ON THE UNIQUENESS OF THE GREEN'S MATRIX ASSOCIATED WITH A PAIR OF SECOND ORDER DIFFERENTIAL EQUATIONS

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The second order matrix differential equation $(L - \lambda F)\phi = 0$, $0 \leqslant x < \infty$, with prescribed boundary conditions at $x = 0$ has been considered and it has been proved that under certain conditions to be satisfied the Green's matrix associated with it is unique.

§1. Let L denote the matrix operator

$$L \equiv \begin{pmatrix} \frac{d}{dx}\left(p_0(x)\frac{d}{dx}\right) + p_1(x) & r(x) \\ r(x) & \frac{d}{dx}\left(q_0(x)\frac{d}{dx}\right) + q_1(x) \end{pmatrix}, \qquad \ldots(1.1)$$

F the symmetric matrix

$$F \equiv F(x) = \begin{pmatrix} F_{11}(x) & F_{12}(x) \\ F_{21}(x) & F_{22}(x) \end{pmatrix} \qquad \ldots(1.2)$$

and $\phi \equiv \phi(x)$ a vector having two components $u \equiv u(x)$ and $v \equiv v(x)$ represented as a column matrix $\begin{pmatrix} u \\ v \end{pmatrix}$

Then consider the matrix equation

$$(L - \lambda F)\,\phi = 0, \; 0 \leqslant x < \infty \qquad \ldots(1.3)$$

We shall assume that

(i) $p_0(x)$, $q_0(x)$ are real valued, possessing continuous derivatives of the first order in $0 \leqslant x < \infty$;

(ii) $p_1(x)$, $q_1(x)$ and $r(x)$ are real valued and continuous in $0 \leqslant x < \infty$;

(iii) $p_0(x)$, $q_0(x) > 0$ for $0 \leqslant x < \infty$;

(iv) the symmetric matrix F is real valued, continuous and positive definite in $0 \leqslant x < \infty$.

The boundary conditions to be satisfied at $x = 0$ by any solution $\phi(x, \lambda)$ of (1.3) are

$$[\phi(x,\lambda)\,\phi_j(0\,|\,x,\lambda)]=0,\ (j=1,2) \qquad \ldots(1.4)$$

where $[\phi_1\phi_2]=0$, ϕ_1, ϕ_2 being the boundary condition vectors at $x=0$.

The Green's matrix $G(x,y;\lambda)=\begin{pmatrix} G_{11} & G_{21} \\ G_{12} & G_{22}\end{pmatrix}$ for our problem has been defined by Bhagat (1969b) as

$$\left.\begin{aligned} G(x,y;\lambda) &= \begin{pmatrix} \Psi_{11}(x,\lambda) & \Psi_{21}(x,\lambda) \\ \Psi_{12}(x,\lambda) & \Psi_{22}(x,\lambda)\end{pmatrix}\begin{pmatrix} u_1(y,\lambda) & v_1(y,\lambda) \\ u_2(y,\lambda) & v_2(y,\lambda)\end{pmatrix}, 0\leqslant y < x \\ &= \begin{pmatrix} u_1(x,\lambda) & u_2(x,\lambda) \\ v_1(x,\lambda) & v_2(x,\lambda)\end{pmatrix}\begin{pmatrix} \Psi_{11}(y,\lambda) & \Psi_{12}(y,\lambda) \\ \Psi_{21}(y,\lambda) & \Psi_{22}(y,\lambda)\end{pmatrix}, x < y < \infty \end{aligned}\right\}\ldots(1.5)$$

The present paper is a sequel to the author's paper (Bhagat 1969b). The object of the present paper is to prove the uniqueness of the Green's matrix (1.5) under certain conditions. The method used here is the same as that of Sears (1950) and Titchmarsh's (1949) and notations used by the author (Bhagat 1969a, b) have been used here.

§2. In order to prove the uniqueness of the Green's matrix (1.5), we require the following two theorems.

