CLAMPED REGULAR CURVILINEAR POLYGONAL PLATES NORMALLY LOADED OVER A CONCENTRIC CIRCLE

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Exact solutions in finite terms are obtained for the small deflections and the clamping couples of certain thin clamped regular curvilinear polygonal plates subject to unsymmetrical loading distributed over the area of a concentric circle. The load considered includes the special case of a linearly varying load and tends to a couple nucleus at the centre as the radius of the loaded circle tends to zero.

1. Introduction

Stevenson (1943) found the deflection of a clamped and uniformly loaded regular curvilinear polygonal plate with n sides. The plate, taken in the z-plane, is mapped on the area inside the unit circle γ in the ζ-plane by the mapping function

$$z(\zeta) = c\zeta(1 + m\zeta^n) \quad \ldots(1)$$

where $z = re^{i\theta}$, $\zeta = \rho e^{i\psi}$, c, m are real positive constants, n is a positive integer and $0 \leqslant m(n+1) \leqslant 1$. When the same plate is acted upon normally by a concentrated load at any point, a couple nucleus at the centre or is symmetrically loaded over the area of a concentric circle the appropriate solutions were derived by Dawoud (1950) and Bassali (1958a, 1959a). Similar results corresponding to plates mapped on the area inside γ by the conformal transformation

$$z(\zeta) = c\zeta/(1 + m\zeta^n), \; c > 0 \quad \ldots(2)$$

$0 \leqslant m(n-1) \leqslant 1$, $n \geqslant 2$ and subject to the same loadings were also obtained by Bassali (1959a, 1959b, 1960). The shapes of some plates corresponding to certain values of m and n are sketched in the foregoing papers and in either case we have a regular curvilinear polygonal plate having n sides and n rounded vertices which become cusps when the critical points of the conformal transformations, given by $z'(\zeta)=0$, fall upon the boundary Γ. Similar problems connected with the more general mapping functions

$$z(\zeta) = c\zeta/(1 + \lambda_1\zeta^n+\lambda_2\zeta^{2n}), \; c > 0 \quad \ldots(3)$$

$$z(\zeta) = c\zeta \sum_{\nu=0}^{m} \lambda_\nu \zeta^{\nu n}, \; c > 0, \; \lambda_0 = 1 \qquad \ldots(4)$$

have been discussed by Bassali (1960), Bassali and Hanna (1961a, b) and Bassali and Barsoum (1966).

This paper is concerned with the transverse flexure of thin plates conformally mapped on the area inside γ by means of (1) or (2) when these plates are acted upon normally by a certain unsymmetrical loading spread over a circle concentric with the plate. The load considered includes the hydrostatic pressure as a special case and tends to a couple nucleus at the centre of the plate as the radius of the loaded circle tends to zero. It is found that the two special cases $n = 1, 2$ require separate treatments and that the case of a clamped circular plate under hydrostatic pressure over an eccentric circle is included as a particular case of (2).

2. Basic Equations and Mathematical Formulation of the Flexural Problem

Let Γ denote the boundary of a clamped thin isotropic plate of thickness $2h$, flexural rigidity D and mapped on the area inside γ by (1) or (2). We assume that the plate is subject to the transverse pressure

$$\left.\begin{aligned} p_1 &= p_0\, r^{s-3} \cos(\theta - \alpha) \quad (s \geqslant 3) \text{ for } r \leqslant b \\ p_2 &= 0 \text{ for } r > b \end{aligned}\right\} \qquad \ldots(5)$$

where p_0, s, α, b are real positive constants and the suffices 1 and 2 refer to the two regions $r \leqslant b$ and $r > b$ respectively (see Fig. 1).

