GENERALISED EXTERIOR DERIVATIONS IN FRAMED MANIFOLDS

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By using an arbitrary endomorphism defined by a vector 1-form h, Hodge theory can be generalised to produce generalised Green's operator on a compact orientable manifold (Eiseman and Stone 1974 b, c). On the other hand, framed manifolds are generalisations of Kaehler manifolds (Goldberg 1972). In the present paper, we establish some results on these manifolds, which eventually lead to generalised Hodge theory.

Preliminaries : Let M be a C^{∞} manifold with f_i-structure of rank $r_i (i = 1, 2)$. Recall that an f-structure of rank r on M means that there exists a $(1, 1)$ tensor field f on M which satisfies

$$f^3 + f = 0 \qquad ...(1)$$

$$\text{rank } f = \text{constant } r \text{ on whole of } M \qquad ...(2)$$

We assume that M is a globally framed manifold with respect to both f_i-structures, thus there exist two sets S_i of linearly independent vectors $\underset{1}{E}_A$ $(A = 1, 2 \ldots\ldots\ldots n-r_1)$ belonging to S_1 and $\underset{2}{E}_a (a = 1, 2 \ldots\ldots n-r_2)$ belonging to S_2 and two more sets S_i of Pfaffian form $\underset{1}{\eta}^A$ and $\underset{2}{\eta}^a$ such that

$$\underset{1}{\eta}^A(\underset{1}{E}_B) = \delta_B{}^A$$

$$\underset{2}{\eta}^a(\underset{2}{E}_b) = \delta_b{}^a$$

and

$$f_1{}^2 = -I + \underset{1}{\eta}^A \otimes \underset{1}{E}_A$$

$$f_2^2 = -I + \underset{2}{\eta}^a \otimes \underset{2}{E}_a$$

It is assumed that $S_1 \cap S_2$ and consequently $s_1 \cap s_2$ is non-void.

We make M into a metric manifold in two distinct ways by choosing the metrics g_i $(i = 1, 2)$ which satisfy

$$\underset{i}{\eta}^A = g_i(\underset{i}{E}_A, \cdot) \quad \ldots(3)$$

and

$$g_i(f_i X, Y) = -g_i(X, f_i Y) \quad \ldots(4)$$

Equality (4) defines two skew-symmetric 2-forms F_i $(i = 1, 2)$ given by

$$F_i(X, Y) = g_i(f_i X, Y)$$

M is called an integrable quasi-symplectic manifold with respect to structure f_i if F_i is closed and parallel along the integral curves of vector fields $\underset{i}{E}_A$ and f_i satisfies the integrability condition, i.e., $[f_i, f_i] = 0$. It is well-known that on a compact integrable quasi-symplectic manifold Hodge theory can be applied (Goldberg 1972).

We assume in the first instance that M is integrable quasi-symplectic manifold with respect to both structures. Corresponding to two structures $f_i (i = 1, 2)$, M has two submanifolds denoted by M_1 and M_2 on which f_1 and f_2 are non-singular endomorphisms. In view of our assumption on S_1, S_2 submanifolds M_1 and M_2 also intersect; we denote $M_1 \cap M_2$ by M, $d_{im}(\mathsf{M}) = m$. $\wedge(M)$, $\wedge(M_i)$ and $\wedge(\mathsf{M})$ denote as usual the exterior algebra of differential forms on these manifolds, where

$$\wedge(M) = \sum_{p=0}^{n} \overset{p}{\wedge}(M)$$

$$\wedge(M_i) = \sum_{p=0}^{r_i} \overset{p}{\wedge}(M_i)$$

$$\wedge(\mathsf{M}) = \sum_{p=0}^{m} \overset{p}{\wedge}(\mathsf{M})$$

Recall that any vector 1-form (a 1-1 tensor field) h on M defines an endomorphism $h^{(q)}: \wedge :^{p}(M)$ given by

$$h^{(q)}(\alpha) = h^{(q)}(\theta^1 \wedge \theta^2 \wedge \ldots\ldots \wedge \theta^p)$$

$$= \frac{1}{q!\,p-q!} \sum_{\pi} |\,\pi\,|\, (h\theta^{\pi(1)} \wedge \ldots\ldots \wedge h\theta^{\pi(q)} \wedge \theta^{\pi(q+1)} \ldots\ldots \theta^{\pi(p)}) \qquad \ldots(5)$$

where summation extends on all possible permutations of $(1,2\ldots, p)$ and $|\,\pi\,|$ stands for the sign of permutation π (Eiseman and Stone 1974*b*).

