ON THE MODULATION OF THERMAL CONVECTION INSTABILITY OF A ROTATING FLUID

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(*Received* 24 *December* 1975)

The authors have investigated the linear stability of a rotating horizontal layer of fluid heated from below, when the temperature gradient has both a steady and time-periodic component. The modulating effects of the oscillating temperature field on the stability characteristics of the basic configuration have been examined by expansion technique in powers of the amplitude of the modulating temperature. The shift in the critical Rayleigh number has been calculated. It is found that the critical Rayleigh numbers at which the instability sets in are enhanced. It is also observed that the modulation is maximum as Ω, the frequency of the oscillating temperature $\to 0$.

1. INTRODUCTION

The effect of modulation on the thermal convection instability of a plane fluid layer heated from below due to the oscillating temperature of the boundaries has been studied by Venezian (1969) and Rosenblat and Tanaka (1971). Venezian treated the problem by a perturbation expansion in powers of the amplitude of oscillation while Rosenblat and Tanaka treated the problem by using the Galerkin technique and the Floquet theory to discuss the stability. Both of them have established that the critical Rayleigh numbers are enhanced by the modulation. Rosenblat and Herbert (1970) have discussed the same problem at low frequency limit with the help of periodicity criterion as well as the amplitude criterion. We have investigated the problem of thermal convection instability to include rotation by keeping the temperature of the upper wall fixed and varying the temperature of the lower wall sinusoidally. We have used the method of Venezian (1969) in expanding in powers of the amplitude. In Section 2 we formulate the problem and in Section 3 we obtain an expression for the shift in the critical Rayleigh number at which instability sets in. In Section 4 we establish the criterion for the least Rayleigh number and in Section 5 we evaluate it numerically and discuss the results.

2. Formulation of the Problem

We consider a plane fluid layer, rotating with an angular velocity Ω, confined between two horizontal walls, a distance h apart. Gravity is acting vertically downwards. The basic equations in the Boussinesq approximation are :

$$\frac{\partial u}{\partial t} + \bar{u}.\ \nabla\ \bar{u} = -\nabla\left(\frac{p}{\rho_m} - \frac{1}{2}\ |\ \bar{\Omega}_1 \times \bar{r}\ |^2\right) + \left(1+\frac{\delta\rho}{\rho_m}\right)\bar{X}$$

$$+\nu\ \nabla^2\bar{u} + 2\ |\ \bar{u} \times \bar{\Omega}_1\ | \qquad ...(1)$$

$$\nabla.\ \bar{u} = 0 \qquad ...(2)$$

$$\frac{\partial T}{\partial t} + \bar{u}.\ \nabla T = k\nabla^2 T \qquad ...(3)$$

$$\rho = \rho_m[1-\alpha(T-T_m)] \qquad ...(4)$$

where

$$\bar{X} = (0,\ 0,\ -g),\ \bar{\Omega}_1 = (0,\ 0,\ \Omega_1) \qquad ...(5)$$

$\bar{u}$, p, T, ρ_m, ν, k respectively denote the velocity vield, the pressure, the temperature, the constant reference density, the kinetic viscosity, and the thermometric conductivity. Ω_1 denotes angular velocity of rotation and $|\ \bar{\Omega}_1 \times \bar{r}^2|$ is the centrifugal force, $\bar{\Omega}_1 \times \bar{u}$, the coriolis acceleration, T_m the reference temperature and α the coefficient of thermal expansion.

We consider the situation in which the wall temperatures have been externally imposed. The upper wall is assumed to be at a fixed temperature and the temperature of the lower wall varies sinusoidally. Thus the thermal boundary conditions are

$$T = \beta h(1 + \varepsilon \sin \omega t) \quad \text{at } z = 0 \qquad ...(6)$$

$$T = 0 \quad \text{at } z = h \qquad ...(7)$$

where β, ε and ω are real, positive constants. Variations of these boundary conditions have been applied by Venezian (1969) and Rosenblat and Tanaka (1971). They assumed the temperature to be oscillating as a cosine function.

The equilibrium state solutions of eqns. (1)-(7) are

$$\bar{u} = 0, \ T = \bar{T}(z, t), \ p = \bar{p} \ (z, t) \qquad ...(8)$$

where $\bar{T}$ is the solution of the conduction equation

$$\frac{\partial \bar{T}}{\partial t} = k \frac{\partial^2 \bar{T}}{\partial z^2} \qquad ...(9)$$

subject to the boundary conditions (6) to (7).

We consider the superposition of infinitesimal disturbances over the equilibrium state solutions (8) and write

$$\bar{u} = (u, v, w), \ T = \bar{T} + \theta', \ p = \bar{p} + p' \qquad ...(10)$$

in eqns. (1)-(4) and linearize them. The linearized equations are

$$\frac{\partial \theta'}{\partial t} + w \frac{\partial \bar{T}}{\partial z} = k \nabla^2 \theta' \qquad ...(11)$$

$$\frac{\partial \zeta}{\partial t} = 2\Omega_1 \frac{\partial w}{\partial z} + \nu \ \nabla^2 \zeta \qquad ...(12)$$

$$\frac{\partial}{\partial t} \nabla^2 w = \alpha g \nabla_1^2 \theta' + \nu \nabla^4 w - 2\Omega_1 \frac{\partial \zeta}{\partial z} \qquad ...(13)$$