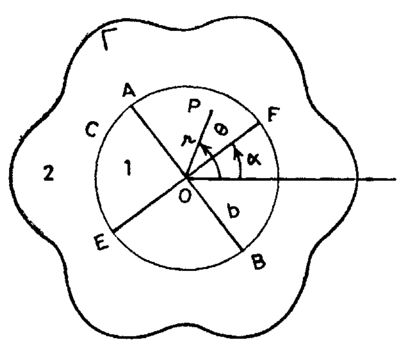

Fig. 1

The moment M of the loading (5) about the diameter AB of the loaded circle is

$$M = \pi\, p_0\, b^s/s \qquad \ldots(6)$$

For $s=4$ we have the case of a linearly varying load distributed over the area of the circle. If $p_0 \longrightarrow \infty$ and $b \longrightarrow 0$ such that $M \longrightarrow G$ we have the case of a couple nucleus G operating at the centre O of the plate considered, the plane of the couple being normal to the plate through EF.

The small deflections w_1 and w_2, measured positively upwards, at any point z of the mid-plane of the plate are given by

$$w_j = \bar{z}\Omega_j(z) + z\bar{\Omega}_j(\bar{z}) + \omega_j(z) + \bar{\omega}_j(\bar{z}) + W_j(z, \bar{z}) \ (j=1, 2) \qquad \ldots(7)$$

where the complex potentials $\Omega_j(z)$, $\omega_j(z)$ $(j=1, 2)$ are regular in their domains and $W_j(z, \bar{z})$ are particular integrals of the biharmonic equation

$$\nabla^4 w_j = 16\partial^4 w_j/\partial z^2 \partial \bar{z}^2 = -p_j(z, \bar{z})/D \qquad \ldots(8)$$

bars being used to denote conjugate complex quantities. It is easily seen that

$$W_1(z, \bar{z}) = -\frac{p_0(z\bar{z})^{s/2}}{2s^2(s^2-4)D}(\bar{\lambda}z + \lambda\bar{z}), \ W_2(z, \bar{z}) = 0 \qquad \ldots(9)$$

where $\quad \lambda = e^{i\alpha} \qquad \ldots(10)$

The conditions for the plate to be clamped along Γ are

$$w = 0, \ \partial w/\partial n = 0 \text{ on } \Gamma \qquad \ldots(11)$$

where $\partial/\partial n$ denotes differentiation along the outward drawn normal. In terms of the complex potentials of region 2 these conditions take the forms

$$\left.\begin{aligned} \bar{z}(\sigma^{-1})\Phi_2(\sigma) + z(\sigma)\bar{\Phi}_2(\sigma^{-1}) + \chi_2(\sigma) + \bar{\chi}_2(\sigma^{-1}) = 0 \\ \bar{z}(\sigma^{-1})\Phi'_2(\sigma) + z'(\sigma)\bar{\Phi}_2(\sigma^{-1}) + \chi'_2(\sigma) = 0 \end{aligned}\right\} \qquad \ldots(12)$$

where

$$\Omega_j(z) = \Phi_j(t), \ \ \omega_j(z) = \chi_j(t) \ (j=1, 2) \qquad \ldots(13)$$

and $\sigma = e^{i\psi}$ is any point on γ.

We have now to determine the four complex potentials (13) which satisfy the boundary conditions (12) and the continuity conditions along the circumference C of the loaded circle. It is shown that these continuity conditions (Bassali 1968b) are satisfied by taking

$$\begin{aligned} \Big[\,\Omega\,\Big]_2^1 &= K\left(\frac{\bar{\lambda}z^2}{2b_1^2} - \lambda \log\frac{z}{b}\right) \\ \Big[\,\omega\,\Big]_2^1 &= -K\left(\frac{2\bar{\lambda}z}{s} + \frac{\lambda b'^2}{2z} + \bar{\lambda}z \log\frac{z}{b}\right) \end{aligned} \qquad \ldots(14)$$

where

$$K = M/(16\ \pi D),\ b'^2 = sb^2/(s+2),\ b_1^2 = (s-2)b^2/s \qquad \ldots(15)$$

We now assume that

$$\Omega_2 = K[F(\zeta) + \lambda \log \zeta],\ \omega_2 = K[cf(\zeta) + \lambda b'^2/2z + \bar{\lambda}\, z \log \zeta] \qquad \ldots(16)$$