$h^{(q)}\alpha = 0$ if $q > p$ and is identity when $q = 0$.

If h and k are two vector 1-forms on M, then the differential concomitant defined as

$$[h, k]\theta = \tfrac{1}{2}\{-[h^{(1)}k^{(1)} - (hk)^{(1)}]d\theta + [h^{(1)}dk\theta + k^{(1)}dh\theta] - [dhk\theta + dkh\theta]\} \qquad \ldots(6)$$

is a mapping from $\wedge^p(M)$ to $\wedge^{p+1}(M)$.

When $h = k$ it simplifies to

$$[h, h]\,\theta = -\,h^{(2)}d\theta + h^{(1)}dh\theta - dh^2\theta$$

If further we put $d_h = h^{(1)}d - dh^{(1)}$ then

$$\left.\begin{aligned} &[h, h] = -h^{(2)}d + d_h h^{(1)} + dh^{(2)} \\ \text{and} \quad &[h, h]d + d[h, h] = d_h \cdot d_h \end{aligned}\right\} \qquad \ldots(7)$$

hence when $[h, h] = 0$, anti-derivation $d_h\colon \wedge^p \to \wedge^{p+1}$ becomes exterior derivation.

Since dual characterisation of $[h, h]$ given by eqns. (7) is the usual Nijenhuis tensor

$$[h, h]\;(X, Y) = h^2[X, Y] + [hX, hY] - h[hX, Y] - h[X, hY]$$

it follows that vanishing of Nijenhuis tensor $[h, h]$ implies the vanishing of its dual and conversely.

§1. Corresponding to f_i $(i = 1, 2)$ we have derivations df_1 and df_2 which we denote by d_1 and d_2.

A p-form is said to be f_i-closed if it is a zero of d_i.

In view of (6) differential concomitant for vector 1-forms f_i and f_j would be

$$[f_i, f_j]\theta = \tfrac{1}{2}\{-[f_i^{(1)}f_j^{(1)} - (f_i f_j)^{(1)}]d\theta + [f_i^{(1)}df_j\theta + f_j^{(1)}df_i\theta] - [df_i f_j\theta + df_j f_i\theta]\} \qquad \ldots(1.1)$$

and from (7) when $i = j$ it would simplify to

$$[f_i, f_i]\theta = -f_i^{(2)}d\theta + f_i^{(1)}df_i\theta - df_i^2\theta \qquad ...(1.2)$$

A p-form θ will be called a conservation law on M with respect to structure f_i if θ and $f_i\theta$ are exact. From (1.1) we have an easily verifiable result, which can be stated as:

Proposition 1.1—A p-form θ is a conservation law with respect to both f_1 and f_2 if and only if it lies in the kernel of $[f_i, f_j]$ and $f_i\theta$ is f_j-closed.

Every form which is a zero of $[f_i, f_j]$ is also a zero of $[f_i, f_i]$ but not conversely.

Since $$d_i \cdot d_i = d[f_i, f_i] + [f_i, f_i]d \qquad ...(1.3)$$

$d_i \cdot d_i = 0$ whenever we have $[f_i, f_i] = 0$.

Again as the dual characterisation of (1.1) is the Nijenhuis tensor for f_i, f_j, it follows that if torsion of structure tensor f_i defined as

$$[f_i, f_i]\ (X, Y) = f_i^2[X, Y] + [f_iX, f_i Y] - f_i[f_iX, Y] - f_i[X, f_i Y] \qquad ...(1.4)$$

for any arbitrary pair of vector fields (X, Y) is zero then $[f_i, f_i]\theta = 0$.

