SOME ELECTROSTATIC PROBLEMS OF TWO STRIPS LYING OUTSIDE A GROUNDED ELLIPTIC CYLINDER

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We present here the solutions of two-dimensional electrostatic boundary value problems involving two equal parallel coplanar strips, charged to equal or equal and opposite constant potentials, when these are placed symmetrically outside a grounded elliptic cylinder. The solution of each problem is first reduced to a Fredholm integral equation of the first kind which is then solved by the regular perturbation technique to obtain approximate expressions for the total charge densities per unit length of the two strips, when the semi-major axis of the elliptic cylinder is small. By taking appropriate limits, we also derive the solutions of the corresponding electrostatic problems when the grounded elliptic cylinder reduces to a circular cylinder or a strip. Even some of these limiting results seem to be new.

1. INTRODUCTION

Tranter (1960) solved the two-dimensional electrostatic problem of two equal parallel coplanar conducting infinite strips charged to potentials ± 1 in a free space by a dual series method. Srivastava and Gupta (1971) further studied the perturbation in the charge density of the two strips when these are placed symmetrically inside or outside a grounded cylinder by applying triple integral equations and finite Hilbert transform techniques by Srivastava and Lowengrub (1970). Goel and Jain (1976a) showed how the solutions of the two electrostatic problems presented by Srivastava and Gupta (1971) can be simplified by following the usual Green's function approach and solving the resulting governing Fredholm integral equation of the first kind by the regular perturbation technique as explained by Jain and Kanwal (1972, 1975a). By the same approach Goel and Jain (1976b) have solved the corresponding electrostatic problem of two strips charged to equal constant potentials, when these are lying inside or outside of a grounded circular cylinder.

We present here, for the first time, the solutions of two 2-dimensional electrostatic boundary value problems of two equal parallel coplanar conducting infinite strips charged to prescribed equal or equal and opposite constant potentials, when the two strips are placed symmetrically outside a grounded elliptic cylinder, in its principal plane, which contains the major axis. Each problem is formulated by the usual Green's function approach and the solution of the problem is reduced to a single governing Fredholm integral equation of the first kind, which embodies the differential equation as well as the boundary conditions of the problem. This governing Fredholm integral equation is further solved by the regular perturbation technique, as explained by Jain and Kanwal (1972, 1975a, 1975b), when the perturbation parameter is small. Approximate expressions for the unknown charge densities of the strips are obtained in each case.

Finally, by taking appropriate limits, we also derive the solutions of the two corresponding electrostatic problems when the grounded elliptic cylinder reduces to a circular cylinder or a strip. The solution of the first limiting case of the first problem agrees with the known result of Srivastva and Gupta (1971), provided the necessry corrections are carried out in their analysis. Moreover, the solution of the first limiting case of the second problem agrees with the known result of Goel and Jain (1976b). As far as the authors know, the solutions of the second limiting case in both the problems seem to be new.

2. Formulation of Electrostatic Problems

We consider here the electrostatic problem of finding the charge densities of two coplanar parallel strips charged to prescribed potentials, when the strips are placed symmetrically outside the grounded elliptic cylinder $\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1$ in the plane $y = 0$. We use cylindrical co-ordinates (r, θ, z) with generators of the cylinder parallel to the z-axis. We now have a two-dimensional boundary value problem of an ellipse $(x^2/a_1^2) + (y^2/b_1^2) = 1$ and two symmetrically placed line segments occupying the regions outside the ellipse. Firstly we nondimensionalize all lengths with the help of the length e_1, so that we have to solve the following two-dimensional boundary value problem for electrostatic potential $\phi(r, \theta)$:

$$\nabla^2\phi(r, \theta) = 0, \text{ in } D \qquad ...(2.1)$$

$$\phi(r, \theta) = 0, 0 \leqslant \theta \leqslant 2\pi, \ (\cos^2\theta/a^2) + (\sin^2\theta/b^2) = \frac{1}{r^2} \qquad ...(2.2)$$

$$\phi(r, 0) = f_1(r), \ d \leqslant r \leqslant 1 \qquad ...(2.3)$$

$$\phi(r, \pi) = f_2(r), \ d \leqslant r \leqslant 1 \qquad ...(2.4)$$

ϕ, $(\partial\phi/\partial r)$ are continuous across the line segments $a < r < d$,
$1 < r < \infty$, $\theta = 0$, $\theta = \pi$ ...(2.5)

where $a = (a_1/e_1)$, $b = (b_1/e_1)$, $d = (d_1/e_1)$, D is the whole region lying outside the ellipse except the two line segments $d \leqslant r \leqslant 1$, $\theta = 0, \pi$ and $f_1(r)$, $f_2(r)$ are the prescribed potentials of the two strips.