where $F(\zeta)$ and $f(\zeta)$ are regular functions of ζ ($|\zeta| \leqslant 1$). Substituting from (16) in (14) leads to

$$\left.\begin{aligned} \Omega_1 &= K[F(\zeta) + (\bar{\lambda} z^2/2b_1^2) - \lambda \log (z/b\zeta)] \\ \omega_1 &= K[cf(\zeta) - (2\bar{\lambda} z/s) - \bar{\lambda} z \log (z/b\zeta)] \end{aligned}\right\} \qquad \ldots(17)$$

With the values (1) or (2) for $z(\zeta)$ the four complex potentials (16) and (17) are regular in their domains 1 and 2. Introducing (9), (16) and (17) in (7) yields

$$\left.\begin{aligned} kw_1 &= kw_0 + \frac{2r}{c} \cos(\theta - \alpha)\ g(r,\ \rho) \\ kw_2 &= kw_0 + \frac{2r}{c} \cos(\theta - \alpha) \log \rho + \nu c\ Re \frac{\lambda}{z} \end{aligned}\right\} \qquad \ldots(18)$$

where

$$k = 8\pi D/cM, \qquad \nu = b'^2/2c^2 \qquad \ldots(19)$$

$$g(r, \rho) = \log \frac{\rho\, b}{r} + \frac{r^2}{4b_1^2} - \frac{1}{s} - \frac{4(r/b)^s}{s(s^2-4)} \qquad \ldots(20)$$

$$kw_0 = Re\left[\, f(\zeta) + \frac{\bar{z}}{c} F(\zeta) \,\right] \qquad \ldots(21)$$

If n and s are unit vectors normal and tangential to any clamped boundary, then the torsional couples vanish round the boundary and the bending couples are given by the simple formulae (Stevenson 1943, p. 109)

$$\widetilde{ns} = -P\nabla^2 w,\ \widetilde{sn} = \eta P \nabla^2 w \qquad \ldots(22)$$

where $P = D/2h$ and η is Poisson's ratio. Substitution for w_2 from (7) gives

$$\widetilde{ns} = -8P\ Re\, \Omega'_2(z) = -8P\ Re\left[\frac{\Phi'_2(\zeta)}{z'(\zeta)}\right]_{\zeta = \sigma} \qquad \ldots(23)$$

The moments and shears at any point of the plate are given by equations (55) p. 501 of Bassali (1960).

The complete solution of the problem now lies in the determination of the two unknown regular functions $F(\zeta)$ and $f(\zeta)$ ($|\zeta| \leqslant 1$) which satisfy the boundary conditions (12). Substituting from (13) and (16) in (12) we get

$$\left.\begin{aligned} &\bar{z}(\sigma^{-1})\,F(\sigma)+z(\sigma)\,\bar{F}(\sigma^{-1})+f(\sigma)+\bar{f}(\sigma^{-1})+\frac{\lambda\nu c^2}{z(\sigma)}+\frac{\bar{\lambda}\nu c^2}{\bar{z}(\sigma^{-1})}=0\\ &z'(\sigma)\bar{F}(\sigma^{-1})+\bar{z}(\sigma^{-1})F'(\sigma)+f'(\sigma)+\frac{\lambda}{\sigma}\bar{z}(\sigma^{-1})+\frac{\bar{\lambda}}{\sigma}z(\sigma)-\frac{\lambda\nu c^2 z'(\sigma)}{z^2(\sigma)}=0 \end{aligned}\right\} \quad \ldots(24)$$

Introducing (1) or (2) in (24) we see that they can be solved for $F(\zeta)$ and $f(\zeta)$ by applying Muskhelishvili's direct method (Muskhelishvili 1933) but it is easier to use the tentative method which was extensively applied by Stevenson (1943) and Bassali (1956–1966).