As a consequence the following results are valid on M:

Proposition 1.2—If M is an integrable framed manifold with structure f_i then the alternating derivation d_i with respect to vector 1-form f_i is an exterior derivation.

Proof is obvious by definition of integrability of M with respect to structures f_i.

Proposition 1.3—On an integrable manifold with structure tensor f_i we always have

$$f_i^{(p+1)}d = d_i f_i^{(p)} \qquad ...(1.5)$$

for any p-form α.

PROOF: For $p = 0$ or $p \geqslant n$ result is trivial and for $1 \leqslant p < n$ the proof follows by induction (Eiseman and Stone 1974*c*).

Proposition 1.4—If M is a compact integrable manifold and M_i is the submanifold with respect to structure f_i then d_i satisfies Poincare's Lemma on M_i, i.e., for any arbitrary p-form α on M_i equation $d_i\alpha = 0$ implies the existence of a $(p-1)$-form β such that $\alpha = d_i\beta$.

PROOF: Vector 1-form f_i restricted to M_i is non-singular. Consequently $f_i^{(p)}$ is non-singular (Eiseman and Stone 1974 c), thus in view of Proposition (1.3) we have

$$f_i^{(p+1)}d(f_i^{(p)})^{-1}\alpha = d_i\alpha = 0$$

or

$$d(f_i^{-1})^{(p)}\alpha = 0$$

which implies that there exists a $(p-1)$- form β such that

$$(f_i^{-1})^{(p)}\alpha = d\beta$$

or

$$\alpha = (f_i)^{(p)}d\beta = d_i(f_i)^{(p-1)}\beta$$

Clearly $f_i^{(p-1)}\beta$ is the required $(p-1)$-form.

§2. Results proved in this section are valid for integrable quasisymplectic manifolds. We drop the suffix i from our structure tensor and treat M as admitting a structure tensor f of rank r.

The f-basis on $T_p(M)$ is an orthonormal basis formed by union of 3 sets $\{X_A\}$, $\{fX_A = X_A^*\}$ and $\{E_a\}(A = 1, 2 ... r/2,\ a = 1, 2 ... n-r)$.

The dual of $[X_A, X_A^*, E_a]$ is denoted by $[W_A, W_A^*, \eta^a]$. Since M is integrable, $[f, f] = 0$ and consequently $d_f \,.\, d_f = 0$. Moreover f is an almost complex structure on the subspace of $T_p(M)$ spanned by $\{X_A, X_A^*\}$ and hence it is non-singular on the corresponding integral submanifold.

Lemma 2.1—In a quasi-symplectic manifold F is d_f-closed.

PROOF: Now

$$d_fF = (f^{(1)}d - df^{(1)})F$$

$$= f^{(1)}dF - df^{(1)})\ (\sum W_A \wedge W_A^*)$$

Since F is closed and

$$f^{(1)}(\sum W_A \wedge W_A^*) = \sum_{A=1}^{r/2}[(fW_A \wedge W_A^*) - (fW_A^* \wedge W_A)]$$

$$= \sum_{A=1}^{r/2}[W_{A^*} \wedge W_{A^*} - W_A \wedge W_A]$$

$$= 0$$

We have: F is closed with respect to d_f.

If F denotes the Kaehler form, the cperator $L = \varepsilon(F)$ and $\wedge = i(F)$ defined on M are dual operators.

Proposition 2.2—On a quasi-symplectic manifold

(i) $Ld_f = d_f L$

(ii) $\wedge \delta_f = \delta_f \wedge$

PROOF: Let α be any form on M, then

$$\begin{aligned} d_f L\alpha &= d_f \varepsilon(F)\alpha \\ &= d_f(F\wedge\alpha) \\ &= d_f F\wedge\alpha + F\wedge d_f\alpha \\ &= F\wedge d_f\alpha \text{ (from Lemma 2.1)} \\ &= Ld_f\alpha \end{aligned}$$

(ii) follows from duality relation of δ_f and $\wedge$ with d_f and L respectively.