where w and ζ are z-component of velocity and vorticity and

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

We assume that the disturbances are of the form

$$f(z, t) \exp \ [i(a_1 x + a_2 y)] \qquad ...(14)$$

and introduce the following non-dimensional quantities

$$z = h\left(Z + \frac{1}{2}\right), \ t = \frac{\tau}{\omega}, \quad a_1^2 + a_2^2 = \frac{a^2}{h^2}$$

$$\bar{T} = \beta h T_0; \ \theta' = \beta h \theta, \ w = \left(\frac{\alpha g \beta h^3 a^2}{\nu}\right) W \qquad ...(15)$$

$$\Omega = \frac{\omega h^2}{k}$$

and define the physical parameters as

$$R = \frac{\alpha g \beta h^4}{k\nu}, \quad \sigma = \frac{\nu}{k}, \quad T_0 = \frac{4\Omega_1{}^2 h^4}{\nu^2} \qquad ...(16)$$

which are respectively the Rayleigh number, the Prandtle number, and the Taylor number. Ω denotes the thickness of the thermal boundary layer at the lower wall.

The perturbation eqns. (11)-(13) can now be written as

$$\Omega \frac{\partial \theta}{\partial \tau} + Ra^2 \left(\frac{\partial T_0}{\partial z} \right) W = \nabla^2 \theta \qquad ...(17)$$

$$\Omega \frac{\partial \zeta}{\partial \tau} = 2\Omega_1 \; Ra^2 \; DW + \sigma \nabla^2 \zeta \qquad ...(18)$$

$$\Omega \frac{\partial}{\partial \tau} \nabla^2 W + \sigma \theta = \sigma \nabla^4 W - \frac{2\Omega_1 \sigma}{\alpha g \beta a^2} D\zeta \qquad ...(19)$$

where $D = \frac{\partial}{\partial z}$ and $\frac{\partial T_0}{\partial z}$ is time-dependent.

Eliminating θ and ζ from eqn. (19) with the help of eqns. (17) and (18), we obtain

$$\left[\left(\Omega \frac{\partial}{\partial \tau} - \nabla^2 \right) \left\{ \left(\Omega \frac{\partial}{\partial \tau} - \sigma \nabla^2 \right)^2 \nabla^2 + \sigma^2 T D^2 \right\} W \right]$$

$$= Ra^2 \sigma \left(\Omega \frac{\partial}{\partial t} - \sigma \nabla^2 \right) \left(W \frac{\partial T_0}{\partial z} \right) \qquad ...(20)$$

The boundary conditions for the free-boundaries are

$$W = D^2 W = D^4 W = \ldots = = 0, \text{ on } \; z = \pm \frac{1}{2} \qquad ...(21)$$

The gradient $\frac{\partial T_0}{\partial z}$ is obtained from the dimensionless solution of (9) subject to the boundary conditions (6) - (7)

$$\frac{\partial T_0}{\partial z} = -1 + \varepsilon f \qquad ...(22)$$

where

$$f = -I_m \left[\frac{\lambda}{\sinh \lambda} \left\{ a(\lambda) e^{\lambda z} + a(-\lambda) e^{-\lambda z} \right\} \exp i\tau \right] \qquad ...(23a)$$

$$\lambda = (i\Omega)^{1/2} = \frac{\Omega^{1/2}}{2}(1+i), \; a(\lambda) = e^{-\lambda/2} \qquad \ldots(23b)$$

We now proceed to solve eqn. (20).

3. The Eigen Functions and The Eigen Values

We shall make use of the perturbation technique to obtain the eigen functions w and the eigen values R of eqns. (20) and (21) for a temperature profile given by (22) which departs from the linear one by order ε as given in (23). Following Venezian (1969), Malkus and Veronis (1958), Schlüter *et al.* (1965), Ingersall (1966), we write

$$\left.\begin{aligned} W &= w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots \\ R &= R_0 + \varepsilon^2 R_2 + \ldots \end{aligned}\right\} \qquad \ldots(24)$$

and substitute in eqn. (20) to obtain

$$Lw_0 = 0 \qquad \ldots(25)$$

$$Lw_1 = -R_0 a^2 \sigma[L_2(w_0 f)] \qquad \ldots(26)$$

$$Lw_2 = -a^2\sigma[R_0 L_2(w_1 f) + R_2 L_2 w_0] \qquad \ldots(27)$$

where

$$L = \left(\Omega\frac{\partial}{\partial\tau} - \nabla^2\right)\left[\left(\Omega\frac{\partial}{\partial\tau} - \sigma\nabla^2\right)^2\nabla^2 + \sigma^2 TD^2\right] + R_0 a^2\sigma(L_2)$$

$$L_2 = \Omega\frac{\partial}{\partial\tau} - \sigma\nabla^2 \qquad \ldots(28)$$

The eigen function w_0 and eigenvalue R_0 are solutions of the problem with $\varepsilon = 0$. These have been taken from Chandrasekhar (1961).

$$w_0 = \sin \pi z \qquad \ldots(29)$$

$$R_0 = \frac{1}{a^2}\left[(\pi^2 + a^2)^3 + \pi^2 T\right] \text{ for } n = 1 \qquad \ldots(30)$$

where T is the Taylor number.