3. First Mapping Function

Case (i)—We now consider the case $n > 2$ of the mapping function

$$z(\zeta)=c\zeta(1+m\zeta^n)\quad (|m|(n+1)\leqslant 1)$$

In this case we assume tentatively that

$$\left.\begin{aligned} F(\zeta)&=a_2\zeta^2+a_n\zeta^n+a_{n+2}\zeta^{n+2}\\ f(\zeta)&=b_1\zeta+b_{n-1}\zeta^{n-1}+b_{n+1}\zeta^{n+1}+\frac{B\zeta^{n-1}}{1+m\zeta^n} \end{aligned}\right\} \quad \ldots(25)$$

where a_2, a_n, a_{n+2}, b_1, b_{n-1}, b_{n+1} and B are constants to be determined. Substituting in (24), clearing out fractions and equating the coefficients of various powers of σ to zero we get a consistent system of equations which gives

$$\left.\begin{aligned} &a_2=N[(n+1)m^2+\nu-1]\bar{\lambda},\ a_n=2mN[1-\nu-(n+1)m^2]\lambda,\ a_{n+2}=-m\bar{\lambda}\\ &b_1=N\{1-(n+2)m^2+2m^4-2\nu[1-(n+1)m^2]\}\,\bar{\lambda}\\ &b_{n-1}=mN[(n+1)m^2+\nu-1]\lambda,\ b_{n+1}=m\bar{\lambda},\ B=m\nu\lambda,\ N=(1-2nm^2)^{-1} \end{aligned}\right\} \quad \ldots(26)$$

When these values are introduced in (25) and the results are inserted in (21), it is found, after some algebraic manipulation, that equations (18) give

$$\begin{aligned} kw_1=&\frac{2r}{c}\cos(\theta-\alpha)\,g(r,\rho)+m\nu c\ Re\,\frac{\lambda\zeta^n}{z}+mt_1\,\rho^{n+1}\cos(\overline{n+1}\,\psi-\alpha)\\ &+\frac{mt_1{}^2\,\rho^{n-1}}{1-2nm^2}\,[(n+1)m^2+\nu-1]\,\cos(\overline{n-1}\,\psi+\alpha)\\ &+\left\{\frac{t_1-2m^2t_n}{1-2nm^2}\,[1-\nu-(n+1)m^2]+m^2\,t_{n+1}-\nu\right\}\rho\,\cos(\psi-\alpha) \end{aligned} \quad \ldots(27a)$$

$$\begin{aligned} kw_2=&\frac{2r}{c}\cos(\theta-\alpha)\,\log\,\rho+mt_1\,\rho^{n+1}\cos(\overline{n+1}\,\psi-\alpha)\\ &+\frac{m\,t_1{}^2\,\rho^{n-1}}{1-2nm^2}\,[(n+1)m^2+\nu-1]\cos(\overline{n-1}\,\psi+\alpha)+ \end{aligned}$$

(*equation continued on p.* 1227)

$$+ \left\{ \frac{t_1 - 2m^2 t_n}{1 - 2n m^2} [1 - \nu - (n+1)m^2] + m^2 t_{n+1} - \nu t_{-1} \right\} \rho \cos(\psi - \alpha) \quad \ldots (27b)$$

where

$$t_n = 1 - \rho^{2n} \quad \ldots (28)$$

For $\nu = 0$ the expression (27b) reduces to that given by Bassali [1959a, p. 134, eqn. (7.6)].

The formula (23) for the clamping couple along Γ now yields

$$\tilde{ns} = -\frac{M}{4\pi ch} \left\{ \frac{n(n+1)(1 + 2m^2 - 2\nu)m^2 + 2\nu - 1 - 2nm^2}{1 - 2nm^2} \cos(\psi - \alpha) \right.$$

$$\left. - m\cos(\overline{n+1}\,\psi - \alpha) + \frac{2m[(n+1)m^2 + \nu - 1]}{1 - 2n m^2} \cos(\overline{n-1}\,\psi + \alpha) \right\} \Big| [1 + m^2(n+1)^2$$