§3. Throughout this section manifold M is integrable. It is known (Goldberg 1972) that on a framed manifold exterior derivation d can be expressed in terms of differential operators d', d'' and d°:

$$d = d' + d'' + d^\circ \qquad ...(3.1)$$

In terms of f-basis given in §2 these operators would be (Goldberg 1972)

$$d' = \sum_A \varepsilon(W_A)D_{X_A},\quad d'' = \sum_A \varepsilon(W_{A^*})\, D_{X_{A^*}},\quad d^\circ = \sum_a \varepsilon(\eta^a)D_{E_a}$$

$$(A = 1, 2..., r/2,\ a = 1, 2..., n-r) \qquad ...(3.2)$$

Similarly, the dual operator δ of d is the sum of differential operators δ', δ'' and δ°, thus

$$\delta = \delta' + \delta'' + \delta^\circ \qquad ...(3.3)$$

where

$$\delta' = -\sum_A i(W_A)D_{X_{A^*}},\quad \delta'' = \sum_A i(W_{A^*})D_{X_A},\quad \delta^\circ = -\sum_a i(\eta^a)D_{E_a} \qquad ...(3.4)$$

It is easy to check that if we set

$$d_f' = f^{(1)}d' - d'f^{(1)},\quad d_f'' = f^{(1)}d'' - d''f^{(1)},\quad d_f^\circ = f^{(1)}d^\circ - d^\circ f^{(1)} \qquad ...(3.5)$$

and

$$\delta_f' = \delta' f_t^{(1)} - f_t^{(1)}\delta', \; \delta_f'' = \delta'' f_t^{(1)} - f_t^{(1)}\delta'', \; \delta_f^\circ = \delta^\circ f_t^{(1)} - f_t^{(1)}\delta^\circ \qquad \dots(3.6)$$

then

$$d_f = d_f' + d_f'' + d_f^\circ \qquad \dots(3.7)$$

and

$$\delta_f = \delta_f' + \delta_f'' + \delta_f^\circ \qquad \dots(3.8)$$

Lemma 3.1—Operators δ_f', δ_f'' and δ_f° are related to the triad (d_f', d_f'', d_f°) by the following relations:

(i) $\delta_f' = - *[d_f'' + d''(trf)\wedge]*$

(ii) $\delta_f'' = - *[d_f' + d'(trf)\wedge]*$

(iii) $\delta_f^\circ = - *[d_f^\circ + d^\circ(trf)\wedge]*$

PROOF: By definition $\delta_f' = \delta' f_t^{(1)} - f_t^{(1)}\delta'$

But

$$\delta' = - *d''* \quad \text{(Goldberg 1972)} \qquad \dots(3.9)$$

and

$$f^{(1)}* + *f_t^{(1)} = (trf)* \quad \text{(Eiseman and Stone 1974c)} \qquad \dots(3.10a)$$

or equivalently

$$*f^{(1)} + f_t^{(1)}* = *(trf) \qquad \dots(3.10b)$$

In view of (3.9) we have

$$\delta_f' = - \{*d''*f_t^{(1)} - f_t^{(1)}*d''*\}$$

Using (3.10*a*) and (3.10*b*)

$$= - \{+ *d''(trf* - f^{(1)}*) - (*trf - *f^{(1)})d''*\}$$

$$= - \{*d''(trf)* - *d''f^{(1)}* + *f^{(1)}d''* - *trfd''*\}$$

Using the second equation of (3.5) and noting that $d''trf - trfd'' = d''(trf)\wedge$ we get

$$= - *\{d_f'' + d''(trf)\wedge\}*.$$

Similarly (ii) and (iii) can be proved

Lemma 3.2 — On manifold M

(i) $d_f' \cdot d_f' = 0, \qquad d_f'd_f'' + d_f''d_f' = 0$

(ii) $d_f''\,.\,d_f''=0, \qquad d_f^\circ d_f'+d_f'd_f^\circ=0$

(iii) $d_f^\circ\,.\,d_f^\circ=0, \qquad d_f^\circ d_f''+d_f''d_f^\circ=0$

PROOF : In view of Proposition 1.1

$$d_f\,.\,d_f=0$$

Consequently from (3.7) we have

$$(d_f'+d_f''+d_f^\circ).\ (d_f'+d_f''+d_f^\circ)$$
$$=d_f'd_f'+d_f''d_f''+d_f^\circ d_f^\circ+(d_f'd_f''+d_f''d_f')$$
$$+(d_f^\circ d_f'+d_f'd_f^\circ)+(d_f^\circ d_f''+d_f''d_f^\circ) \qquad ...(3.11)$$

By comparing tri-degrees in (3.11) we have the required results.