Theorem 2.1 — Let $g \equiv g(x)=\begin{pmatrix} g_1 \\ g_2\end{pmatrix}$ be a solution of (1.3) such that $g^TF\bar{g}$ belongs to $L(0,\infty)$ and let the parameter λ be complex. Let $p_0(x)$, $q_0(x)$, $p_1(x)$, $q_1(x)$, $r(x)$ and F satisfy

(i) $|p_1(x)|$, $|q(x)|$, $|r(x)|\leqslant Q(x)$, where $Q(x)\geqslant\delta>0$;

(ii) $Q'(x)$ exists and is continuous;

(iii) $Q'(x)$ and $Q(x)$ satisfy

$$\underset{x\to\infty}{\mathrm{Lt}}\left|\frac{Q'(x)}{Q^c(x)}\right|<\infty \qquad (0<c\leqslant\frac{3}{2});$$

(iv) $p_0(x)$, $q_0(x)\geqslant 1$ and $\dfrac{p_0(x)}{p_0'(x)}$, $\dfrac{q_0(x)}{q_0'(x)}\geqslant\eta>0$;

(v) $F_{ij}'(x)$ exists and $\underset{x\to\infty}{\mathrm{Lt}}\dfrac{F_{ij}'(x)}{F_{ij}(x)}$ is finite $(1\leqslant i,\ j\leqslant 2)$;

(vi) $t(x) \leqslant F_{ij}(x) \leqslant S(x)$ $(1 \leqslant i, j \leqslant 2)$ and $\frac{S(x)}{t(x)}$ tends to a finite limit not zero as x tends to infinity;

(vii) $\frac{Q(x)}{S(x)} \to \infty$ as $x \to \infty$.

Then

$$\int_0^\infty \frac{\bar{g}'^T(x) F \bar{g}'(x)}{Q(x)} dx < \infty$$

We have by (1.3) and integration by parts

$$\int_0^R \left(1 - \frac{x}{R}\right) \frac{1}{Q(x)} \left[\frac{1}{p_0} (F_{11}\bar{g}_1 + F_{21}\bar{g}_2)(\lambda F_{11} g_1 + (\lambda F_{12} g_2 - p_1 g_1 - r g_2) \right.$$

$$\left. + \frac{1}{q_0} (F_{12}\bar{g}_1 + F_{22}\bar{g}_2)(\lambda F_{21} g_1 + \lambda F_{22} g_2 - q_1 g_2 - r g_1) \right] dx$$

$$= -\frac{1}{Q(0)} \bar{g}^T(0) F g'(0) - \int_0^R \left(1 - \frac{x}{R}\right) \frac{1}{Q(x)} g'^T F \bar{g}' dx + \frac{1}{R} \int_0^R \frac{1}{Q(x)} \bar{g}^T F g' dx$$

$$+ \int_0^R \left(1 - \frac{x}{R}\right) \frac{Q'(x)}{Q^2(x)} \bar{g}^T F g' dx + \int_0^R \left(1 - \frac{x}{R}\right) \frac{1}{Q(x)} \left[F_{11} \frac{p_0'}{p_0} \bar{g}_1 g_1' + F_{12} \frac{q_0'}{q_0} \bar{g}_1 g_2' \right.$$

$$\left. + F_{21} \frac{p_0'}{p_0} \bar{g}_2 g_1' + F_{22} \frac{q_0'}{q_0} \bar{g}_2 g_2' \right] dx - \int_0^R \left(1 - \frac{x}{R}\right) \frac{1}{Q(x)} \bar{g}^T F' g' dx \qquad \ldots (2.1)$$

Let $g_k = \alpha_k + i\beta_k$, $\lambda = \mu + i\nu$, where $i = \sqrt{-1}$ and let

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \alpha_* = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \beta_* = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}, \Upsilon = \begin{pmatrix} \frac{p_0'}{p_0} F_{11} & \frac{q_0'}{q_0} F_{12} \\ \frac{p_0'}{p_0} F_{21} & \frac{q_0'}{q_0} F_{22} \end{pmatrix}$$

Then equating real parts from both sides of (2.1) we have

$$\int_0^R \left(1-\frac{x}{R}\right)\frac{1}{Q(x)}\, g'^T F \bar{g}'\, dx = -re\frac{1}{Q(0)}\bar{g}^T(0)F(0)g'(0)$$

$$+\frac{1}{R}\int_0^R \frac{1}{Q(x)}(\alpha^T F\alpha' + \beta^T F\beta')dx + \int_0^R \left(1-\frac{x}{R}\right)\frac{Q'(x)}{Q^2(x)}(\alpha^T F\alpha' + \beta^T F\beta')\, dx$$

$$+\int_0^R \left(1-\frac{x}{R}\right)\frac{1}{Q(x)}(\alpha^T \Upsilon\alpha' + \beta^T \Upsilon\beta')\, dx$$

$$-\int_0^R \left(1-\frac{x}{R}\right)\frac{1}{Q(x)}(\alpha^T F'\alpha' + \beta^T F'\beta')\, dx$$