3. Reduction of the Electrostatic Problem

The integral representation formula for $\phi(r, \theta)$ follows from the usual Green's function approach. Indeed, the potential ϕ satisfying (2.1), (2.2) and (2.5) is

$$\phi(r,\theta) = \int_d^1 \sigma(r_0, 0) g(r, \theta | r_0, 0) dr_0 + \int_d^1 \sigma(r_0, \pi) g(r, \theta \mid r_0, \pi) dr_0 \qquad \ldots(3.1)$$

where $\sigma(r_0, 0)$ and $\sigma(r_0, \pi)$ are unknown charge densities of the two strips and Green's function g can be constructed by applying the conformal mapping theorem as given by Mackie (1965) and Stakgold (1963). Finally, we use the boundary conditions (2.3), (2.4) in (3.1) and obtain a pair of simultaneous Fredholm integral equations of the first kind :

$$\int_d^1 \sigma(r_0,0) g(r, 0|r_0, 0) dr_0 + \int_d^1 \sigma(r_0, \pi) g(r, 0|r_0, \pi) dr_0 = f_1(r)\,, \quad 1 \leqslant r \leqslant d \quad \ldots(3.2)$$

$$\int_d^1 \sigma(r_0,0) g(r, \pi|r_0, 0) dr_0 + \int_d^1 \sigma(r_0, \pi) g(r_0, \pi|r_0,\pi) dr_0 = f_2(r), 1 \leqslant r \leqslant d. \quad \ldots(3.3)$$

The required particular values of the Green's function g are:

$g(r, 0|r_0, 0) = g(r, \pi|r_0, \pi) =$

$$-\frac{1}{2\pi} \log \left| \frac{(a+b)\{[r+(r^2-c^2)^{1/2}]^{-1} - [r_0+(r_0^2-c^2)^{1/2}]^{-1}}{1-\{(a+b)^2/[r+(r^2-c^2)^{1/2}][r_0+(r_0^2-c^2)^{1/2}]\}} \right| \qquad \ldots(3.4)$$

$g(r, 0|r_0, \pi) = g(r, \pi|r_0, 0) =$

$$-\frac{1}{2\pi} \log \left| \frac{(a+b)\{[r+(r^2-c^2)^{1/2}]^{-1} + [r_0+(r_0^2-c^2)^{1/2}]^{-1}}{1+\{(a+b)^2/[r+(r^2-c^2)^{1/2}][r_0+(r_0^2-c^2)^{1/2}]\}} \right| \qquad \ldots(3.5)$$

where $c^2 = a^2 - b^2$. We now solve the two simultaneous Fredholm integral equations (3.2) and (3.3) for the two simple cases:

(I) $f_1(r) = 1, f_2(r) = -1$, i.e. when the two strips are charged to potentials $\pm$ 1.

(II) $f_1(r) = f_2(r) = b_0$ (const.), i.e. when the two strips are charged to an unknown constant potential b_0. This unknown constant is determined by using the condition that the total charge per unit length on each strip is given to be unity.

We now give solutions of these two boundary value problems when the perturbation parameter a is very small.

4. Solution to Problem I

The electrostatic boundary value problem I is defined by eqns. (2.1) to (2.5) when $f_1(r) = 1$, $f_2(r) = -1$. Solution of this problem is governed by simultaneous Fredholm integral equations (3.2) and (3.3). We solve these equations by the regular perturbation technique when $a << d < 1$. In this case, we have from (3.2) to (3.5)

$$\sigma(r_0, 0) = -\sigma(r_0, \pi) = h(r_0^2) \qquad \text{...(4.1)}$$

where the unknown function $h(r_0^2)$ satisfies

$$\int_d^1 h(r_0^2)\, K(r, r_0) dr_0 = 1, \quad d \leqslant r \leqslant 1 \qquad \text{...(4.2)}$$

