PERIODIC BOUNDARY VALUE PROBLEM FOR A SYSTEM OF NONLINEAR SECOND ORDER EQUATIONS

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An existence theorem for the solutions of a periodic boundary value problem for a system of second order equations is established.

Consider the vector boundary value problem

$$x'' + f(t, x, x') = 0 \qquad \ldots(1)$$

$$x(0) - x(\omega) = x'(0) - x'(\omega) = 0, \; \omega \in [0, T] \qquad \ldots(2)$$

where $x = (x_1, \ldots, x_n)$ is an n-dimensional vector;

$$f(t, x, y) = (f_1(t, x_1, \ldots, x_n, y_1, \ldots, y_n), \ldots, f_n(t, x_1, \ldots, x_n, y_1, \ldots, y_n))$$

is a vector-valued function defined for $0 \leqslant t \leqslant T$, $x, y \in R^n$.

As the norm of $x = (x_1, \ldots, x_n)$ and of $A = (a_{ik})$ it will be taken $\|x\| = \sum_i |x_i|$ and $\|A\| = \sum_{i,k} |a_{ik}|$ respectively.

We also assume that:

(A_1) $f(t, x, x')$ is a vector-valued, continuous function with domain $[0, T] \times R^{2n}$,

(A_2) there exists a matrix $A = (a_i \delta_{ik})_1^n$, $a_i < 0$ for all i (δ_{ik} is the Kronecker delta), and a function $H(t, r)$ with the following properties :

(i) $H(t, r)$ is piecewise continuous in $t \geqslant 0$, $r \geqslant 0$, continuous in $t \geqslant 0$, and nondecreasing (for fixed t) with respect to $r \geqslant 0$,

(ii) $\| Ax - f(t, x, x') \| \leqslant H(t, \|x\| + \|x'\|)$, $0 \leqslant t \leqslant T$, $(x, x') \in R^{2n}$,

(iii) $$\int_0^T H(s, C)\, ds < 2\left\| \left(\sqrt{|A|} \right)^{-1} \right\|^{-1} C \, \frac{\exp(aT) - 1}{\exp(aT) + 1}$$

for some constant $C > 0$, where $a = \operatorname{Min}_i \sqrt{|a_i|}$.

The problem (1), (2) is equivalent to

$$x'' + Ax = Ax - f(t, x, x') \qquad \text{...(3)}$$

$$x(0) - x(\omega) = x'(0) - x'(\omega) = 0 \qquad \text{...(4)}$$

Problem (3), (4) is equivalent to

$$x(t) = \int_0^\omega G(t, x)\Big[\ A\,x\,(s) - f\,\Big(s, x(s), x'(s)\Big)\Big]\,ds \qquad \text{...(5)}$$

where $G(t, s)$ is Green's matrix for the problem (3), (4)

$$G(t, s) = \begin{cases} 2^{-1}\Big(\sqrt{|A|}\Big)^{-1}[E - \exp\sqrt{|A|}\,\omega]^{-1}\{\exp[-\sqrt{|A|}(t-s)] \\ \qquad \times \exp(\sqrt{|A|}\,\omega) + \exp[\sqrt{|A|}\,(t-s)]\} \text{ for } 0 \leqslant s \leqslant t \leqslant \omega \\ 2^{-1}\Big(\sqrt{|A|}\Big)^{-1}[E - \exp\sqrt{|A|}\,\omega]^{-1}\{\exp[-\sqrt{|A|}\,(s-t)] \\ \qquad \times \exp(\sqrt{|A|}\,\omega) + \exp[\sqrt{|A|}\,(s-t)]\} \text{ for } 0 \leqslant t \leqslant s \leqslant \omega \end{cases} \qquad \text{...(6)}$$

$[0, \omega] \subseteq [0, T] \subseteq [0, \infty)$ and the matrix function $\exp[\sqrt{|A|}\,t]$ and $\exp[-\sqrt{|A|}\,t]$ are defined by the matrix series (cf. Gantmakher 1966)

$$\exp[\sqrt{|A|}\,t] = \sum_{p=0}^{\infty} \frac{(\sqrt{|A|})^p}{p!}\,t^p \qquad \text{...(7)}$$

$$\exp[-\sqrt{|A|}\,t] = \sum_{p=0}^{\infty} (-1)^p \frac{(\sqrt{|A|})^p}{p!} \qquad \text{...(8)}$$

We use a theorem (stated below) due to Bihair (1967) to show that under the above assumptions, (5) has at least one solution.