$$-\mu\int_0^R \left(1-\frac{x}{R}\right)\frac{1}{Q(x)}\Bigg[\left(\frac{F_{11}^2}{p_0}+\frac{F_{12}^2}{q_0}\right)\alpha_*^T\alpha_* + \left(\frac{F_{12}^2}{p_0}+\frac{F_{22}^2}{q_0}\right)\beta_*^T\beta_*$$

$$+ 2\left(\frac{F_{11}F_{12}}{p_0}+\frac{F_{21}F_{22}}{q_0}\right)\alpha_*^T\beta_*\Bigg]dx$$

$$+\int_0^R \left(1-\frac{x}{R}\right)\frac{1}{Q(x)}\left[\frac{p_1}{p_0}F_{11}\alpha_*^T\alpha_* + F_{12}\left(\frac{p_1}{p_0}+\frac{q_1}{q_0}\right)\alpha_*^T\beta_* + \frac{q_1}{q_0}F_{22}\beta_*^T\beta_*\right]dx$$

$$+\int_0^R \left(1-\frac{x}{R}\right)\frac{r}{Q(x)}\left[\left(\frac{F_{11}}{p_0}+\frac{F_{22}}{q_0}\right)\alpha_*^T\beta_* + \frac{F_{12}}{q_0}\alpha_*^T\alpha_* + \frac{F_{21}}{p_0}\beta_*^T\beta_*\right]dx$$

$$= A + I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 \text{ (say)}. \qquad \ldots(2.2)$$

On integration by parts

$$I_1 = \frac{1}{2R}\left\{\frac{g^T(R)F\bar{g}(R)}{Q(R)} - \frac{g^T(0)F(0)\bar{g}(0)}{Q(0)}\right\} + \frac{1}{2R}\int_0^R \frac{Q'(x)}{Q^2(x)}\, g^T F\bar{g}\, dx$$

$$-\frac{1}{2R}\int_0^R \frac{1}{Q(x)} g^T F' \bar{g}\, dx = I_{11} + I_{12} + I_{13} \text{ (say)}.$$

As $g^T F \bar{g}$ belongs to $L[0,\infty)$, $g^T F \bar{g} < B$, where B is some constant.

Also $Q(R) \geqslant \delta > 0$.

Hence $I_{11} = O\left(\frac{1}{R}\right)$

By condition (iii)

$$I_{12} = \frac{1}{2R}\int_0^R \frac{Q'(x)}{Q^c(x)} \cdot \frac{1}{Q^{2-c}(x)} g^T F \bar{g}\, dx = O\left(\frac{1}{R}\right)$$

By condition (v)

$$I_{13} = O\left\{\frac{1}{2R}\int_0^R g^T F \bar{g}\, dx = O\left(\frac{1}{R}\right)\right.$$

Thus altogether

$$I_1 = O\left(\frac{1}{R}\right) \qquad \ldots(2.3)$$

Applying the inequality

$$|\, a^T F b \,| \leqslant \frac{1}{2}(a^T F a + b^T F b) \qquad \ldots(2.4)$$

where $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$,

we have

$$|\, I_2 \,| < \frac{1}{2}\int_0^R \left(1 - \frac{x}{R}\right)\frac{Q'^2(x)}{Q^{2c}(x)} g^T F \bar{g}\, dx + \frac{1}{2}\int_0^R \left(1 - \frac{x}{R}\right)\frac{1}{Q^{4-2c}(x)} g'^T F \bar{g}'\, dx$$

Now since

$$\int_0^R \left(1 - \frac{x}{R}\right) g^T F \bar{g} \; dx < \int_0^R \left(1 - \frac{x}{R'}\right) g^T F \bar{g} \; dx, \text{ for } R' > R,$$

we obtain by making first R' tend to infinity and then R tend to infinity

$$\underset{R \to \infty}{Lt} \int_0^R \left(1 - \frac{x}{R}\right) g^T F \bar{g} \; dx > \infty \qquad \ldots(2.5)$$

Therefore, by condition (iii)

$$\int_0^R \left(1 - \frac{x}{R}\right) \frac{Q'^2(x)}{Q^{2c}(x)} g^T F \bar{g} \; dx = O(1)$$

Thus, as $Q^{3-2c}(x) \geqslant \delta^{3-2c}$, $0 < c \leqslant \frac{3}{2}$, we have

$$| I_2 | < \frac{1}{2\delta^{3-2c}} \int_0^R \left(1 - \frac{x}{R}\right) \frac{g'^T F \bar{g}'}{Q(x)} dx + O(1) \qquad \ldots(2.6)$$