and the kernel $K(r, r_0)$ is given by

$$K(r, r_0) = g(r, 0|r_0, 0) - g(r, \pi|r_0, 0) \qquad \text{...(4.3)}$$

We now present an approximate solution of (4.2) when $a << d < 1$. Firstly, we give the expansion of the values of $g(r, 0|r_0, 0)$ and $g(r, \pi|r_0, 0)$ defined by (3.4) and (3.5)

$$g(r, 0|r_0, 0) = -\frac{1}{2\pi}\Big[\log 2a(1+\beta) = \log|r - r_0| - \log (4rr_0)$$

$$+ \frac{a^2(1+\beta)^2}{4rr_0} + \frac{a^2(1-\beta^2)}{4}\left(\frac{1}{r_0^2} + \frac{1}{rr_0} + \frac{1}{r^2}\right) + \frac{a^4(1-\beta^2)^2}{16}\Bigg\{\frac{r^2 + rr_0 + r_0^2}{r^3 r_0^3}$$

$$+\left(\frac{1}{r^4} + \frac{1}{r_0^4} + \frac{1}{r^2 r_0^2}\right) - \frac{1}{2}\frac{1}{r^2 r_0^2} + \frac{1}{2}\left(\frac{1}{r^2} + \frac{1}{r_0^2}\right)^2\Bigg\} + \frac{a^4(1+\beta)^4}{32 r^2 r_0^2} + \frac{a^4(1+\beta)^2(1-\beta^2)}{16 rr_0}$$

$$\times\left(\frac{1}{r^2} + \frac{1}{r_0^2}\right) + O(a^6)\Big] \qquad \text{... (4.4)}$$

$$g(r, 0|r_0, \pi) = -\frac{1}{2\pi}\Big[\log 2a(1+\beta) + \log (r + r_0) - \log (4rr_0) - \frac{a^2(1+\beta)^2}{4rr_0}$$

$$+ \frac{a^2(1-\beta^2)}{4}\left(\frac{1}{r_0^2} - \frac{1}{rr_0} + \frac{1}{r^2}\right) + \frac{a^4(1-\beta^2)^2}{16}\Bigg\{\left(\frac{1}{r^4} + \frac{1}{r_0^4} - \frac{1}{r^2 r_0^2}\right) - \left(\frac{r^2 - rr_0 + r_0^2}{r^3 r_0^3}\right)$$

$$- \frac{1}{2}\frac{1}{r^2 r_0^2} + \frac{1}{2}\left(\frac{1}{r^2} + \frac{1}{r_0^2}\right)^2\Bigg\} + \frac{a^4(1+\beta)^4}{32 r^2 r_0^2} -$$

(eqn. contd. next page)

$$-\frac{a^4(1+\beta)^2(1-\beta^2)}{16rr_0}\left(\frac{1}{r^2}+\frac{1}{r_0^2}\right)+O(a^6)\Big],\ \beta=b/a \qquad ...(4.5)$$

When we substitute the above expansions in (4.3) we obtain

$$K(r,r_0)=\frac{1}{2\pi}\left[\log\left|\frac{r+r_0}{r-r_0}\right|-\frac{Aa^2}{rr_0}-\frac{Ba^4(r^2+r_0^2)}{r^3r_0^3}+O(a^6)\right] \qquad ...(4.6)$$

where $A=(1+\beta)$, $B=\frac{1}{4}(1+\beta)(1-\beta^2)$

The expansion of $K(r,r_0)$ as given by (4.6) suggests that the Fredholm integral equation (4.2) can be solved by setting

$$h(r_0^2)=h_0(r_0^2)+a^2h_2(r_0^2)+a^4h_4(r_0^4)+O(a^6) \qquad ...(4.7)$$

When we substitute the expansions (4.6) and (4.7) in equation (4.2) and equate on either side the coefficients of equal powers of the perturbation parameter a, we obtain

$$\int_d^1 h_0(r_0^2)\log\left|\frac{r+r_0}{r-r_0}\right|dr_0=2\pi,\ d\leqslant r\leqslant 1 \qquad ...(4.8)$$

$$\int_d^1 h_2(r_0^2)\log\left|\frac{r+r_0}{r-r_0}\right|dr_0=\frac{A}{r}\int_d^1\frac{h_0(r_0^2)}{r_0}dr_0,\ d\leqslant r\leqslant 1 \qquad ...(4.9)$$