$$+ 2m(n+1)\cos n\psi] \quad \ldots (29)$$

Case (ii)—For $n = 1$ we have

$$z(\zeta) = c\zeta(1 + m\zeta) \quad (|m| \leqslant \tfrac{1}{2})$$

$$F(\zeta) = \zeta(a_0 + a_1\zeta + a_2\zeta^2), \quad f(\zeta) = b_0 + b_1\zeta + b_2\zeta^2 + \frac{B}{1 + m\zeta} \quad \ldots (30)$$

where a_0 and b_0 may be taken as real. The boundary conditions (24) along Υ lead to

$$\left. \begin{aligned} a_0 &= \frac{2m\cos\alpha}{1 - 2m^2}(1 - \nu - 2m^2), \; a_1 = \frac{(m^2-1)\bar{\lambda} - m^2\lambda}{1 - 2m^2}(1 - \nu - 2m^2), \; a_2 = -m\bar{\lambda} \\ b_0 &= \frac{m\cos\alpha}{1 - 2m^2}(2m^2 + \nu - 1), \; b_1 = (1 - 2\nu - m^2)\bar{\lambda}, \; b_2 = m\bar{\lambda}, \; B = m\nu\lambda \end{aligned} \right\} \quad \ldots (31)$$

The deflections w_1 and w_2 are given by

$$kw_1 = \frac{2r}{c}\cos(\theta - \alpha)\, g(r, \rho) + m\nu c\, Re\, \frac{\lambda\zeta}{z} + m\left[\frac{\nu}{1 - 2m^2} - 1\right] t_1^2 \cos\alpha$$

$$+ m\,\rho^2 t_1 \cos(2\psi - \alpha) + [(1 - \nu - m^2 t_1)t_1 - \nu]\,\rho\cos(\psi - \alpha) \quad \ldots (32a)$$

$$kw_2 = \frac{2r}{c}\cos(\theta - \alpha)\log\rho + t_1\left\{ mt_1\left[\frac{\nu}{1 - 2m^2} - 1\right]\cos\alpha + m\,\rho^2\cos(2\psi - \alpha) \right.$$

$$\left. + \rho[1 + (\nu/\rho^2 - m^2)t_1]\cos(\psi - \alpha) \right\} \quad \ldots (32b)$$

Setting $\nu = 0$ in (32*b*) gives an expression which agrees with equation (7.10) of Bassali (1959a). The clamping couple along Pascal's limacon Γ in this case is

$$\widetilde{ns} = -\frac{M}{4\pi ch(1+4m^2+4m\cos\psi)}\left[2m\left[\frac{\nu}{1-2m^2}-1\right]\cos\alpha + (2\nu-1-2m^2)\cos(\psi-\alpha) - m\cos(2\Psi-\alpha)\right] \quad \ldots(33)$$

Case (iii)—For $n = 2$ we have

$$z(\zeta) = c\zeta(1+m\zeta^2)\left(|m| \leqslant \frac{1}{3}\right)$$

Assuming that

$$F(\zeta) = \zeta^2(a_0 + a_2\zeta^2),\ f(\zeta) = \zeta\left(b_0 + b_2\zeta^2 + \frac{B}{1+m\zeta^2}\right) \quad \ldots(34)$$

we find that the resulting identities (24) are satisfied by

$$\left.\begin{aligned} a_0 &= \frac{\bar{\lambda} - 2m\lambda}{1-4m^2}(3m^2+\nu-1),\ a_2 = -m\bar{\lambda} \\ b_0 &= \bar{\lambda} + \frac{2m^4\bar{\lambda} + m\nu\lambda + (3m^2-1)(m\lambda+2\nu\bar{\lambda})}{1-4m^2},\ b_2 = m\bar{\lambda},\ B = m\nu\lambda \end{aligned}\right\} \quad \ldots(35)$$

and the corresponding expressions for the deflections and the clamping couples along Γ are