The critical Rayleigh number R_{0c} can be obtained from the above for $x = \frac{a^2}{\pi^2}$ where x is a root of the equation

$$2x^3+3x^2-1-\frac{1}{\pi^2}=0 \qquad \ldots(31)$$

To solve eqn. (26), we require

$$L_2\left[I_m(w_0 f)\right]=I_m\left[L_2\ (f\ \sin\ \pi z)\right]$$

which is obtained with the help of eqn. 23 as

$$I_m\left[L_2(f\ \sin\ \pi z)\right]=I_m\left[\ L_c(f\sin\ \pi z)-2\sigma\ DW_0Df\ \right] \qquad \ldots(32)$$

where

$$L_c=i\Omega(1-\sigma)+\sigma(\pi^2+a^2) \qquad \ldots(33)$$

and

$$I_m[DW_0Df]=-I_m\left[\frac{\lambda^2 e^{i\tau}}{2\sinh\lambda}\left\{a(\lambda)e^{\lambda z}-a(-\lambda)e^{-\lambda z}\quad \cos\pi z\right]\right. \qquad \ldots(34)$$

Equation (26) with the help of (33) and (34) can be written as

$$LW_1=R_0a^2\sigma I_m\left[\ L_c\frac{\lambda e^{i\tau}}{2\sinh\lambda}\left\{\ a(\lambda)e^{\lambda z}+a(-\lambda)e^{-\lambda z}\right\}\sin\pi z\right.$$

$$\left.-2\sigma\pi\left\{\frac{\lambda^2 e^{i\tau}}{2\sinh\lambda}(a(\lambda)e^{\lambda z}-a(-\lambda)e^{-\lambda z})\right\}\cos\pi z\right] \qquad \ldots(35)$$

Equation (35) will be solved for W_1 by making use of the Fourier series expansions for the right hand side. We expand the following in Fourier series

$$e^{\lambda z}\ \sin\ m\pi z=\sum_{1}^{\infty} g_{nm}\sin n\pi z \qquad \ldots(36)$$

$$e^{\lambda z}\ \cos m\pi z=\sum_{1}^{\infty} h_{nm}\cos n\pi z \qquad \ldots(37)$$

g_{nm} and h_{nm} are obtained by multiplying eqns. (36) and (37) by $\sin n\pi z$ and $\cos n\pi z$ respectively and integrating between $\pm\frac{1}{2}$

$$g_{nm}(\lambda)=2\int_{-1/2}^{1/2} e^{\lambda z}\ \sin\ m\pi z\ \sin\ n\pi z\ dz$$

$$= 2\cosh\frac{\lambda}{2}\left[(-1)^{\frac{m+n+1}{2}}\frac{2(m+n)\pi}{Dr1}+\frac{2\pi(n-m)}{Dr2}(-1)^{\frac{n-m-1}{2}}\right] \quad \ldots(38)$$

for $m+n$ odd

$$= 2\lambda\sinh\frac{\lambda}{2}\left[(-1)^{\frac{m+n+2}{2}}\frac{1}{Dr1}+\frac{(-1)}{Dr2}^{\frac{n-m-2}{2}}\right] \quad \ldots(39)$$

for $m+n$ even

and

$$h_{nm}(\lambda) = 2\int_{-1/2}^{1/2} e^{\lambda z}\cos m\pi z\cos n\pi z\, dz$$

$$= 2\cosh\frac{\lambda}{2}\left[(-1)^{\frac{m+n-1}{2}}\frac{2(m+n)\pi}{Dr1}+\frac{2\pi(n-m)}{Dr2}(-1)^{\frac{n-m-1}{2}}\right] \quad \ldots(40)$$

for $n+m$ odd

$$= 2\lambda\sinh\frac{\lambda}{2}\left[\frac{(-1)}{Dr1}^{\frac{m+n}{2}}+\frac{(-1)}{Dr2}^{\frac{n-m}{2}}\right] \quad \text{for } n+m \text{ even} \ldots(41)$$

where

$Dr1 = \lambda^2+(m+n)^2\pi^2$, $Dr2 = \lambda^2+(n-m)^2\pi^2$

Equation (35) is then written as

$$Lw_1 = R_0a^2\sigma I_m\left[L_c\frac{\lambda}{2\sinh\lambda}\sum_1^{\infty}A_n(\lambda)e^{i\tau}\sin n\pi z\right.$$

$$\left.-2\sigma\pi\left\{\frac{\lambda^2}{2\sinh\lambda}\sum_1^{\infty}B_n(\lambda)e^{i\tau}\cos n\pi z\right\}\right] \quad \ldots(42)$$

where

$$A_n(\lambda) = a(\lambda)\, g_{n1}(\lambda)+a(-\lambda)\, g_{n1}(-\lambda)$$

$$B_n(\lambda) = a(\lambda)\, h_{n1}(\lambda)-a(-\lambda)\, h_{n1}(-\lambda) \quad \ldots(43)$$

We also define

$$L(\Omega,\ n)=-(n^2\pi^2+a^2)\,[-\Omega^2\{1+2\sigma(n^2\pi^2+a^2)\}+\sigma^2(n^2\pi^2+a^2)^3 + \sigma^2 T n^2\pi^2 - R_0 a^2\sigma] - i\ \Omega\{(n^2\pi^2+a^2)\,[-\Omega^2+\sigma^2(n^2\pi^2+a^2)^2 + 2\sigma(n^2\pi^2+a^2)^2] - R_0 a^2\sigma + \sigma^2 T n^2\pi^2\} \quad ...(44)$$

Then it can be easily verified that

$$\left.\begin{aligned} L(\sin n\pi z\ e^{i\tau}) &= e^{i\tau}\ \sin\ n\pi z\ L(\Omega,\ n) \\ L(\cos n\pi z\ e^{i\tau}) &= e^{i\tau}\ \cos\ n\pi z\cdot L(\Omega,\ n) \end{aligned}\right\} \quad ...(45)$$