$$\int_d^1 h_4(r_0^2)\log\left|\frac{r+r_0}{r-r_0}\right|dr_0=\frac{A}{r}\int_d^1\frac{h_2(r_0^2)}{r_0}dr_0+B\int_d^1 h_0(r_0^2)\left[\frac{1}{rr_0^3}+\frac{1}{r^3r_0}\right]dr_0,$$
$$d\leqslant r\leqslant 1 \qquad ...(4.10)$$

and so on. We invert these integral equations successively by the usual technique as given by Srivastva and Lowengrub (1970), Jain and Kanwal (1972), Goel and Jain (1976a,b), and readily obtain

$$h_0(r_0^2)=2/TK(d) \qquad ...(4.11)$$

$$h_2(r_0^2)=\frac{A}{Td^2K(d)}\left[\frac{d^2}{r_0^2}+\frac{E(d)}{K(d)}-1\right] \qquad ...(4.12)$$

$$h_4(r_0^2)=\frac{1}{2\pi d^3T}\left[(M-2L)d^2-M+\{M+(2L+M)d^2\}\frac{E(d)}{K(d)}\right.$$
$$\left.+\frac{\{2Ld^2-3M(1+d^2)\}d^2}{r_0^2}+\frac{6Md^4}{r_0^4}\right] \qquad ...(4.13)$$

where

$$\left.\begin{aligned} T &= [(r_0^2 - d^2)(1 - r_0^2)]^{1/2} \\ L &= \frac{\pi}{4d^3K(d)}\left[A^2\left(\frac{2E(d)}{K(d)} - 1 + d^2\right) + 2B(1+d^2)\right] \\ M &= \frac{\pi B}{dK(d)} = \frac{\pi}{4}\,\frac{(1+\beta)(1-\beta^2)}{dK(d)} \end{aligned}\right\} \quad \ldots(4.14)$$

$K(d) = F\left(\frac{\pi}{2}, d\right)$ and $E(d) = E\left(\frac{\pi}{2}, d\right)$ are the complete elliptic integrals of the first and second kind respectively and further, we have used some standard formulae given in the Appendix A.

Finally, the relations (4·1), (4·7), (4·11) to (4·13) yield

$$\sigma(r_0, 0) = -\sigma(r_0, \pi)$$

$$= \frac{2}{TK(d)} + \frac{Aa^2}{Td^2K(d)}\left\{\frac{d^2}{r_0^2} + \frac{E(d)}{K(d)} - 1\right\} + \frac{a^4}{2\pi d^3 T}\Big[(M - 2L)d^2 - M +$$

$$\{2Ld^2 + M(1+d^2)\}\frac{E(d)}{K(d)} + \frac{2Ld^4 - 3M(1+d^2)d^2}{r_0^2} + \frac{6Md^4}{r_0^4}\Big] + O(a^6) \quad \ldots(4\cdot15)$$

Moreover, approximate expressions for the total charges Q_j, $j = 1, 2$ per unit length on the two strips can be readily obtained by using the equation (4·15) in the relations

$$Q_1 = \int_d^1 \sigma(r_0, 0)\,dr_0, \quad Q_2 = \int_d^1 \sigma(r_0, \pi)\,dr_0 = -Q_1$$

For the special case of a circular cylinder, $x^2 + y^2 = \alpha^2$, $\beta \to 1$ so in this case

$$A = 2, \quad B = 0 = M, \quad L = \frac{\pi}{d^3K(d)}\left\{\frac{2E(d)}{K(d)} + d^2 - 1\right\}$$

$$\sigma(r_0, 0) = -\sigma(r_0, \pi) = \frac{2}{TK(d)}\Big[1 + \frac{a^2}{d^2}\left\{\frac{d^2}{r_0^2} + \frac{E(d)}{K(d)} - 1\right\}$$

$$\times\left\{1 + \frac{a^2}{2d^2}\left(\frac{2E(d)}{K(d)}\right) + d^2 - 1\right\} + O(a^6) \quad \ldots(4\cdot16)$$

The expression for the charge densities (4·16) agrees with the result of Srivastava and Gupta (1971) when the minor corrections are carried out in their analysis.