$$kw_1 = \frac{2r}{c}\cos(\theta-\alpha)\,g(r,\rho) + m\nu c\,Re\,\frac{\lambda\zeta^2}{z} + \frac{m\rho(3m^2-1+\nu)t_1^2}{1-4m^2}\cos(\psi+\alpha) + \left\{m^2t_3-\nu+\frac{3m^2-1+\nu}{1-4m^2}(2m^2t_2-t_1)\right\}\rho\cos(\psi-\alpha) + m\rho^3t_1\cos(3\psi-\alpha) \ldots(36a)$$

$$kw_2 = \frac{2r}{c}\cos(\theta-\alpha)\log\rho + \rho t_1\left[\frac{m(3m^2-1+\nu)t_1}{1-4m^2}\cos(\psi+\alpha) + \left\{\frac{\nu}{\rho^2}+\frac{3m^2-1+\nu}{1-4m^2}(2m^2+2m^2\rho^2-1) + (1+\rho^2+\rho^4)m^2\right\}\cos(\psi-\alpha) + m\rho^2\cos(3\psi-\alpha)\right] \quad \ldots(36b)$$

$$\widetilde{ns} = -\frac{M}{4\pi ch(1+9m^2+6m\cos 2\psi)} \times$$

$$\left[\frac{2m(3m^2+\nu-1)\cos(\psi+\alpha) + [12m^4+2m^2-1+2\nu(1-6m^2)]\cos(\psi-\alpha)}{1-4m^2} - \right.$$

(*equation continued on p. 1231*)

$$- m \cos (3\psi - \alpha)\Big] \qquad \ldots(37)$$

4. Second Mapping Function

We now turn our attention to the conformal transformation

$$z(\zeta) = c\zeta/(1 + m\zeta^n), \; |m|(n-1) \leqslant 1, \; n \geqslant 2$$

Case (i)—For $n > 2$ we have

$$F(\zeta) = a_0 + a_2\zeta^2 + \frac{A_0 + A_2\zeta^2}{1 + m\zeta^n}, \quad f(\zeta) = b_1\zeta + b_{n-1}\zeta^{n-1} + \frac{B\zeta^{n-1}}{1 + m\zeta^n} \qquad \ldots(38)$$

and it is found that the boundary conditions (24) are satisfied by

$$\left.\begin{array}{l} a_0 = (2\nu - 1)\lambda, \; a_2 = \nu\bar{\lambda} \\ A_0 = [2 + (n-2)m^2 - 2(1 + \overline{n-1}\, m^2)\,\nu]\lambda, \; A_2 = [nm^2(2\nu - 1) + m^2 - 1]\bar{\lambda} \\ b_1 = -2\nu\bar{\lambda}, \; b_{n-1} = -m\nu\lambda, \; B = m(1-\nu)\lambda. \end{array}\right\} \ldots(39)$$

The deflections and clamping couple along Γ are furnished by

$$kw_1 = \frac{2r}{c} \cos (\theta - \alpha)\, g(r, \rho) - \nu[2\rho \cos (\psi - \alpha) + m\rho^{n-1} \cos (\overline{n-1}\,\psi + \alpha)]$$

$$+ \frac{r^2}{\rho c^2} \Big\{ [(1 + \overline{n - 2\nu n - 1}\, m^2)t_1 - m^2(1-\nu)t_{n-1} + \nu(\rho^2 + m^2)] \cos (\psi - \alpha)$$

$$+ m(\overline{1 - 2\nu}\, t_1 + \nu)\rho^{n-2} \cos (\overline{n-1}\,\psi + \alpha) + m\nu\rho^{n+2} \cos (\overline{n+1}\,\psi - \alpha) \Big\} \ldots(40a)$$

$$kw_2 = \frac{2r}{c} \cos (\theta - \alpha) \log \rho + \frac{r^2}{\rho c^2} \Big\{ [(1 + \overline{n - 2\nu n - 1}\, m^2 - \nu t_{n-1})t_1$$