Again applying the inequality (2.4) and condition (iv) and (2.5)

$$| I_3 | < \frac{1}{2\eta} \int_0^R \left(1 - \frac{x}{R}\right) \frac{1}{Q(x)} g'^T F \bar{g}' \; dx + O(1) \qquad \ldots(2.7)$$

By condition (v) and inequality (2.4)

$$| I_4 | < \frac{M}{2} \int_0^R \left(1 - \frac{x}{R}\right) \frac{1}{Q(x)} g^T F \bar{g} \; dx + \frac{M}{2} \int_0^R \left(1 - \frac{x}{R}\right) \frac{1}{Q(x)} g'^T F \bar{g}' \; dx$$

$$= O(1) + \frac{M}{2} \int_0^R \left(1 - \frac{x}{R}\right) \frac{1}{Q(x)} g'^T F \bar{g}' \; dx, \text{ by condition (i) and (2.5).} \qquad \ldots(2.8)$$

By (2.5) and conditions (iv), (vi), (vii)

$$| I_5 | \leqslant | \mu | \int_0^R \left(1 - \frac{x}{R} \right) \left| \frac{S(x)}{Q(x)} \right| g^T F \bar{g} \, dx + | \mu | \int_0^R \left(1 - \frac{x}{R} \right)$$

$$\times \left| \frac{S(x)}{Q(x)} \cdot \frac{S(x)}{l(x)} \right| g^T F \bar{g} \, dx < | \mu | \, \varepsilon \int_0^R \left(1 - \frac{x}{R} \right) g^T F \bar{g} \, dx$$

$$+ | \mu | \, \varepsilon K \int_0^R \left(1 - \frac{x}{R} \right) g^T F \bar{g} \, dx = O(1) \qquad \ldots(2.9)$$

Finally by (2.5) and conditions (i), and (iv)

$$| I_6 | = O(1) \qquad \ldots(2.10)$$

and by conditions (i) and (vi)

$$| I_7 | \leqslant \int_0^R \left(1 - \frac{x}{R} \right) \left| \frac{S(x)}{l(x)} \right| g^T F \bar{g} \, dx = O(1) \qquad \ldots(2.11)$$

Now substituting for $I_1, I_2, \ldots, I_7$ in (2.2), we have

$$\int_0^R \left(1 - \frac{x}{R} \right) \frac{1}{Q(x)} g'^T F \bar{g}' dx = O(1) \qquad \ldots(2.12)$$

Now

$$\int_0^R \left(1 - \frac{x}{R} \right) \frac{1}{Q(x)} g'^T F \bar{g}' \, dx = \left\{ \int_0^{\frac{1}{2}R} + \int_{\frac{1}{2}R}^R \right\} \left(1 - \frac{x}{R} \right) \frac{1}{Q(x)} g'^T \bar{g}' dx$$

$$\geqslant \frac{1}{2} \int_0^{\frac{1}{2}R} \frac{g'^T F \bar{g}'}{Q(x)} dx$$

Thus

$$\int_0^{\frac{1}{2}R} \frac{g'^T F \bar{g}'}{Q(x)} dx \leqslant 2 \int_0^R \left(1 - \frac{x}{R} \right) \frac{1}{Q(x)} g'^T F \bar{g}' \, dx = O(1)$$

Hence the result follows by making $R \to \infty$.

In particular, if $\det F \geqslant 1$,

$$\int_0^\infty \frac{|g'|^2}{Q(x)} dx < \infty \qquad \ldots(2.13)$$

Theorem 2.2—Let $h \equiv h(x) = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ and $g \equiv g(x) = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ be solutions of (1.3) for two distinct values $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively of λ such that $g^T F \bar{g}$ and $h^T F \bar{h}$ both belong to $L[0, \infty)$ and g and h both satisfy the boundary conditions (1.4) at $x = 0$. Let

$$P(x) = \int_0^x \left\{ Q(x) \right\}^{-\frac{1}{2}} dt$$

(where $Q(x)$ is defined in Theorem 2.1) be divergent. Also let

$$\det F(x) \geqslant \{ \max(p_0(x), q_0(x)) \}^2$$

Then

$$\int_0^\infty h^T L g \, dx = \int_0^\infty g^T L h \, dx \qquad \ldots(2.14)$$