Another special case of a particular interest follows when $\beta \to 0$. This is the electrostatic boundary value problem of three strips:

$\phi(r, 0) = 1,\ d \leqslant r \leqslant 1;\ \phi(r, \pi) = -1,\ d \leqslant r \leqslant 1;\ \phi(r, \theta) = 0,\ 0 \leqslant r \leqslant a,\ \theta = 0,\ \pi$

In this case

$$A = 1,\ B = \tfrac{1}{2},\ L = \frac{\pi}{8d^3K(d)}\left\{ \frac{4E(d)}{K(d)} + 3d^2 - 1 \right\},\quad M = \frac{\pi}{4dK(d)}$$

$$\sigma(r_0, 0) = -\sigma(r_0, \pi) = \frac{2}{TK(d)}\Big[1 + \frac{a^2}{2d^2}\left\{ \frac{E(d)}{K(d)} + \frac{d^2}{r_0^2} - 1 \right\} + \frac{a^4}{4d^2}$$

$$\times\left[\left(1 + \frac{E(d)}{d^2K(d)} \right)\frac{E(d)}{K(d)} - \frac{1}{2}\left\{ 1 + \frac{2E(d)}{d^2K(d)} \right\} + \frac{1}{r_0^2}\left\{ \frac{E(d)}{K(d)} - 1 \right\} + \frac{3d^2}{2r_0^4}\right] + O(a^6) \qquad \ldots(4{\cdot}17)$$

5. Solution to Problem II

The electrostatic boundary value problem II is defined by equations (2.1) to (2.5) when $f_1(r) = f_2(r) = b_0$ (const.). This unknown constant is determined by using the condition that the total charge per unit length on each strip is given to be unity. In this case, it is also assumed that $a << d < 1$. The governing integral equations (3.2) and (3.3) lead to

$$\int_d^1 \frac{1}{r_0} k(r_0^2)\ L(r, r_0) dr_0 = b_0, \quad d \leqslant r \leqslant 1 \qquad \ldots(5.1)$$

where

$$\sigma(r_0, 0) = \sigma(r_0, \pi) = r_0^{-1}\ k(r_0^2) \qquad \ldots(5.2)$$

$$L(r, r_0) = g(r,0\,|\,r_0,0) + g(r,0\,|\,r_0,\pi) \qquad \ldots(5.3)$$

$$\int_d^1 \sigma(r_0,0) dr_0 = \int_d^1 \sigma(r_0,\pi) dr_0 = 1 \qquad \ldots(5.4)$$

Now we substitute the value of $g(r,0|r_0, 0)$ and $g(r,0|r_0,\pi)$ from equations (4.4) and (4.5) in (5.3) to obtain an approximate expansion of the kernel $L(r, r_0)$ defined by (5.3). On substituting this expansion of the kernel $L(r, r_0)$ in integral equation (5.1) we readily obtain

$$\int_d^1 r_0^{-1}\ k(r_0^2)\ M(r, r_0)\ dr_0 = -2\pi B, \qquad d \leqslant r \leqslant 1 \qquad \ldots(5.5)$$

where

$$M(r, r_0) = \log \left| 1 - \frac{r_0^2}{r^2} \right| + \frac{a^2(1-\beta^2)}{2}\left(\frac{1}{r^2} + \frac{1}{r_0^2}\right) + \frac{a^4(1+\beta)^2}{16}$$

$$\times \left[\frac{2(1+\beta^2)}{r^2 r_0^2} - 3(1-\beta)^2\left(\frac{1}{r^4} + \frac{1}{r_0^4}\right)\right] + O(a^6) \qquad (5.6)$$

$$B = b_0 + \frac{1}{\pi} \log \left[\frac{a(1+\beta)}{2}\right] - \frac{1}{\pi} \int_d^1 \sigma(r_0, 0) (\log r_0) dr_0 \qquad \ldots(5.7)$$

We process equation (5.5) as explained in the last section by setting

$$k(r_0^2) = k_0(r_0^2) + a^2 k_2(r_0^2) + a^4 k_4(r_0^2) + O(a^6) \qquad \ldots(5.8)$$

$$B = B_0 + a^2 B_2 + a^4 B_4 + O(a^6) \qquad \ldots(5.9)$$

and consequently obtain

$$\sigma(r_0, 0) = \sigma(r_0, \pi) = r_0^{-1} k(r_0^2)$$

$$= \frac{1}{r_0 T}\Bigg[-\frac{4B_0 d}{\lambda} + 4a^2\left\{\left(\frac{(1+d)^2}{\lambda^2} + \frac{1+d^2}{\lambda}\right)\left(\frac{1-\beta^2}{4d}\right)B_0 - \frac{d}{\lambda}B_2\right.$$

