ON JOINT NUMERICAL RADIUS

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The main object of this paper is to establish certain relations between joint normaloidity, joint spectraloidity and joint convexoidity of tensor products of several operators.

§1. The concept of the joint spectrum of a family of elements in a commutative Banach algebra B with identity was first defined by Arens and Calderon (1955) as follows : If $a_1, a_2, \ldots, a_n$ are elements of B, the joint spectrum $\sigma(a_1, a_2, \ldots, a_n)$ relative to B is the set of all points $(z_1, z_2, \ldots, z_n)$ of the n-dimensional complex space C^n such that $a_1-z_1, a_2-z_2, \ldots, a_n-z_n$ belong to the same proper maximal ideal in B. Recently, Dash (1973) has suitably modified this definition for a commuting family of operators on a complex Hilbert space H. Let $A = (A_1, \ldots, A_n)$ be an n-tuple of commuting operators on H. Let U be the double commutant of the set $\{A_1, A_2, \ldots, A_n\}$ i. e. the set of all operators on H which commute every operator that commutes with all of $A_1, A_2, \ldots, A_n$. Then U is a commutative Banach algebra with identity containing the set $\{A_1, A_2, \ldots, A_n\}$. According to Dash (1973), a point $z = (z_1, \ldots, z_n)$ of C^n is in the joint spectrum $\sigma(A_1, \ldots, A_n)$ relative to U if for all $B_1, B_2, \ldots, B_n$ in U

$$\sum_{i=1}^{n} B_i (A_i - z_i) \neq I$$

Let $H_1, \ldots, H_n$ be complex Hilbert spaces, I_k the identity operator and A_k an arbitrary bounded operator on H_k, $1 \leqslant k \leqslant n$. Consider the tensor product space $H_1 \otimes H_2 \otimes, \ldots, \otimes H_n$ and the operators T_k $(1 \leqslant k \leqslant n)$ defined by

$$T_1 = A_1 \otimes I_2 \otimes \ldots\ldots \otimes I_n$$

$$T_2 = I_1 \otimes A_2 \otimes \ldots\ldots \otimes I_n$$

and, in general, $T_k = I_1 \otimes I_2 \otimes \ldots \otimes I_{k-1} \otimes A_k \otimes I_{k+1} \otimes \cdots \otimes I_n$

T_k's are commuting operators on $H_1 \otimes H_2 \otimes \ldots \otimes H_n$.

We shall need the following definitions :

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Definition 1 (Dash 1973a)—The joint numerical range $W(A)$ of an n-tuple $A=(A_1, \ldots, A_n)$ of operators on H is the set of all points $z=(z_1, \ldots, z_n)$ of C^n such that for some f in H with $\|f\|=1$,

$$W(A)=\{<Af, f> = (<A_1f, f>, <A_2f, f>, \ldots, <A_nf, f>)\}$$

Definition 2 (Dash 1973)—A complex vector $\lambda=(\lambda_1, \cdots, \lambda_n)$ is said to be in the joint approximate point spectrum $\sigma\pi$ (A) of $A=(A_1, A_2, \ldots, A_n)$ if there exists a sequence $\{f_n\}$ of unit vectors in H such that $\|(A_i-\lambda_i)f_n\| \to 0$, $i=1, 2, \ldots, n$.

Definition 3 (Dash 1973)—A point $\lambda=(\lambda_1, \cdots, \lambda_n)$ of C^n is in the joint point spectrum $\sigma_p(A)$ if and only if there exists f in H such that

$$A_if=\lambda_if, \quad 1 \leqslant i \leq n$$

§2. Set

$$\text{(i)} \quad |\lambda| = \left(\sum_{i=1}^{n} |\lambda_i|^2\right)^{1/2}$$

$$\text{(ii)} \quad \|A\| = \left(\sum_{i=1}^{n} \|A_i\|^2\right)^{1/2}$$

and (iii) $H(A_1, \cdots, A_n)=\overline{W(A_1, \ldots A_n)} \cap \{\lambda=(\lambda_1, \cdots, \lambda_n): |\lambda| = \|A\|\}$

We shall prove the following theorem (see Hildebrandt 1964).