Making use of eqns. (44) and (45), eqn. (43) yields

$$W_1 = R_0 a^2 \sigma I_m\Big[\left(\frac{\lambda e^{i\tau}}{2\sinh\lambda}\right)\Big\{\pounds_c \sum_1^{\infty} \frac{A_n(\lambda)\ \sin n\pi z}{L(\Omega,\ n)} - 2\sigma\pi\lambda \sum_1^{\infty} \frac{B_n(\lambda)\ \cos n\pi z}{L(\Omega,\ n)}\Big\}\Big] \quad ...(46)$$

Now we proceed to solve eqn. (27) in which we have

$$L_2(w_1 f) = \pounds_n(w_1 f) - \sigma D w_1\ Df \quad ...(47)$$

where

$$\pounds_n = i\Omega(1-\sigma) + \sigma(n^2\pi^2+a^2) \quad ...(48)$$

We also have

$$DW_1 = n\pi\ (P\cos n\pi z - Q\ \sin n\pi z)$$

$$Df = -\frac{\lambda^2 e^{i\tau}}{2\sinh\lambda}\Big[\ a(\lambda)e^{\lambda z} - a(-\lambda)e^{-\lambda z}\Big] \quad ...(49)$$

where P and Q are coefficients of $\sin n\pi z$ and $\cos\ n\pi z$ in eqn. (46). Eqn. (27) with the help of above relations can be written as

$$Lw_2 = -R_0 a^2 \sigma I_m[\pounds_n(w_1 f) - 2\sigma DW_1\ Df] - R_2 a^2 \sigma^2(\pi^2+a^2)\sin\pi z \quad ...(50)$$

The solubility condition for eqn. (50) is that its right hand side is orthogonal to the null space of the operator L which requires that $\sin\pi z$ must be orthogonal to the time independent part of the RHS. Multiplying eqn. (50) by $\sin\ \pi z$ and integrating between $\pm\ \frac{1}{2}$, we obtain

$$R_2 = \frac{-2R_0}{\sigma\cdot(\pi^2+a^2)} I_m \left[\pounds_n \int_{-1/2}^{1/2} \overline{f\, w_1} \sin \pi z \, dz \right.$$

$$\left. - 2\sigma \int_{-1/2}^{1/2} \overline{DW_1}\ \overline{Df} \sin \pi z \, dz \right] \qquad \ldots(51)$$

where bar denotes the time average.

We also observe that

$$I_m(f \sin \pi z) = - I_m \left[\frac{\lambda e^{i\tau}}{2\sinh \lambda} A_n(\lambda) \sin n\pi z \right]$$

$$Df(\sin \pi z) = - I_m \left[\frac{\lambda^2 e^{i\tau}}{2\sinh \lambda} C_n(\lambda) \sin n\pi z \right]$$

where

$$C_n(\lambda) = a(\lambda)\, g_{n1}(\lambda) - a(-\lambda)\, g_{n1}(-\lambda) \qquad \ldots(52)$$

Equation (51) can now be written as

$$R_2 = \frac{R_0{}^1 a^2}{2(\pi^2+a^2)} R_e \left\{ \left|\frac{\lambda}{2\sinh\lambda}\right|^2 \sum_1^{\infty} \frac{\pounds_c \pounds_n \mid A_n(\lambda) \mid^2 L^*(\Omega, n)}{\mid L(\Omega, n) \mid^2} \right.$$

$$\left. - 4\sigma^2\pi^2 \left|\frac{\lambda^2}{2\sinh\lambda}\right|^2 \sum_1^{\infty} \frac{n\, B_n(\lambda)\, C_n{}^*(\lambda)\, L^*(\Omega, n)}{\mid L(\Omega, n) \mid^2} \right\} \qquad \ldots(53)$$

In obtaining eqn. (53), use is made of the relations

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{FC}\, dt = \frac{1}{2}[F^*C] = \frac{1}{2}[C^*F]$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} I_m(F\, C)\, dt = \frac{1}{2\pi}\int_0^{2\pi} R_c(F\, C)\, dt$$

We shall also need

$$R_e[\pounds_c \pounds_n{}^* L^*(\Omega, n)] = a_1[\Omega(1-\sigma)^2 + \sigma^2(n^2\pi^2 + a^2)(\pi^2 + a^2)]$$
$$+ b_1 \Omega \sigma (1-\sigma)(n^2-1)\pi^2 \qquad \ldots(54)$$

where a_1 and b_1 are the real and imaginary parts of $L(\Omega, n)$ given in eqn. (44). From eqns. (38) – (41), one can easily evaluate that when n is odd

$$\left.\begin{aligned} a_{n1} &= 4\lambda \sinh\frac{\lambda}{2}(-1)^{\frac{n-1}{2}} \frac{\lambda^2+(n^2+1)\pi^2}{Dr1\cdot Dr2} \\ b_{n1} &= 8\lambda\sinh\frac{\lambda}{2}(-1)^{\frac{n-1}{2}} \frac{n\pi^2}{Dr1\cdot Dr2} \end{aligned}\right\} \qquad ...(55)$$

where

$$Dr1 = \lambda^2+(n+m)^2\pi^2,\ Dr2 = \lambda^2+(n-m)^2\pi^2$$

and for n even

$$\left.\begin{aligned} a_{n1} &= -4\pi n \cosh\frac{\lambda}{2}(-1)^{n/2} \frac{\lambda^2+(n^2-1)\pi^2}{Dr1\cdot Dr2} \\ b_{n1} &= 4\pi \cosh\frac{\lambda}{2}(-1)^{n/2} \frac{\lambda^2+(-n^2+1)\pi^2}{Dr1\cdot Dr2} \end{aligned}\right\} \qquad ...(56)$$