$$\left. - \frac{(1-\beta^2)}{2r_0^2}\frac{d}{\lambda}B_0\right\} + 2a^4\left\{AB_0 + \frac{1-\beta^2}{2d}\left(\frac{(1+d)^2}{\lambda^2} + \frac{1+d^2}{\lambda}\right)B_2\right.$$

$$- 2\frac{d}{\lambda}B_4 + \frac{1}{r_0^2}\left\{\frac{(1+\beta)^2(1+d^2)}{4d}\left(\frac{1+\beta^2-3\beta}{\lambda} + \frac{(1-\beta)^2}{\lambda^2}\frac{(1+d)^2}{1+d^2}\right)B_0\right.$$

$$\left.\left. - \frac{(1-\beta^2)d}{\lambda}B_2\right\} - \frac{3(1-\beta^2)^2}{4}\frac{d}{\lambda}\frac{B_0}{r_0^4}\right\} + O(a^6)\Bigg] \qquad \ldots(5.10)$$

where

$$\lambda = \log\,[(1-d)/(1+d)]$$

$$A = \frac{(1+\beta)^2(1+\beta^2)(1+d^2)^2}{16\lambda d^3}\left\{1 + \frac{3(1-\beta)^2}{2(1+\beta^2)}\left(\frac{1-d^2}{1+d^2}\right)^2\right\}$$

$$+ \frac{(1+\beta)^4(1+d^2)}{16\lambda^2 d^2}\left\{1 + \frac{5(1+d^4) - 18d^2}{4d(1+d^2)}\left(\frac{1-\beta}{1+\beta}\right)^2\right\}$$

$$- \frac{(1-\beta^2)^2}{8\lambda^3}\frac{(1+d)^4}{d^3}$$

and we have used some formulae of definite integrals given in Appendix B.

In order to obtain the values of the unknown constants B_0, B_2, B_4 occurring in the above expression, we substitute (5.10) in (5.4) and obtain

$$B_0 = -\lambda/2\pi, \; B_2 = -(1-\beta^2)(1+d)^2/8\pi d^2$$

$$B_4 = -\frac{(1+\beta)^2}{64\pi d^3}\left[\{5(1+\beta^2)-6\beta\}(1+d^2) + \frac{(1-\beta)^2\{13(1+d^4)-2d^2\}}{4d}\right] \qquad ...(5.11)$$

Finally we discuss two limiting cases whose solutions can be obtained from the above analysis. For the case of a circular cylinder $x^2+y^2=a^2$, $\beta \to 1$. So in this case

$$B_0 = -\lambda/2\pi, \; B_2 = 0, \; B_4 = -(1+d^2)/4\pi d^3$$

$$\sigma(r_0, 0) = \sigma(r_0, \pi)$$

$$= \frac{2d}{\pi r_0 T}\left[1 - \frac{(1+d^2)^2}{4d^4}\left\{1 - \frac{2d^2}{1+d^2}\frac{1}{r_0^2}\right\}a^4 + O(a^6)\right] \qquad ...(5.12)$$

which agrees with the known result of Goel and Jain (1976b). When the elliptic cylinder reduces to the strip, $-a \leqslant x \leqslant a$, $y=0$, $\beta \to 0$ and we have in this case

$$\left.\begin{aligned} B_0 &= -\lambda/2\pi, \; B_2 = -(1+d)^2/8\pi d^2 \\ B_4 &= -(13d^4+20d^3-2d^2+20d+13)/256\pi d^4 \end{aligned}\right\} \qquad ...(5.13)$$

$$\sigma(r_0, 0) = \sigma(r_0, \pi)$$

$$= \frac{2d}{\pi r_0 T}\left[1 - \frac{a^2}{2}\left(\frac{1+d^2}{2d^2} - \frac{1}{r_0^2}\right) - a^4\left\{A_1 + \frac{(1+d)^2}{16d^4}\left(\frac{1+d^2}{\lambda} + \frac{(1+d)^2}{\lambda^2}\right) + A_2 + \frac{1+d^2}{8d^2}\frac{1}{r_0^2} - \frac{3}{8}\frac{1}{r_0^4}\right\} + O(a^6)\right] \qquad ...(5.14)$$

where

$$A_1 = \frac{(1+d^2)^2}{32d^4}\left\{1 + \frac{3}{2}\left(\frac{1-d^2}{1+d^2}\right)^2\right\} + \frac{1+d^2}{32\lambda d^3}\left\{1 + \frac{5(1+d^4)-18d^2}{4d(1+d^2)}\right\} - \frac{(1+d)^4}{16\lambda^2 d^4}$$

$$A_2 = 2\pi B_4/\lambda$$

As far as the authors know, even the limiting result (5.14) seems to be new.