Lemma 3.3 — On an integrable quasi-symplectic manifold

(i) $d_f'L=Ld_f',\ d_f''L=Ld_f'',\ d_f^\circ L=Ld_f^\circ$

(ii) $\delta_f'\Lambda=\Lambda\delta_f',\ \delta_f''\Lambda=\Lambda\delta_f'',\ \delta_f^\circ\Lambda=\Lambda\delta_f^\circ$

PROOF : From Lemma (2.2) we have

$$d_fL=Ld_f$$

From eqn. (3.7) we get

$$(d_f'+d_f''+d_f^\circ)L=L(d_f^\circ+d_f'+d_f'') \qquad ...(3.12)$$

By comparing tri-degrees in (3.12), (i) is obtained.

The relations (ii) are the dual of the corresponding formulas in (i) hence, the lemma is established.

Lemma 3.4 — On manifold M we have:

$$\delta_f'\,.\,\delta_f'=0,\ \delta_f''\delta_f''=0,\ \delta_f''\delta_f'+\delta_f''\delta_f'=0$$
$$\delta_f^\circ\,.\,\delta_f^\circ=0,\ \delta_f^\circ\delta_f'+\delta_f'\delta_f^\circ=0,\ \delta_f^\circ\delta_f''+\delta_f''\delta_f^\circ=0 \qquad ...(3.13)$$

PROOF is obvious from duality considerations of Lemma 3.2.

Lemma 3.5 — In an integrable quasi-symplectic manifold the Laplace-Beltrami operator Δ_f satisfies:

$$\Delta_f=(d_f'\delta_f'+\delta_f'd_f')+(d_f''\delta_f''+\delta_f''d_f'')+(d_f^\circ\delta_f^\circ+\delta_f^\circ d_f^\circ)$$

and

(i) $d_f'\delta_f'' + \delta_f''d_f' = 0$

(ii) $d_f'\delta_f^\circ + \delta_f^\circ d_f' = 0$

(iii) $d_f''\delta_f' + \delta_f'd_f'' = 0$

(iv) $d_f''\delta_f^\circ + \delta_f^\circ d_f'' = 0$

(v) $d_f^\circ\delta_f' + \delta_f'd_f^\circ = 0$

(vi) $d_f^\circ\delta_f'' + \delta_f''d_f^\circ = 0$

PROOF : By definition the Laplace-Beltrami operator $\Delta_f = d_f\delta_f + \delta_f d_f$ for f-derivation, is a mapping which maps $\wedge^{\lambda,\mu,\nu}$ to $\wedge^{\lambda,\mu,\nu}$.

After substituting the values of d_f and δ_f, and multiplying them we have :

$$\Delta_f = (d_f' + d_f'' + d_f^\circ)(\delta_f' + \delta_f'' + \delta_f^\circ) + (\delta_f' + \delta_f'' + \delta_f^\circ)(d_f' + d_f'' + d_f^\circ)$$

$$= (d_f'\delta_f' + \delta_f'd_f') + (d_f''\delta_f'' + \delta_f''d_f'') + (d_f^\circ\delta_f^\circ + \delta_f^\circ d_f^\circ)$$

$$+ (d_f'\delta_f'' + \delta_f''d_f') + (d_f'\delta_f^\circ + \delta_f^\circ d_f') + (d_f''\delta_f' + \delta_f'd_f'')$$

$$+ (d_f''\delta_f^\circ + \delta_f^\circ d_f'') + (d_f^\circ\delta_f' + \delta_f'd_f^\circ) + (d_f^\circ\delta_f'' + \delta_f''d_f^\circ)$$

We note that only first three terms of this sum conform to the pattern of mapping defined for $\triangle_f$, i.e. each one of them maps a form of tri-degree (λ,μ,ν) to a similar form, whereas the other six terms in view of their types contribute nothing when actual $\triangle_f$ operation is carried out; and since they are all of different degree they vanish separately giving us equalities (i) through (vi).