It is easily seen that for all n (odd or even)

$$\left.\begin{aligned} a_{n1}(\lambda) &= a_{n1}(-\lambda) \\ b_{n1}(\lambda) &= b_{n1}(-\lambda) \end{aligned}\right\} \qquad ...(57)$$

With the help of eqns. (55) – (57) we obtain [cf. eqns. (43) and (52)] for n odd

$$\left.\begin{aligned} A_n(\lambda) &= 4\lambda \sinh\lambda(-1)^{\frac{n-1}{2}} \frac{\lambda^2+(n^2+1)\pi^2}{Dr1\cdot Dr2} \\ B_n(\lambda) &= 8\lambda \sinh\lambda(-1)^{\frac{n-1}{2}} \frac{\pi^2 n}{Dr1\cdot Dr2} \\ C_n(\lambda) &= 8\lambda \sinh^2\frac{\lambda}{2}(-1)^{\frac{n-1}{2}} \frac{\lambda^2+(n^2+1)\pi^2}{Dr1\cdot Dr^2} \end{aligned}\right\} \qquad ...(58)$$

and for n even

$$\left.\begin{aligned} A_n(\lambda) &= (-1)^{n/2+1}\, 8\pi n \cosh^2\frac{\lambda}{2} \frac{\lambda^2+(n^2-1)\pi^2}{Dr1\cdot Dr2} \\ B_n(\lambda) &= (-1)^{n/2}\, 8\pi \cosh^2\frac{\lambda}{2} \frac{\lambda^2+(1-n^2)\pi^2}{Dr1\cdot Dr2} \\ C_n(\lambda) &= 4\pi n \sinh\lambda(-1)^{n/2} \frac{\lambda^2+(n^2-1)\pi^2}{Dr1\cdot Dr2} \end{aligned}\right\} \qquad ...(59)$$

To evaluate R_2, we need the following for n odd

$$\frac{A_n(\lambda)^2}{2\sinh\lambda} = \frac{4\Omega^2[\Omega^2+(n^2+1)^2\pi^4]}{|Dr1|^2\ |Dr2|^2} \qquad ...(60)$$

where

$$|Dr1|^2 = [\Omega^2 + (n+1)^4 \pi^4], \quad |Dr2|^2 = [\Omega^2 + (n-1)^4 \pi^4]$$

and

$$\left|\frac{\lambda^2}{2\sinh\lambda}\right| = \frac{\Omega^2}{4(\sinh^2 2x \cos^2 2x + \cosh^2 2x \sin^2 2x)}$$

$$x = \frac{\Omega^{1/2}}{2\sqrt{2}}$$

$$B_n(\lambda)C_n^*(\lambda) = \frac{128\, n\pi^2\, D\Omega}{|Dr1|^2\, |Dr2|^2}[A_3 + i\, B_3] \qquad \ldots(61)$$

$$A_3 = (n^2+1)\pi^2 \sinh x \cosh x \cos 2x - \Omega \sin x \cos x \cosh 2x$$

$$B_3 = -\Omega \sinh x \cosh x \cos 2x - (n^2+1)\pi^2 \sin x \cos x \cosh 2x$$

and when n is even these quantities are

$$\left|\frac{\lambda A_n(\lambda)}{2\sinh\lambda}\right| = \frac{4\pi^2 n^2 \Omega(\cosh^2 x \cos^2 x + \sinh^2 x \sin^2 x)[\Omega^2 + (n^2-1)^2\pi^4]}{(\sinh^2 x \cos^2 x + \cosh^2 x \sin^2 x)\, |Dr1|^2\, |Dr2|^2} \qquad \ldots(62)$$

also

$$B_n(\lambda)C_n^*(\lambda) = \frac{64\pi^2 n\, |\cosh \lambda/2|^2}{|Dr1|^2\, |Dr2|^2}(A_5 + iB_5) \qquad \ldots(63)$$

where $A_5 = [\Omega^2 - (n^2-1)^2\pi^4] \sinh x \cosh x \cos 2x$

$$+ 2\Omega(n^2-1)\pi^2 \sin x \cos x \cosh 2x$$

$$B_5 = \sin x \cos x \cosh 2x [\Omega^2 + (n^2-1)^2\pi^4]$$

$$+ 2\Omega(n^2-1)\pi^2 \sinh x \cosh x \cos 2x$$

4. Minimum Rayleigh Number for Convection Instability

The value of R is the eigenvalue corresponding to the function w which is oscillating and remains bounded in time. In general R is a function of the wave number a and the amplitude of perturbation, ε. Thus we write

$$R(a, \varepsilon) = R_0(a) + \varepsilon^2 R_2(a) + \ldots \qquad \ldots(64)$$

The minimum value of R as a function of a can be obtained by equating $\frac{\partial R}{\partial Q}$ to zero, that is when

$$\left[\frac{\partial R_0}{\partial Q} + \varepsilon^2 \frac{\partial R_2}{\partial Q} + \dots\right]_{a = a_c} = 0 \qquad \dots(65)$$

Let this least value of R which we call R_c occur at a_c (the critical wave number). We assume that a_c can also be expanded in powers of ε, i.e.

$$a_c = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \dots \qquad \dots(66)$$