Bihair (1967) considered the integral equation

$$x(t) = z(t) + \int_0^t k_1(t, s) f_1\Big(s, x(s)\Big)\,ds + \int_t^\infty k_2(t, s) f_2\Big(s, x(s)\Big)\,ds \qquad \text{...(9)}$$

where $k_1(t, s)$, $k_2(t, s)$ are real $n \times n$ matrices, continuous on $0 \leqslant s \leqslant t < \infty$ and $0 \leqslant t \leqslant s < \infty$ respectively, and $f_i(t, x)$, $x(t)$, $z(t)$ are vectors in R^n, and proved the following theorem :

Theorem B—Let the following conditions be satisfied :

(a) $k_1(t, s)$ and $k_2(t, s)$ are continuous and bounded in $0 \leqslant s \leqslant t < \infty$ and $0 \leqslant t \leqslant s < \infty$ respectively, i.e. $\| k_i (t, s) \| \leqslant K_i$, $i = 1, 2$,

(b) $f(t, x)$ continuous for $t \geqslant 0$, $x \in R^n$ and

$$\|f(t, x)\| \leqslant H(t, \|x\|), \quad t \geqslant 0, \quad x \in R^n$$

where $H(t, r)$ is piecewise continuous in $t \geqslant 0$, $r \geqslant 0$ and non-decreasing (for fixed t) with respect to r,

$$\text{(c)} \quad \Upsilon + K_1 \int_0^t H(s, h(s))\, ds + K_2 \int_t^\infty H(s, h(s))\, ds \leqslant h(t)$$

for certain function $h(t) \geqslant 0$, continuous for $t \geqslant 0$ and arbitrary constant $\Upsilon > 0$,

(d) $z(t)$ is bounded continuous for $t \geqslant 0$ ($z(t) \in C(R_+)$), then equation (9) has at least one solution $x(t)$ continuous for $t \geqslant 0$ satisfying $\| x(t) \| \leqslant h(t)$, $t \geqslant 0$.

We are now ready to state and prove our main result.

Theorem 1—Under the assumptions (A_1) and (A_2), there exists a positive real number ω_0 such that for every ω, $\omega_0 \leqslant \omega \leqslant T$ the problem (1), (2) has at least one solution $x(t)$ continuous for $0 \leqslant t \leqslant \omega$ satisfsying $\| x(t) \| \leqslant C$, $0 \leqslant t \leqslant \omega$.

PROOF : We note that

$$\| G(t, s) \| \leqslant \sum_{i=1}^{n} \frac{1}{2\sqrt{|a_i|}} \cdot \frac{\exp[\sqrt{|a_i|}\,\omega] + 1}{\exp[\sqrt{|a_i|}\,\omega] - 1}$$

$$\leqslant \frac{1}{2} \cdot \frac{\exp[a\,\omega] + 1}{\exp[a\,\omega] - 1} \|(\sqrt{|A|})^{-1}\| = \phi(\omega)$$

Let $B = \int_0^T H(s, C)\, ds$ and let

$$\omega_0 = \frac{1}{a} \log \left(\frac{2C \| (\sqrt{|A|})^{-1} \|^{-1} + B}{2C \| (\sqrt{|A|})^{-1} \|^{-1} - B} \right),$$

then by (iii), $0 < \omega_0 \leqslant T$. Furthermore, it is easily seen that for every ω, $\omega_0 \leqslant \omega \leqslant T$ we have $\phi(\omega) \leqslant \phi(\omega_0)$ so that

$$\| G(t, s) \| \leqslant \frac{C}{B}$$

Thus $G(t, s)$ is continuous, bounded in $0 \leqslant s \leqslant t \leqslant \omega$ and in $0 \leqslant t \leqslant s \leqslant \omega$ for all ω, $\omega_0 \leqslant \omega \leqslant T$, and hence the first hypothesis of Theorem B is satisfied.