A form which is a zero of Δ_f is called f-harmonic on an integrable quasi-symplectic manifold. There always exist non-trivial f-harmonic forms. We prove below the main result of our section.

Theorem 3.6 — On a compact integrable quasi-symplectic manifold the forms $F^p = F\wedge F\ldots\wedge F$ (p-times) are f-harmonic of degree $2p$ for every integer $p \leqslant r/2$ (Goldberg 1972).

For proving the theorem, we need a simple result which we establish in the form of a lemma.

Lemma 3.7 — On an integrable quasi-symplectic manifold, Kaehler form F is f-co-closed.

PROOF: From definition of δ_f we have

$$\delta_f F = (\delta_f' + \delta_f'' + \delta_f^\circ)F \qquad \ldots(3.6)$$

Now

$$\delta_f' F = \sum_{A=1}^{r/2} i(W_A) D_{X_A *} F$$

$$\delta_f'' F = \sum_{A=1}^{r/2} i(W_A *) D_{X_A} F$$

and since X_A, X_A* are both horizontal vectors

$$D_{X_A *} F = 0 = D_{X_A} F \quad \text{(Goldberg 1972)} \qquad \text{...(3.7)}$$

Again $\delta^\circ F = \sum_{a=1}^{n-r} i(\eta^a) D_{E_a} F$

and as F is parallel along the integral curves of E_a

$$D_{E_a} F = 0 \qquad \text{...(3.8)}$$

Using (3.7) and (3.8) we have the result that

$$\delta_f F = 0$$

PROOF OF THEOREM: We prove the result by induction. To begin with we note that F is f-harmonic, for it is easy to check that any form α which is f-closed and f-co-closed is a zero of Δ_f, and by Lemmas 3.3 and 3.7 we know that F is such a form.

Now suppose that F^{p-1} is f-harmonic, then

$$\begin{aligned} \Delta_f(F^p) &= \Delta_f(F \wedge F^{p-1}) \\ &= \Delta_f F \wedge F^{p-1} + F \wedge \Delta_f F^{p-1} \\ &= 0 \end{aligned}$$

If $\wedge_{H_f}$, $\wedge_{d_f}$, $\wedge_{\delta_f}$ denote the exterior algebras of f-harmonic, f-closed and f-co-closed forms respectively, then the inclusion relations

$$\wedge_{H_f} \subset \wedge_{d_f}$$

$$\wedge_{H_f} \subset \wedge_{\delta_f}$$

are obvious.

Using these concepts we (Prakash and Lata 1976) obtained orthogonal decomposition of $\wedge (M)$ and formulated generalised Green's operator on framed manifolds.

§ 4.In this section we consider the p-forms of M which are conservation laws for f_1 and f_2.

A p-form α on M can be put as:

$$\alpha = \alpha_h \wedge \alpha_v$$

where α_h and α_v are horizontal and vertical components of α; α_h is of bi-degree (λ, μ) and α_v of degree v. In terms of f-basis they can simply be:

$$\alpha_h = W_{A_1} \wedge \ldots \wedge W_{A_\lambda} \wedge W_{B_1}^* \wedge \ldots \wedge W_{B_\mu}^*$$

$$\alpha_v = \overset{a}{\eta}{}^1 \wedge \ldots \overset{a}{\eta}{}^v$$

Proposition 4.1 — On a manifold M admitting closed Pfaffian forms every vertical p-form is f-closed.

PROOF : For

$$d_f\alpha = d_f\alpha_v$$

$$= (f^{(1)}d - df^{(1)})\,(\overset{a}{\eta}{}^1 \wedge \ldots \wedge \overset{a}{\eta}{}^v) \qquad \ldots(4.1)$$

Since $\overset{a}{\eta}{}^i f = 0$ and $d\overset{a}{\eta}{}^i = 0$ for each a_i, the result is obvious.