Obviously under the given conditions the integral on each side exists. By Green's theorem and Theorem 3.1 of Bhagat (1969a), we have

$$\int_0^x (h^T L g - g^T L h) dt = [gh](x) \qquad \ldots(2.15)$$

Let $H(x) = \max(p_0(x), q_0(x))$

Then

$$\begin{aligned} |[gh](x)| &= |p_0(g_1' h_1 - g_1 h_1') + q_0(g_2' h_2 - g_2 h_2')| \\ &\leqslant H(x) \, | g_1' h_1 - g_1 h_1' + g_2' h_2 - g_2 h_2' | \\ &\leqslant \{H(x)(g^T \bar{g} + h^T \bar{h}) H(x)(g'^T \bar{g}' + h'^T \bar{h}')\}^{1/2}, \text{ by Cauchy's inequality,} \\ &\leqslant (g^T F \bar{g} + h^T F \bar{h})^{\frac{1}{2}} (g'^T F \bar{g}' + h'^T F \bar{h}')^{\frac{1}{2}} \end{aligned}$$

(See Mirsky 1955, prob. 37, p. 426).

Therefore, by Schwarz's inequality, we have

$$\int_0^R P'(x)[gh](x)\,dx \leqslant \left\{ \int_0^R (g^T F\bar{g} + h^T F\bar{h})\,dx \int_0^R \frac{1}{Q(x)}(g'^T F\bar{g}' + h'^T F\bar{h}')\;dx \right\}^{\frac{1}{2}}$$

$$= O(1)$$

Hence

$$\int_0^\infty P'(x)\,[gh]\,(x)\,dx < \infty \qquad \ldots(2.16)$$

Now multiplying (2.15) by $P'(x)$ and integrating between the limits 0 to R, we obtain by (2.16)

$$\int_0^R (P(R) - P(t))(h^T Lg - g^T Lh)\,dt = O(1) \qquad \ldots(2.17)$$

Let $\theta(x) = h^T Lg - g^T Lh$.
Then it is clear that $\theta(x)$ belongs to $L[0,\infty)$
We have, if $R' < R$,

$$\left| \int_0^R \frac{P(x)}{P(R)}\theta(x)dx \right| \leqslant \int_0^{R'} \frac{P(x)}{P(R)} \mid \theta(x) \mid dx + \int_{R'}^R \frac{P(x)}{P(R)} \mid \theta(x) \mid dx$$

$$\leqslant \frac{P(R')}{P(R)} \int_0^{R'} \mid \theta(x) \mid dx + \int_{R'}^R \mid \theta(x) \mid dx$$

We can make the second integral on the right as small as we please by properly choosing R' and after choosing R', the first term tends to zero as R tends to infinity. Thus

$$\int_0^R \frac{P(x)}{P(R)}\theta(x)\,dx \to 0 \text{ as } R \to \infty \qquad \ldots(2.18)$$

Since $P(R) \to \infty$, dividing (2.17) by $P(R)$ we obtain

$$\underset{R\to\infty}{\text{Lt}} \int_0^R \left(1 - \frac{P(x)}{P(R)} \right)\theta(x)\,dx = 0 \qquad \ldots(2.19)$$

Hence the result follows by (2.18) and (2.19).

§3. We now prove the uniqueness of the Green's matrix (1.5).

Theorem — If all the conditions of Theorems 2.1 and 2.2 are satisfied, then the Green's matrix $G(x,y;\lambda)$ defined by (1.5) is unique.

If possible, let us suppose that there are two Green's matrices $G^{(1)}(x,y;\lambda)$ and $G^{(2)}(x,y;\lambda)$ satisfying the properties of a Green's matrix, where the conditions laid down in the theorem are satisfied. Then the matrix $f(x,\lambda)$ given by

$$f(x,\lambda) = G^{(1)}(x,y;\lambda) - G^{(2)}(x,y;\lambda)$$

y given and λ not real, is different from the zero matrix.

The vectors

$$f_k(x,\lambda) = G_k^{(1)}(x,y;\lambda) - G_k^{(2)}(x,y;\lambda) \qquad (k = 1,\ 2)$$

satisfy the equation (1.3) and the boundary conditions (1.4) at $x = 0$ and are such that $f_k^T F \bar{f}_k$ belongs to $L[0,\infty)$.

Putting $g = f_k$ and $h = \bar{f}_k$ in (2.14) we get

$$(\lambda - \bar{\lambda}) \int_0^\infty f_k^T F \bar{f}_k \, dx = 0$$

Hence

$$\int_0^\infty f_k^T F \bar{f}_k \, dx = 0$$

and consequently f_k vanish identically, since F is positive definite. Hence f is a zero matrix.

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