Equation (65) with the help of (66) reduces to

$$\frac{\partial R_0}{\partial a_0} + \varepsilon \left(\frac{\partial^2 R_0}{\partial a_0^2}\right) a_1 + \varepsilon^2 \left[\frac{1}{2}\left(\frac{\partial^3 R_0}{\partial a_0^3}\right) a_1^2 + \left(\frac{\partial^2 R_0}{\partial a_0^2}\right) a_2 + \frac{\partial R_2}{\partial a_c}\right] \dots = 0 \qquad \dots(67)$$

giving

$$\left.\begin{aligned} \frac{\partial R_0}{\partial a_0} &= 0 \\ a_1 &= 0 \\ a_2 &= -\left(\frac{\partial R_2}{\partial a_0}\right) \frac{\partial^2 R_0}{\partial a^2_0} \end{aligned}\right\} \qquad \dots(68)$$

From the first we obtain

$$a_0 = \pi \sqrt{x}$$

where x is a root of the cubic eqn. (31). We can write a similar expansion for R_c:

$$\begin{aligned} R_c(\varepsilon) &= R_c(a_c, \varepsilon) \\ &= R_c(a_0) + \varepsilon^2 R_2(a_0) + \dots \end{aligned} \qquad \dots(69)$$

To order ε^2, R_c is obtained by evaluating R_0 and R_2 at $a = a_0$. If one needs to evaluate R_c up to order ε^4, the value of a_2 must be taken into account. In the next section, we evaluate only the lowest order in R_c due to the oscillating temperature of the lower boundary.

5. Numerical Evaluation and Discussion

The effect of modulation in certain limiting situations is found to be:

(a) T, the Taylor number is sufficiently large: when T is large R_0 is accordingly large. Thus the percentage modulation is always finite and is higher for higher T values.

(b) Ω, the frequency of the modulating temperature $\rightarrow\infty$, $R_{2c}\rightarrow 0$.
(c) $\Omega\rightarrow 0$. In this limit eqn. [53] yields

$$R_{2c} = \frac{1}{2} R_0{}^2 a^2 \Bigg/ \left\{ \frac{R_0 a^2}{\sigma} - T\pi^2 - (\pi^2 + a^2)^3 \right\}$$

The percentage modulation has been numerically calculated and is found to be larger than the modulation for non-zero Ω.

The modulation has been computed for various values of the triad (T, a_0, R_c) on percentage basis, viz. $\frac{R_{2c}}{R_0} \times 100$ for $\sigma = 1, 5$. Table I shows the maximum percentage modulation with the frequency at which these occur. It is to be noted that for fixed σ maximum modulation occurs at $\Omega\rightarrow 0$, say 10^{-4} or 10^{-5}; it goes to almost zero at $\Omega = 0.1$ and then again increases going to another maximum at some frequency and then again decreases going to zero at $\Omega\rightarrow 0$. It is also found that the maximum modutation at $\Omega\rightarrow 0$ is much larger than the maximum for $\Omega \neq 0$ $(\Omega >> 0)$.

Figures 1—3 show the percentage modulation as a function of frequency for a few values of field parameter Q and σ. It is observed that the modulation increases with $\sigma = \nu/K$. It has also been observed that modulation has small effect at low frequencies, say 0.1—1. This is physically justified as explained by Venezian (1969).

These calculations have been made under the assumption that the amplitude of modulation is small compared to the steady temperature $(\Delta T/T_0 << 1)$, and the convection currents are neglected, i.e. non-linear effects are not taken into account.

ACKNOWLEDGEMENT

One of the authors (K. M. S.) expresses his thanks to the Humboldt Foundation for the award of a senior research fellowship during the tenure of which this work was completed. Thanks are also due to Mr. Giesen for numerical evaluation. The authors are grateful to Dr. Kever and the Directors of the Institute of Plasma Physics for constant interest and encouragement.

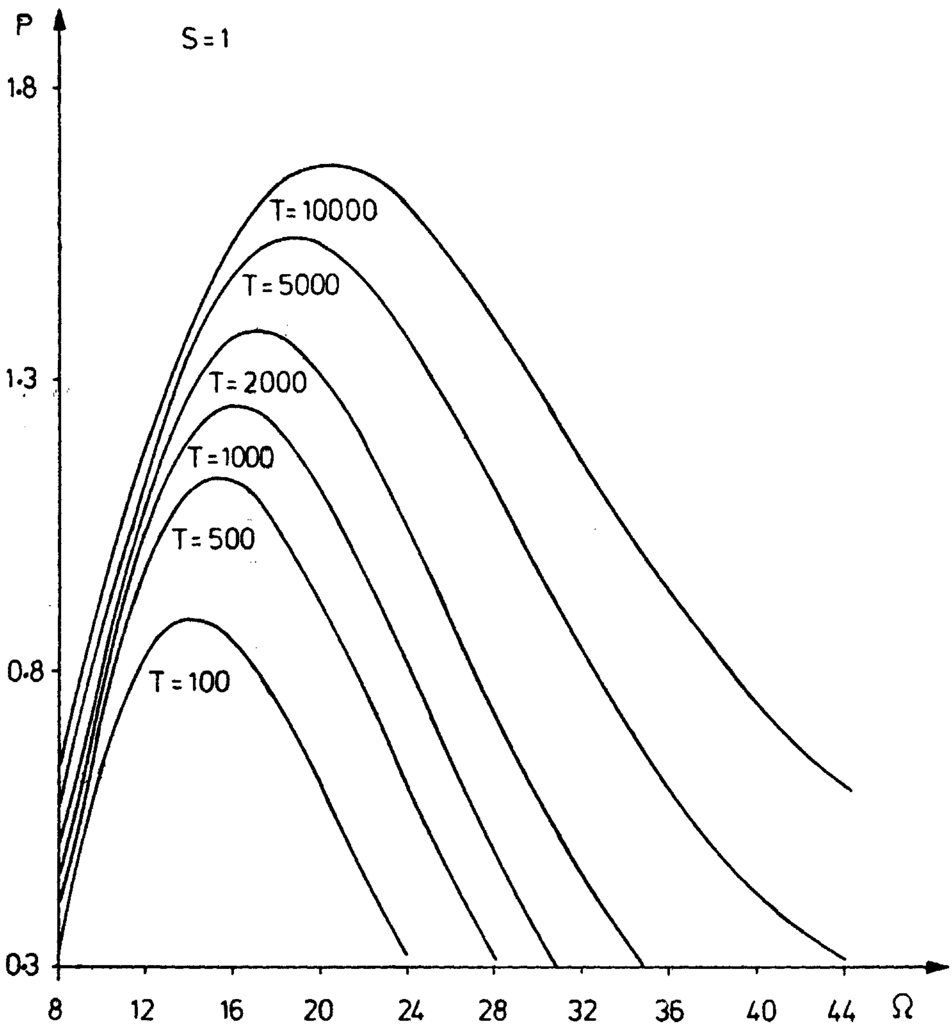
P
S = 1
1.8
1.3
0.8
0.3
T = 10000
T = 5000
T = 2000
T = 1000
T = 500
T = 100
8
12
16
20
24
28
32
36
40
44
Ω