$$+ m^2(2\nu t_n - t_{n-1})] \cos (\psi - \alpha) + mt_1 \rho^{n-2} [\cos (\overline{n-1}\,\psi + \alpha)$$

$$+ \nu t_1 \cos (\overline{n+1}\,\psi - \alpha)] \Big\} \qquad \ldots(40b)$$

$$\tilde{ns} = -\frac{M}{4\pi ch} \Big\{ C_1 \cos (\psi - \alpha) + mC_2 \cos (\overline{n+1}\,\psi - \alpha) + mC_3 \cos (\overline{n-1}\,\psi + \alpha)$$

$$+ 2\nu m^2 \cos (\overline{2n+1}\,\psi - \alpha) + m^2 \cos (\overline{2n-1}\,\psi + \alpha) \Big\}$$

$$\Big/ [1 + m^2(n-1)^2 - 2m(n-1) \cos n\psi] \qquad \ldots(41)$$

where

$$\left.\begin{aligned} C_1 &= -1 + (n-1)(n-2)(1+m^2)m^2 + 2\nu[1-(n+1)(n-2)m^2 - n(n-1)m^4] \\ C_2 &= -1 + (n-1)(n-2)m^2 + 2\nu[2 + (1+n-n^2)m^2] \\ C_3 &= (3-3n+n^2)m^2 + 2\nu[1-n(n-1)m^2] \end{aligned}\right\} \quad \ldots(42)$$

For $\nu = 0$ it is verified that (40b) reduces to equation (6.6) of Bassali (1959a, p. 132).

Case (ii)—For $n = 1$ we have the bilinear transformation

$$z(\zeta) = \frac{c\zeta}{1+m\zeta} \quad (|m| < 1),$$

and Γ is a circle in the z-plane with centre $\left(-\frac{cm}{1-m^2}, 0\right)$ and radius $c/(1-m^2)$.

In this case we assume that

$$F(\zeta) = a_2\zeta^2 + \frac{A_0 + A_1\zeta + A_2\zeta^2}{1+m\zeta}, \qquad f(\zeta) = b_0 + b_1\zeta \qquad \ldots(43)$$

where A_1, b_0 are real. We easily find that the boundary conditions (24) lead to

$$\left.\begin{aligned} &a_2 = \nu\bar{\lambda},\ A_0 = (1-m^2)\lambda + (\nu-1)m^2\bar{\lambda},\ A_1 = m\,[(2\nu-1) + (\nu-1)m^2]\cos\alpha \\ &A_2 = (2\nu m^2 - 1)\bar{\lambda},\ b_0 = m(1-2\nu)\cos\alpha,\ b_1 = -2\nu\bar{\lambda} \end{aligned}\right\} \ldots(44)$$

and the deflections and clamping couple along the circular edge are

$$kw_1 = \frac{2r}{c}\cos(\theta-\alpha)\,g(r,\rho) + \frac{r^2}{\rho^2c^2}\Big[m(t_1 - 2\nu - \nu m^2\rho^2)\cos\alpha - \nu m^2\rho\cos(\psi+\alpha)$$

$$+ \rho(t_1 - 2\nu + \nu\rho^2 - 2\nu m^2)\cos(\psi-\alpha) + m\nu\rho^2(\rho^2-2)\cos(2\psi-\alpha)\Big] \quad \ldots(45a)$$

$$kw_2 = \frac{2r}{c}\cos(\theta-\alpha)\log\rho + \frac{r^2t_1}{\rho^2c^2}\Big[m\cos\alpha + (\rho + \nu t_1/\rho)\cos(\psi-\alpha)$$

$$+ m\nu t_1\cos(2\psi-\alpha)\Big] \quad \ldots(45b)$$

$$\overline{ns} = -\frac{M}{4\pi ch}\Big[m(2\nu+m^2)\cos\alpha + (2\nu-1+4\nu m^2)\cos(\psi-\alpha) + m^2\cos(\psi+\alpha)$$

$$+ m(4\nu-1+2\nu m^2)\cos(2\psi-\alpha) + 2\nu m^2\cos(3\psi-\alpha)\Big] \quad \ldots(46)$$

For $\alpha = 0$ and $s+1$ instead of s in (5) an equivalent solution of the problem in this case was given in a previous paper (Bassali 1956) where the bending of an elastically restrained circular plate normally loaded over an eccentric circle was discussed by means of another procedure.