Appendix A

We give here some standard formulae which have been used in our analysis of Problem I.

$$\int_d^1 U(r,r_0)\,\frac{dr_0}{T} = \pi K(d),\ d \leqslant r \leqslant 1 \qquad \text{...(A-1)}$$

$$\int_d^1 r_0^2 U(r,r_0)\frac{dr_0}{T} = \pi\{r + K(d) - E(d)\},\ d \leqslant r \leqslant 1 \qquad \text{...(A-2)}$$

$$\int_d^1 r_0^4 U(r,r_0)\frac{dr_0}{T} = \pi\{\tfrac{1}{2}(1+d^2)r + \tfrac{1}{3}r^3 + \tfrac{1}{3}[(2+d^2)K(d) - 2(1+d^2)\,E(d)]\},$$

$$d \leqslant r \leqslant 1 \qquad \text{...(A-3)}$$

$$\int_d^1 U(r,r_0)\,\frac{dr_0}{Tr_0^2} = \frac{\pi}{d^2}\,\{\frac{d}{r} - E(d) + K(d)\},\quad d \leqslant r \leqslant 1 \qquad \text{...(A-4)}$$

$$\int_d^1 U(r,r_0)\,\frac{dr_0}{Tr_0^4} = \frac{\pi}{2d}\left\{(1+\frac{1}{d^2})\frac{1}{r} + \frac{2}{3}\frac{1}{r^3}\right\} + \frac{\pi}{3d^4}\left\{(2+d^2)K(d)\right.$$

$$\left. - 2(1+d^2)E(d)\right\},\ d \leqslant r \leqslant 1 \qquad \text{...(A-5)}$$

where $U(r,r_0) = \log\left|\dfrac{r+r_0}{r-r_0}\right|$

Appendix B

We give here some standard formulae which have been used in our analysis of Problem II.

$$\int_d^1 r_0 V(r,r_0)\frac{dr_0}{T} = \pi\{\tfrac{1}{2}\log(1-d^2) - \log 2r\},\ d \leqslant r \leqslant 1 \qquad \text{...(B-1)}$$

$$\int_d^1 r_0^3 V(r,r_0)\frac{dr_0}{T} = \frac{\pi}{2}\{(1+d^2)[\tfrac{1}{2} + \tfrac{1}{2}\log(1-d^2) - \log 2r] - r^2\},\ d \leqslant r \leqslant 1 \quad \text{(B-2)}$$

$$\int_d^1 r_0^5 V(r,r_0)\frac{dr_0}{T} = \frac{\pi}{32}\{7(1+d^4) + 10d^2\} - \frac{\pi}{4}\{(1+d^2)r^2 + r^4\}$$

$$+ \frac{\pi}{8}\{3(1+d^4) + 2d^2\}\{\tfrac{1}{2}\log(1-d^2) - \log 2r\},\ d \leqslant r \leqslant 1 \quad \text{(B-3)}$$

$$\int_d^1 V(r, r_0) \frac{dr_0}{Tr_0} = \frac{\pi}{2d} \lambda, \qquad d \leqslant r \leqslant 1, \qquad \text{...(B-4)}$$

$$\int_d^1 V(r, r_0) \frac{dr_0}{Tr_0^3} = \frac{\pi}{2d^2} \left\{ 1 + \frac{1+d^2}{2d} \lambda - \frac{d}{r^2} \right\}, \; d \leqslant r \leqslant 1 \qquad \text{...(B-5)}$$

$$\int_d^1 V(r, r_0) \frac{dr_0}{Tr_0^5} = \frac{\pi}{4d} \left\{ \frac{3}{2} \frac{1+d^2}{d^3} + \frac{3(1+d^4)+2d^2}{4d^4} \lambda - \frac{1+d^2}{d^2} \frac{1}{r^2} - \frac{1}{r^4} \right\}$$

$$d \leqslant r \leqslant 1 \qquad \text{...(B-6)}$$

where $V(r, r_0) = \log \mid 1 - (r_0^2/r^2) \mid$ and $\lambda = \log \left(\frac{1-d}{1-d} \right)$

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