FIG. 1

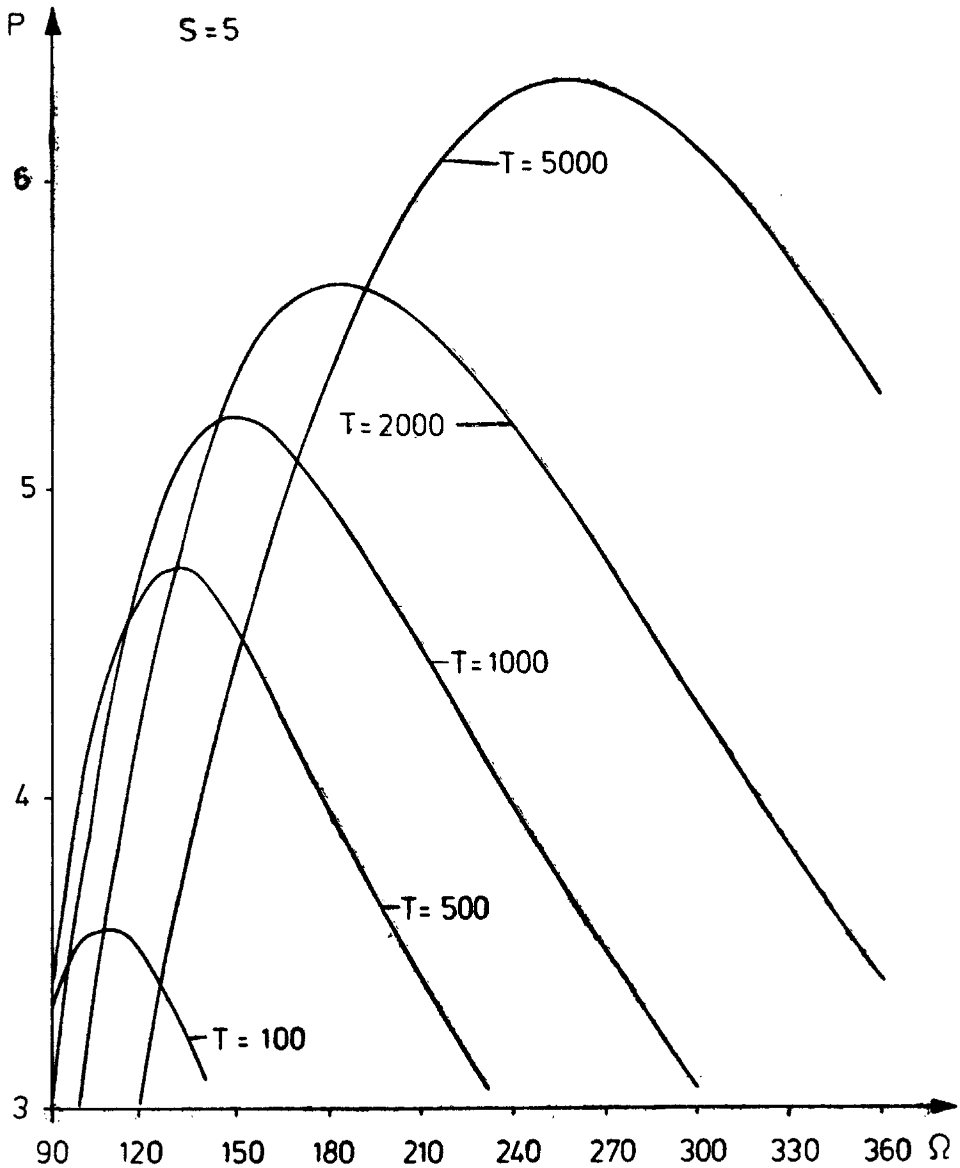
P
S = 5
T = 5000
T = 2000
T = 1000
T = 500
T = 100
6
5
4
3
90
120
150
180
210
240
270
300
330
360
Ω