Theorem 1—Let $A=(A_1 \ldots, A_n)$ be an n-tuple of commuting operators on a complex Hilbert space H. Then

(i) $H(A_1, \ldots, A_n) \subset \sigma_\pi(A_1, \cdots, A_n) \cap \sigma_\pi(A_1^*, \cdots, A_n^*)^*$
$\subset \sigma(A_1, \cdots, A_n)$

(ii) $H(A_1, \cdots, A_n) \cap \sigma_p(A_1, \cdots, A_n) = H(A_1, \cdots, A_n) \cap W(A_1, \cdots, A_n)$

Proof: Let $\lambda=(\lambda_1,+ \cdots, \lambda_n) \in H(A_1, \ldots, A_n)$. Then $|\lambda| = \|A\|$ and there exists a sequence $\{f_k\}$ of unit vectors such that

$$\operatorname*{Lt}_{k\to\infty} \lambda_i^{(k)} = \lambda_i$$

where $\lambda_i^{(k)}=(A_i f_k, f_k)$, $1 \leqslant i \leqslant n$

Since

$$\|(A_1-\lambda_1)f_k\|^2+\ldots\ldots+\|(A_n-\lambda_n)f_k\|^2$$
$$=\|A_1f_k\|^2-\bar{\lambda}_1\lambda_1^{(k)}-\lambda_1\bar{\lambda}_1^{(k)}+|\lambda_1f_k|^2 \cdots+\|A_nf_k\|^2-\bar{\lambda}_n\lambda_n^{(k)}-\lambda_n\bar{\lambda}_n^{(k)}+|\lambda_nf_k|^2$$
$$\leqslant\left(\sum_{i=1}^{n}\|A_i\|^2-\bar{\lambda}_1\lambda_1^{(k)}-\lambda_1\bar{\lambda}_1^{(k)}-\ldots\ldots\bar{\lambda}_n\lambda_n^{(k)}-\lambda_n\bar{\lambda}_n^{(k)}+\sum_{i=1}^{n}|\lambda_i|^2\right)$$
$$=\left(\sum_{i=1}^{n}|\lambda_i|^2-\bar{\lambda}_1\lambda_1^{(k)}-\lambda_1\bar{\lambda}_1^{(k)}\ldots\ldots-\bar{\lambda}_n\lambda_n^{(k)}-\lambda_n\bar{\lambda}_n^{(k)}+\sum_{i=1}^{n}|\lambda_i|^2\right)$$
$$\to 0 \text{ as } k\to\infty$$

Therefore

$$\|(A_i-\lambda_i)f_k\|\to 0 \text{ as } k\to\infty,\ 1\leqslant i\leqslant n$$

Using the fact $|\lambda_i|=|\lambda_i^*|$ and $\|A_i^*\|=\|A_i\|$, we can similarly prove that

$$\|(A_i^*-\lambda_i^*)f_k\|\longrightarrow 0 \text{ as } k\longrightarrow\infty,\ 1\leqslant i\leqslant n$$

As $\sigma_\pi(A_1,\ldots,A_n)\subset\sigma(A_1,\ldots,A_n)$ and $\sigma(A_1^*,\ldots,A_n^*)=\sigma(A_1,\ldots,A_n)^*$ (Dash 1973), this proves that

$$\lambda=(\lambda_1,\ldots,\lambda_n)\in\sigma_\pi(A_1,\ldots,A_n)\cap\sigma_\pi(A_1^*,\ldots,A_n^*)^*$$
$$\subset\sigma(A_1,,\ldots,A_n)$$

To prove part (ii), let $\lambda=(\lambda_1,\ldots,\lambda_n)$ be an arbitrary vector in $H(A_1,\ldots,A_n)\cap W(A_1,\ldots,A_n)$. Then for some f in H with $\|f\|=1$, $\lambda_i=\langle A_if,f\rangle$, $1\leqslant i\leqslant n$ and $|\lambda|=\|A\|$.

