjk}\ v^s{}_{\backslash h} + K^i_{hsk}\ v^s{}_{\backslash j} + K^i_{hjs}v^s{}_{jk} - K^s_{hjk}v^i{}_{\backslash s} = 0 \qquad \text{...(3.3)}$$

Hence, if $v^i{}_{|j} = c\,\delta^i_j$, $c = c(x) \neq 0$, then $K^i_{hjk} = 0$, i.e. Fn^* becomes a flat space Bn. Thus, we have the following:

Theorem 3.1 — If an Fn^* admits an affine motion of the form $\overline{x^i} = x^i + v^i(x)\ dt$, where v^i spans a vector field defined by $v^i{}_{\backslash j} = c\delta^i_j$, this space becomes a flat one, say an arbitrary affine space Bn.

Considering a contraposition of the above Theorem, we get

Theorem 3.2 — If a non-flat Fn^* admits an affine motion of the form

$$\overline{x^i} = x^i + v^i(x)\ dt,\ v^i{}_{\backslash j} = c\delta^i_j$$

$c(x)$ has to vanish, say v^i has to span a field of parallel contravariant vectors (a contrafield).

4. General Case

In view of (1.5), the Lie-derivative of the curvature tensor field $K^i_{hjk}(x, \dot{x})$ is given by

$$Lv\, K^i_{hjk}(x, \dot{x}) = K^i_{hjk|s}\, v^s + (\dot{\partial}_s K^i_{hjk}) v^s_{|r}\, \dot{x}^r + K^i_{sjk}\, v^s_{|h} + K^i_{hsk}\, v^s_{|j} + K^i_{hjs}\, v^s_{|k} - K^s_{hjk}\, v^i_{|s} \quad ...(4.1)$$

Introducing (1.9) and (2.2) into the above equation, we get

$$\beta_s v^s K^i_{hjk} + (\dot{\partial}_s K^i_{hjk})\, v^s_{|r}\, \dot{x}^r + K^i_{sjk}\, v^s_{|h} + K^i_{hsk}\, v^s_{|j} + K^i_{hjs}\, v^s_{|h} - K^s_{hjk}\, v^i_{|s} = 0 \quad ...(4.2)$$

In Fn^*, let us consider an affine motion of the following form

$$\overline{x^i} = x^i + v^i(x)\, dt, \quad v^i_{|j} = c\delta^i_j \quad ...(4.3)$$

where $c(x) \neq 0$.

In view of (4.3), eqn. (4.2) yields

$$(v^s\beta_s + 2c)\, K^i_{hjk} = 0 \quad ...(4.4)$$

Since $K^i_{hjk} \neq 0$, from this, we get

$$c = -\tfrac{1}{2} v^s \beta_s \quad ...(4.5)$$

Hence the motion (4.3) takes the following form

$$\overline{x^i} = x^i + v^i(x)dt, \quad v^i_{|j} = -\tfrac{1}{2} v^s\, \beta_s\, \delta^i_j \quad ...(4.6)$$

Substituting the latter part of (4.6) into (2.1), we obtain

$$K^i_{jkh}\, v^h = \tfrac{1}{2}\{2\beta_{m|k} - \beta_m \beta_k\}\, v^m \delta^i_j \quad ...(4.7)$$

We have the following identities (Rund 1959):

$$K^i_{hjk|m} + K^i_{hkm|j} + K^i_{hmj|k} = -\dot{x}^s\, \{(\dot{\partial}_l \Gamma^{*i}_{hj})\, K^l_{skm} + (\dot{\partial}_l \Gamma^{*i}_{hm}) K^l_{sjk} + (\dot{\partial}_l\, \Gamma^{*i}_{hk})\, K^l_{smj}\} \quad ...(4.8)$$

and

$$K^i_{hjk} + K^i_{jkh} + K^i_{khj} = 0 \quad ...(4.9)$$

Introducing (1.9) into (4.8), we get

$$\beta_m K^i_{hjk} + \beta_j K^i_{hkm} + \beta_k K^i_{hmj} = -\dot{x}^s\, \{(\dot{\partial}_l \Gamma^{*i}_{hj})\, K^l_{skm} + (\dot{\partial}_l \Gamma^{*i}_{hm}) K^l_{sjk} + (\dot{\partial}_l \Gamma^{*i}_{hk})\, K^l_{smj}\} \quad ...(4.10)$$

Multiplying this identity by v^m and summing up with respect to m, we obtain

$$K^i_{hjk}\,\beta_m v^m = K^i_{hjm} v^m \beta_k - \beta_j K^i_{hkm}\, v^m - \dot{x}^s\ \{ (\dot{\mathfrak{D}}_l \Gamma^{*i}_{hj})\, K^l_{skm}\, v^m$$

$$+ (\dot{\mathfrak{D}}_l \Gamma^{*i}_{hm})\, K^l_{sjk} v^m - (\dot{\mathfrak{D}}_m \Gamma^{*i}_{hk})K^l_{sjm} v^m \} \qquad \ldots(4.11)$$

where we have used $K^i_{hjk} = -K^i_{hkj}$

Introducing (4.7) into the right-hand side of (4.11), we get

$$K^i_{hjk}\beta_m v^m = \tfrac{1}{2}\{ \beta_{r|j}\beta_k - \beta_{r|k}\beta_j \} v^r \delta^i_h - \dot{x}^s (\dot{\mathfrak{D}}_l \Gamma^{*i}_{hm})\, K^l_{sjk}\, v^m \qquad \ldots(4.12)$$