FIG. 2

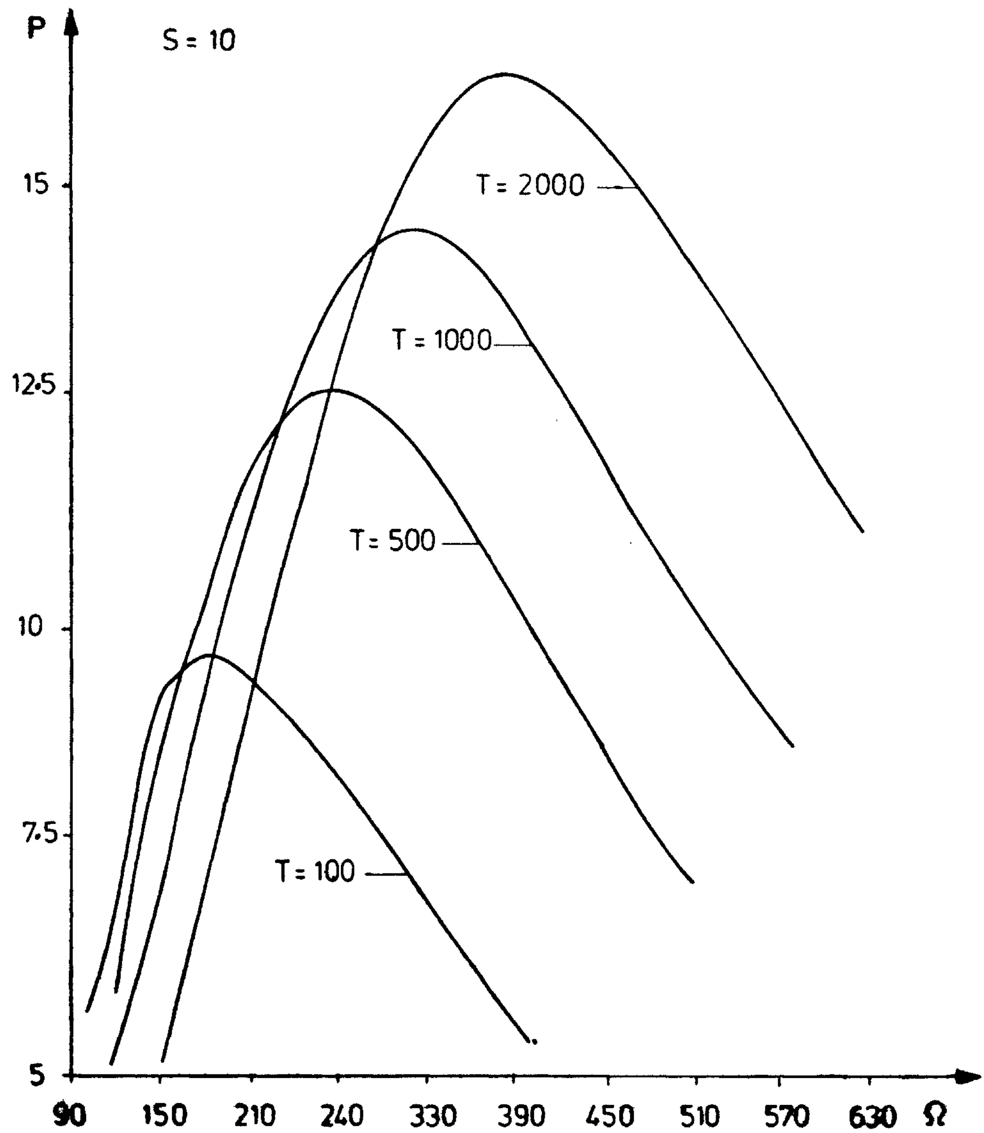
P
S = 10
15
12·5
10
7·5
5
T = 2000
T = 1000
T = 500
T = 100
90
150
210
240
330
390
450
510
570
630
Ω

FIG. 3

TABLE 1

Showing the maximum percentage shift in the critical Rayleigh number

T	a_c	R_c	$\frac{R_{2c}}{R_c} \times 100$		$\frac{R_{2c}}{R_c} \times 100$		$\frac{R_{2c}}{R_c} \times 100$	
			$\sigma = 1$	Ω	$\sigma = 5$	Ω	$\sigma = 10$	Ω
0	2.233	6.575×10^2	0.74	14	2.81	100	7.70	150
10	2.270	6.771×10^2	0.76	14	2.90	100	7.90	150
10^2	2.594	8.263×10^2	0.89	14	3.60	110	9.20	180
5×10^2	3.278	1.215×10^3	1.13	15	4.70	130	11.90	270
10^3	3.770	1.676×10^3	1.25	16	5.20	150	13.50	320
2×10^3	4.221	2.299×10^3	1.38	17	5.70	180	15.10	390
5×10^3	5.011	3.670×10^3	1.54	19	6.30	260	16.90	510
10^4	5.698	5.377×10^3	1.66	20	6.90	320	18.00	600
3×10^4	6.961	1.021×10^4	·1.84	24	7.60	450	20.01	930
10^5	8.626	2.131×10^4	2.06	30	8.10	680	20.14	1160
3×10^5	10.450	4.257×10^4	2.31	40	8.54	980	21.46	1950
10^6	12.860	9.222×10^4	2.85	70	9.26	1420	22.04	2960
10^7	19.020	4.147×10^5	3.57	110	8.73	3100		
10^8	28.020	1.877×10^6	4.01	150				
10^9	41.200	8.746×10^6	4.21	220				
10^{10}	60.520	4.047×10^7	4.32	300				

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