Case (iii)—For $n=2$ we have

$$z(\zeta)=\frac{c\zeta}{1+m\zeta^2}\left(|m|\leqslant\frac{1}{3}\right)$$

$$F(\zeta)=a_2\zeta^2+\frac{A_0+A_2\zeta^2}{1+m\zeta^2},\qquad f(\zeta)=b_1\zeta \qquad \text{...(47)}$$

$$\left.\begin{array}{l} a_2=\nu\lambda,\ A_0=\lambda+m\bar{\lambda}-\nu m(2m\lambda+\bar{\lambda}),\ A_2=\nu m\,(2\lambda+3m\bar{\lambda})-m\lambda-\bar{\lambda} \\ b_1=-\nu(m\lambda+2\bar{\lambda}) \end{array}\right\} \qquad \text{...(48)}$$

$$kw_1=\frac{2r}{c}\cos(\theta-\alpha)\,g(r,\rho)+\frac{r^2}{\rho c^2}\Big[m(t_1-2\nu-\nu m^2\rho^4)\cos(\psi+\alpha)$$

$$+(t_1+\nu\rho^2-2\nu m^2-2\nu+2\nu m^2\rho^2t_1)\cos(\psi-\alpha)$$

$$-\nu m\rho^2\left\{m\cos(3\psi+\alpha)+(2-\rho^2)\cos(3\psi-\alpha)\right\}\Big] \qquad \text{...(49a)}$$

$$kw_2=\frac{2r}{c}\cos(\theta-\alpha)\log\rho+\frac{r^2t_1}{c^2\rho}\Big[m\cos(\psi+\alpha)+m\nu t_1\cos(3\psi-\alpha)$$

$$+\left\{1+\nu t_1(\rho^{-2}-2m^2)\right\}\cos(\psi-\alpha)\Big] \qquad \text{...(49b)}$$

$$\tilde{ns}=-\frac{M}{4\pi ch(1+m^2-2m\cos 2\psi)}\Big[C_1\cos(\psi-\alpha)+mC_2\cos(\psi+\alpha)$$

$$+mC_3\cos(3\psi-\alpha)+m^2\cos(3\psi+\alpha)+2\nu m^2\cos(5\psi-\alpha)\Big] \qquad \text{...(50)}$$

where

$$C_1=2\nu-1-4\nu m^4,\ C_2=m^2+2\nu-4\nu m^2,\ C_3=2\nu(2-m^2)-1. \qquad \text{...(51)}$$

5. The Special Case $m=0$

Setting $m=0$ in the previous results of the six cases considered we get the following formulae for the deflections and clamping couple in the case of a clamped circular plate of radius c subject to the normal loading (5) over the area of a concentric circle of radius b.

$$kw_1=\rho\cos(\theta-\alpha)\left[2\log\frac{b}{c}+\frac{r^2}{2b_1^2}-\frac{2}{s}-\frac{8(r/b)^s}{s(s^2-4)}+t_1-2\nu+\nu\rho^2\right] \qquad \text{...(52)}$$

$$kw_2=\rho\cos(\theta-\alpha)\left[2\log\rho+t_1(1-\nu t_{-1})\right] \qquad \text{...(53)}$$

$$\tilde{ns} = \frac{(1-2\nu)M}{4\pi ch} \cos(\theta-\alpha) \quad ...(54)$$

Detailed results concerning the deflections, moments and shearing forces at any point of an elastically restrained circular plate normally loaded over a concentric circle are given at the end of Bassali's paper (1956) and complete agreement with equations (52), (53) and (54) is checked on noting the difference in notation.

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