Multiplying the identity (4.9) by $\beta_m v^m$, we have

$$K^i_{hjk}\,\beta_m v^m + K^i_{jkh}\,\beta_m v^m + K^i_{khj}\,\beta_m v^m = 0 \qquad \ldots(4.13)$$

Substituting (4.12) into the above identity, we obtain

$$\tfrac{1}{2} v^r [\delta^i_h\, (\beta_k \beta_{r|j} - \beta_{r|k}\beta_j) + \delta^i_j\, (\beta_{r|k}\beta_h - \beta_{r|h}\beta_k) + \delta^i_k\, (\beta_{r|h}\beta_j - \beta_{r|j}\beta_h)]$$

$$-\dot{x}^s v^m\ \{ (\dot{\mathfrak{D}}_l \Gamma^{*i}_{hm})\, K^l_{sjk} + (\dot{\mathfrak{D}}_l \Gamma^{*i}_{jm})\, K^l_{skh}\, (\dot{\mathfrak{D}}_l \Gamma^{*i}_{km})\, K^l_{shj}\} = 0 \qquad \ldots(4.14)$$

Contracting the above equation with respect to the indices i and h, we get

$$\frac{(n-2)}{2}(\beta_{r|j}\beta_k - \beta_{r|k}\beta_j)\, v^r - \dot{x}^s v^m \{ (\dot{\mathfrak{D}}_l \Gamma^{*i}_{im})\, K^l_{sjk} + (\dot{\mathfrak{D}}_l \Gamma^{*i}_{jm})\, K^l_{ski}$$

$$+ (\dot{\mathfrak{D}}_l \Gamma^{*i}_{km})\, K^l_{sij} \} = 0 \qquad \ldots(4.15)$$

Comparing this result with (4.12) and after little simplification we obtain

$$(n-2)\, K^i_{hjk}\beta_m v^m = \dot{x}^s v^m \{(n-1)\, (\dot{\mathfrak{D}}_l \Gamma^{*i}_{hm})\, K^l_{sjk} + (\dot{\mathfrak{D}}_l \Gamma^{*i}_{jm}) K^l_{skh}$$

$$+ (\dot{\mathfrak{D}}_l \Gamma^{*i}_{km})\, K^l_{shj} \} \qquad \ldots(4.16)$$

Thus, we have the following theorem.

Theorem 4.1 — If an Fn^* admits an affine motion of the form (4.3) then the equation (4.16) holds.

In a general affinely connected Finsler space Fn, if $v^i_{\ |j} = 0$, the vector v^i determines a contrafield. We shall also consider a contrafield spanned by $v^i(x)$ in our Fn^*.

In Fn^*, we shall consider to find a necessary and sufficient condition upon the possibility of a special affine motion of the form

$$\overline{x^i} = x^i + v^i(x) dt v^i_{\ |j} = 0 \qquad \ldots(4.17)$$

Now, if it will be the case, the eqn. (2.1) of motion becomes

$$Lv\Gamma^{*i}_{jk} = K^i_{jkh}\, v^h = 0 \qquad \ldots(4.18)$$

and the integrability condition of (2.1) becomes

$$Lv\, K^i_{hik} = K^i_{hjk}\, \beta_s\, v^s = 0 \qquad \ldots(4.19)$$

So, owing to the non-flatness of the space, from (4.19), we get

$$\beta_s v^s = 0 \qquad ...(4.20)$$

In view of (4.18) and (4.20), we can conclude as follows :

If an Fn^* space admits an affine motion of the form (4.17), it is necessary that we have

$$v^s\beta_s = 0,\ K^i_{jkh}v^h = 0. \qquad ...(4.21)$$

Conversely if we have (4.21), using the formula (4.9) and the latter part of (4.21), we get

$$K^i_{hjk}v^h = 0$$

From the general theory of parallel vectors (Knebelman 1945), v^i determines a contrafield in Fn^*, $v^i_{\,)j} = 0$, when and only when v^i satisfy the equation (4.21). Thus, we may regard $v^i(x)$ as contrafield in Fn^*. In such a case $LvK^i_{hjk} = 0$ holds good and which is the integrability condition of $Lv\Gamma^{*i}_{jk} = 0$. Because, if $v^i_{\,|j} = 0$, by virtue of its definition, LvK^i_{hjk} is written as $v^s\beta_s K^i_{hjk}$. However, we have the former of (4.21) so we get $LvK^i_{hjk} = 0$, consequently $Lv\Gamma^{*i}_{jk} = 0$ are integrable. Thus, (4.21) is also a sufficient condition for Fn^* admitting (4.17).

Thus, we have the following theorem.

Theorem 4.2 — The necessary and sufficient condition that a non-flat Fn^* admits an infinitesimal transformation $\overline{x^i} = x^i + v^i(x)\ dt$ in order that the vector $v^i(x)$ span a contrafield in Fn^* is that eqn. (4.21) be valid.

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