From our assumptions it follows that the functions

$$\begin{cases} F(t, x, x') = Ax - f(t, x, x'), \\ H(t, \|x\| + \|x'\|), \end{cases} \quad t \in [0, \omega],\ (x, x') \in R^{2n}$$

satisfy the second hypothesis of Theorem B. We also have

$$\frac{C}{B} \int_0^{\omega} H(s, C)\, ds \leqslant C$$

which shows that the third hypothesis of Theorem B is satisfied.

Finally, taking $z(t) \equiv 0$, we have shown that all the hypotheses of Theorem B are satisfied, and hence at least one solution $x(t)$ of (5) (and hence of (1), (2)) with the desired properties exists.

Corollary 1—Under the assumptions (i), (ii) and the assumption

(iv) $\int_0^{\infty} H(s, C)\, ds < 2 \| (\sqrt{|A|})^{-1} \|^{-1} C$, for some $C > 0$, there

exists a real number ω_0 such that for every $\omega, \omega \geqslant \omega_0$, (1), (2) has at least one solution $x(t)$ satisfying $\| x(t) \| \leqslant C$, $0 \leqslant t \leqslant \omega$.

Corollary 2—If the vector-valued function $f(t, x, x')$ satisfies the hypothesis of uniqueness with a given initial condition, and if the hypotheses of Theorem 1 are satisfied, then there exists an $\omega_0 > 0$ such that if $f(t, x, x')$ is periodic (in t) of period ω, $\omega_0 \leqslant \omega \leqslant T$, (1), (2) has a periodic solution of period ω.

Remark: Some authors (e.g. Cesari and Hale 1957, Ehrman 1957, Pliss 1964) have given sufficient conditions for the existence of periodic solutions of systems of differential equations of the form

$$x_i''+\sigma_i^2 x_i = \varepsilon f_i (x_1, \ldots, x_n, x_1', \ldots, x_n', \varepsilon, t), \; i= 1, 2, \ldots, n \quad \ldots(10)$$

where ε is a small (in absolute value) real parameter, $\sigma_1, \sigma_2, \ldots, \sigma_n$ are real positive constants, and each f_i is a real valued function periodic in t. Using our Theorem 1 we can establish the existence of periodic solutions of the systems of differential equations of the form

$$x_i'' + \sigma_i^2 x_i = \varepsilon f_i (x_1, \ldots, x_n, x_1', \ldots, x_n', \varepsilon, t), \; i = 1, 2, \ldots, n \quad \ldots(11)$$

In example 1 (below) we will consider a system of this type.

Notation : For the vectors $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n)$ we define

$$ab = (a_1 b_1, a_2 b_2, \ldots, a_n b_n)$$

and

$$a^2 = aa$$

Example 1—Consider the system of differential equations

$$x_i'' - \sigma_i^2 x_i = -\varepsilon x_i'(1-x_i^2) + \mu p_i(t), \; i = 1, 2, \ldots, n, \quad \ldots(12)$$

where $p_i(t)$, $i = 1, 2, \ldots, n$ are continuous in t, ε, μ are small (in absolute value) parameters, and $\sigma_1, \ldots, \sigma_n$ are real constants. Then there exists ω_0, $0 \leqslant \omega_0 \leqslant T$ such that if $p(t) = (p_1(t), \ldots, p_n(t))$ is periodic of period ω, $\omega_0 \leqslant \omega \leqslant T$, (12) has an ω-periodic solution.