Since

$$\|A_1f-\lambda_1f\|^2+\ldots\ldots+\|A_nf-\lambda_nf\|^2$$
$$=\|A_1f\|^2-|\lambda_1|^2-|\lambda_1|^2+|\lambda_1|^2+\ldots+\|A_nf\|^2-|\lambda_n|^2-|\lambda_n|^2+|\lambda_n|^2$$
$$\leqslant\sum_{i=1}^{n}\|A_i\|^2-\sum_{i=1}^{n}|\lambda_i|^2=0$$

therefore

$$A_1f=\lambda_1f,\ \ldots,\ A_nf=\lambda_nf$$

and thus $\lambda = (\lambda_1, \ldots, \lambda_n) \in \sigma_p(A_1, \ldots, A_n)$. Conversely, let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \sigma_p(A_1, \ldots, A_n)$, then for some f in H with $\|f\| = 1$, $A_i f\ \lambda_i = f$, $1 \leqslant i \leqslant n$ and thus $<A_i f, f> = \lambda_i$. Hence $\lambda = (\lambda_1, \ldots \lambda_n) \in W(A_1, \ldots, A_n)$. This completes the proof.

Now we define the joint numerical radius $w(T)$ of T by

$$w(T) = \sup \{ |\lambda| : |\lambda \in W(T)\}$$

$$= \sup \left\{ \left(\sum_{i=1}^{n} |\lambda_i|^2 \right)^{\frac{1}{2}} : \lambda \in W(T) \right\}$$

where $T = (T_1, \ldots, T_n)$.

Clearly $w(T) < \infty$. Also (i) $w(T) \geqslant 0$, $w(T) = 0$ if and only if $T = 0$, (ii) $w(\alpha T) = |\alpha| w(T)$, for each scalar α, (iii) $w(T+S) \leqslant w(T) + w(S)$ where $S = (S_1, \ldots, S_n)$ (using Cauchy-Schwarz's inequality), $S_i = I_1 \otimes \ldots \otimes I_{i-1} \otimes B_i \otimes I_{i+1} \otimes \ldots \otimes I_n$, B_i's being bounded operators on H_i's, (iv) $w(T^*) = w(T)$, where $T^* = (T_1^*, \ldots, T_n^*)$. Therefore $w(T)$ defines a norm on the vector space of all n-tuples $T = (T_1, \ldots, T_n)$.

Dash (1973*a*) has proved that $\sigma(T_1, \ldots, T_n) \subset \overline{W(T_1, \ldots T_n)}$ and $W(T_k) = W(A_k)$, $1 \leqslant k \leqslant n$, where $W(T_k)$ denotes the numerical range of T_k (Halmos 1967, p. 165). Thus we have the following corollary :

Corollary 1 — $r(T) = \|T\|$ if and only if $w(T) = \|T\|$, where $T = (T_1, \ldots, T_n)$.

PROOF: Let $r(T) = \|T\|$

Then $\|T\| = r(T) \leqslant w(T)$

$$= \sup_{\|f\|=1} \left(\sum_{i=1}^{n} |<T_i f, f>|^2 \right)^{\frac{1}{2}}$$

$$\leqslant \left(\sum_{i=1}^{n} \|T_i\|^2 \right)^{\frac{1}{2}}$$

$$= \|T\|$$

and thus $w(T) = \|T\|$.

Conversely, let $w(T) = \|T\|$

Then by Theorem 1 (i), we have

$$\|T\| = |W(T)| \leqslant |\sigma(T)|$$
$$\leqslant |W(T)|$$
$$= \|T\|$$

Therefore $|\sigma(T)| = r(T) = \|T\|$.