Corollary 4.2 — If in addition to the hypothesis of proposition 4.1 we take M as compact, then every vertical form α as well as $f\alpha$ is exact.

PROOF: Since α is a vertical form it can be put as:

$$\alpha = \overset{a}{\eta}{}^1 \wedge \ldots \wedge \overset{a}{\eta}{}^v$$

and

$$d\overset{a}{\eta}{}^1 = 0 \text{ by hypothesis.}$$

Hence by Poincare's Lemma there exists a form β such that $\eta = d\beta$ which shows that η is exact.

Again since by Proposition 4.1 α is f-closed, from eqn. (4.1) we have

$$df^{(1)}\alpha = 0 = df\alpha$$

which implies, by similar reasoning, that $f\alpha$ is exact.

Since $f^{(1)}F$ is identically zero we have a slightly stronger result.

Proposition 4.3 — If Pfaffian forms η^a satisfy:

$$d\eta^a = K^a F\,(a = 1, 2, \ldots, n-r)$$

for some constant K^a, then conclusion is same as that of Proposition 4.1.

PROOF: Since

$$f^{(1)}\overset{a}{\eta}{}^i = \eta \cdot \overset{a}{}{}^i f = 0$$

$$d_f \alpha = d_f \alpha_v$$

$$= (f^{(1)}d - df^{(1)}) \, (\overset{a}{\eta}{}^1 \wedge \overset{a}{\eta}{}^2 ... \overset{a}{\eta}{}^v)$$

$$= f^{(1)}\{d\overset{a}{\eta}{}^1 \wedge \overset{a}{\eta}{}^2 ... \overset{a}{\eta}{}^v - \overset{a}{\eta}{}^1 \wedge d(\overset{a}{\eta}{}^2 \wedge ... \wedge \overset{a}{\eta}{}^v)\}$$

$$= f^{(1)}(\overset{a}{K}{}^1 F > \overset{a}{\eta}{}^2 ... \overset{a}{\eta}{}^v) - f^{(1)}(\overset{a}{\eta}{}^1 \wedge d(\overset{a}{\eta}{}^2 \wedge ... \wedge \overset{a}{\eta}{}^v)$$

First of these terms equals:

$$= \frac{1}{v-1!}\{\overset{a}{K}{}^1 f^{(1)}F \wedge \overset{a}{\eta}{}^2 ... \overset{a}{\eta}{}^v - \overset{a}{K}{}^1 F \wedge f^{(1)}\overset{a}{\eta}{}^2 ...$$

$$\wedge \overset{a}{\eta}{}^v ... (-1)^{v-1}\overset{a}{K}{}^1 F \wedge \overset{a}{\eta}{}^2 ... \wedge f^{(1)}\overset{a}{\eta}{}^v\}$$

Each terms in the above expression is zero for $f^{(1)}F$ and $f^{(1)}\eta^i = \eta^i \cdot f$ are both zero.

Similarly second term which equals

$$f^{(1)}\overset{a}{\eta}{}^1 \wedge d(\overset{a}{\eta}{}^2 \wedge ... \wedge \overset{a}{\eta}{}^v) + \overset{a}{\eta}{}^1 \wedge f^{(1)}d(\overset{a}{\eta}{}^2 \wedge ... \wedge \overset{a}{\eta}{}^v)$$

can be shown to be zero.

Proposition 4.4 — If M is compact and $d\eta^a = K^a F$ then every vertical form α as well as $F\alpha$ is exact.

PROOF: Obvious from Prop. 4.3 and Cor. 4.2.

Remarks: The set of p-forms which are conservation laws with respect to structure f_i form a vector space say V_i^p. It is easy to check that

$$V_i^p \subset \wedge^p_{d_f}$$

In view of Lemma 2.2 we also note that if V^p denotes the vector space of p-forms which are conservation laws with respect to both structures, then

$$V^p \subset \wedge^p(M) \subset \wedge(M).$$

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