To show this we choose $A = (a_i \delta_{ik})_1^n$, $a_i < 0$ so that

$$\| A_1 + A \| < 2 \| (\sqrt{| A |})^{-1} \|^{-1} \frac{e^{aT} - 1}{T(e^{aT} + 1)} \quad \ldots(13)$$

where $a = \operatorname*{Min}_i \sqrt{| a_i |}$ and $A_1 = (\sigma_i^2 \delta_{ik})_1^n$ [such an A always exists; an obvious choice for example is $A = (-\sigma_i^2 \delta_{ik})_1^n$]. Now let

$$H(t, \| x \| + \| x' \|) = \| A_1 + A \| (\| x \| + \| x' \|) + | \varepsilon | (\| x \| + \| x' \|) [1 + (\| x \| + \| x' \|)^2] + | \mu | B$$

where $B = \operatorname*{Max}_{t \in [0, T]} \| p(t) \|$. Then with the proper choices of C, $| \varepsilon |$, and $| \mu |$, it can easily be shown that all the hypotheses of Theorem 1 are satisfied. Let ω_0 be as in the conclusion of Theorem 1. If $p(t)$ is periodic of period ω, $\omega_0 \leqslant \omega \leq T$, then by Corollary 2, (12) has a periodic solution of period ω.

Example 2. Consider the vector boundary value problem

$$\left.\begin{aligned} x'' - \sigma_1^2 x &= \varepsilon(\alpha x + Axp_1(t) + \beta x^3 + \gamma xy^2) \\ y'' - \sigma_2^2 y &= \varepsilon(\delta y + Byp_2(t) + \mu y^3 + \nu x^2 y) \end{aligned}\right\} \quad \ldots(1)$$

$$\left.\begin{aligned} x(0) - x(\omega) &= x'(0) - x'(\omega) = 0 \\ y(0) - y(\omega) &= y'(0) - y'(\omega) = 0 \end{aligned}\right\} \qquad \omega \in [0, T] \quad \ldots(15)$$

where ε is a small (in absolute value) parameter; $\alpha, \beta, \delta, \mu, \nu, A, B, \sigma_1, \sigma_2$ are real constants, $p_1(t)$, $p_2(t)$ are continuous in t. Then there is an ω_0, $0 < \omega_0 \leq T$ such that for each ω, $\omega_0 \leq \omega \leq T$, the problem (14), (15) has a solution.

Example 3—Consider the following system

$$x'' + f(x)\, x'^k - A_1 x = \mu\, p(t), \; k \geqslant 2 \quad \ldots(16)$$

where k is an integer, all coefficients are continuous, $f_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$ are locally Lipschitzian in $x = (x_1, x_2, \ldots, x_n)$, $| f_i(x_1, x_2, \ldots x_n) | \leqslant b$ for all $i = 1, 2, \ldots, n$ and for all $x = (x_1, x_2, \ldots, x_n)$, $A_1 = (\sigma_i^2 \delta_{ik})_1^n$, and $| \mu |$ is sufficiently small. Then there exists ω_0, $0 < \omega_0 \leqslant T$ such that if $p(t) = (p_1(t), \ldots, p_n(t))$ is periodic of period ω, $\omega_0 \leqslant \omega \leq T$, (16) has an ω-periodic solution.

To show this, we let A be as in example 1 and let

$$H(t, \|x\| + \|x'\|) = \|A_1 + A\|(\|x\| + \|x'\|) + b(\|x\| + \|x'\|)^k + | \mu | B$$

where $B = \underset{t \in [0, T]}{\text{Max}} \|p(t)\|$. Now with the proper choices of C and $| \mu |$, it can easily be seen that all the hypotheses of Theorem 1 are satisfied.

References

Bihair, I. (1967). Notes on non-linear integral equation. *Studia Scient. Math. Hungarica*, **2**, 1-6.

Cesari, L., and Hale, J. K. (1957). A new sufficient condition for periodic solutions of weakly nonlinear differential systems. *Proc. Am. math. Soc.*, **8**, 757-64.

Ehrman, E. (1957). *Arch. ration. Analysis*, **1**, 2.

Gantmakher, F. R. (1966). The Theory of Matrices [in Russian]. Nauka, Moscow.

Pliss, V.A. (1964). Nonlocal Problems Concerning Oscillations [in Russian].