We shall call $T=(T_1,...,T_n)$ to be jointly normaloid if $w(T) = \|T\|$, jointly spectraloid if $w(T) = r(T)$ and jointly convexoid if $Co\sigma(T) = \overline{W(T)}$ where $Co\sigma(T)$ denotes the convex hull of $\sigma(T)$.

Corollary 2—Every jointly normaloid $T=(T_1, \ldots, T_n)$ is jointly spectraloid.

PROOF : This follows obviously from Corollary 1.

We state the following result without proof.

Proposition—Every jointly convexoid $T=(T_1, \ldots, T_n)$ is also jointly spectraloid.

For a single operator A, it is known that A is normaloid if and only if $\|A^n\|=\|A\|^n$, for all positive integers n. However, we observe here that the analogous result need not be true for a jointly normaloid $T=(T_1, \ldots, T_n)$. For simplicity we shall prove the result for $n=2$. The general case follows on the similar lines. Firstly we prove the following.

Theorem 2 — If $T=(T_1,T_2)$ is jointly normaloid, then so is $T^2=(T_1^2,T_2^2)$.

PROOF : Since $r(T)=\|T\|$, there exists $z=(z_1,z_2)$ in $\sigma(T)$ such that $|z| = \|T\|$ or $|z_1|^2+|z_2|^2 = \|T_1\|^2+\|T_2\|^2$. Since $z_i \in \sigma(T_i)$ and $\|T_i\| \geqslant |z_i|$, $\|T_i\|^2 - |z_i|^2 \geqslant 0$, and hence $|z_1|^2+|z_2|^2 = \|T_1\|^2+\|T_2\|^2$ implies that $\|T_i\| = |z_i|$. It follows that there exists a sequence $\{x_k\}$ of unit vectors such that

$\|(T_i - z_i)x_k\| \to 0$ as $k \to \infty$, $i=1,2$, and hence $\|(T_i^2 - z_i^2)x_k\| \to 0$. Thus $z^2 \in \sigma(T^2)$. Since $|z_i| = \|T_i\|$ and T_i's are normaloid, we have $|z_i^2| = \|T_i^2\|$. In consequence

$$\left(\sum_{i=1}^{n} |z_i^2|^2\right)^{1/2} = \left(\sum_{i=1}^{n} \|T_i^2\|^2\right)^{1/2}$$ or $|z^2| = \|T^2\|$ showing that $r(T^2) = \|T^2\|$.

Theorem 3 — Let $T=(T_1,T_2)$. If $\|T^2\| = \|T\|^2$, then either $T_1=0$ or $T_2=0$.

PROOF: By definition

$$\|T^2\| = (\|T_1^2\|^2 + \|T_2^2\|^2)^{1/2}$$

and

$$\|T\|^2 = \|T_1\|^2 + \|T_2\|^2$$

Therefore, we have

$$\|T_1^2\|^2 + \|T_2^2\|^2 = (\|T_1\|^2 + \|T_2\|^2)^2$$

or

$(\|T_1\|^4 - \|T_1^2\|^2) + (\|T_2\|^4 - \|T_2^2\|^2) + 2\|T_1\|^2\|T_2\|^2 = 0$. Since each term is non-negative, we have $2\ \|T_1\|^2\|T_2\|^2 = 0$. Hence either $T_1 = 0$ or $T_2 = 0$. Now we come to our main conclusion.

Theorem 4.—If $T = (T_1, T_2)$ is jointly normaloid, where T_1 and T_2 are non-zero operators, then $r(T^2) \neq r(T)^2$.

PROOF: Since T^2 is also jointly normaloid by Theorem 2, $r(T^2) = \|T^2\|$. Therefore if $r(T^2) = r(T)^2$, then $\|T^2\| = \|T\|^2$ and hence by Theorem 3, either $T_1 = 0$ or $T_2 = 0$, a contradiction.

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