ON CONVERGENCE FIELDS OF STRONG FUNCTIONAL NÖRLUND METHODS

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In this paper relations between the convergence fields of 'strong functional' Nörlund summability methods with index 1 have been established.

1. INTRODUCTION

Peyerimhoff (1956) and Miesner (1965) have obtained relations between the convergence fields of the Nörlund methods (N, p_n) and (N, q_n), where $q(z) = p(z)\, r(z)$ and $r(z)$ satisfies appropriate restrictions. The results for integral transforms analogous to those of Peyerimhoff (1956) and Miesner (1965) (for ordinary summability) have been obtained by Choudhary (1970). Recently, Kuttner and Thorpe (1972) have established the results analogous to those of Peyerimhoff and Miesner for strong Nörlund summability with index 1.

Our aim in this paper is to obtain relations between the convergence fields of the strong functional Nörlund summability methods with index 1.

2. STRONG FUNCTIONAL NÖRLUND SUMMABILITY $[N, p, \beta]_\lambda$, $\lambda > 0$.

Let $\mathfrak{F}$ be the class of functions $f(t)$ which is an indefinite integral of some Lebesgue integrable function, say $a(t)$, *i.e.*,

$$f(t) = f(0) + \int_0^t a(u)\, du.$$

We suppose throughout that $p(t)$, $q(t)$, $r(t)$ are Lebesgue integrable in any (relevant) finite interval. We write

$$p_1(t) = \beta + \int_0^t p(u)\, du$$

and similarly for other letters in place of p. We assume that $p_1(t) \neq 0$ for all t.

Definition—Let $p(t)$ satisfy the following: For given $T > 0$, there exists $\mu = \mu(T) > 0$ such that

$$|\, p(t)\, | \geqslant \mu \quad (0 \leqslant t \leqslant T). \qquad \ldots(2{\cdot}1)$$

We shall say that $f(t)$ ($f(t) \in \mathfrak{J}$) is strongly summable (N, p, β) with index $\lambda > 0$ to s, and write $f(t) \rightarrow s[N, p, \beta]_\lambda$, if

$$\int_0^t |p(u)|^{1-\lambda} \, |Y(u) - (s - f(0))\, p(u)|^\lambda \, du = o(|p_1(t)|), \qquad \ldots(2\cdot2)$$

where

$$Y(t) = \beta a(t) + \int_0^t p(t-u)\, a(u)\, du. \qquad \ldots(2\cdot3)$$

As remarked by Kumar (1974), $p(t)$ should be assumed to satisfy (2·1) only in the case when $\lambda > 1$.

We recall that, with the usual notation for convolutions, the integral in (2·3) can be written $(p^*a)_t$. We shall make use of the result that the convolutions are associative and commutative.

3. The Lemmas

In order to prove our theorem we need a few lemmas. In most of the lemmas we will assume†

$$\int_0^t |dp(u)| = o(|p_1(t)|). \qquad \ldots(3\cdot1)$$

Firstly we show that if (3·1) holds, then

$$|p_1(t)| \rightarrow \infty \text{ as } t \rightarrow \infty \qquad \ldots(3\cdot2)$$

and

$$p(t) = o(|p_1(t)|). \qquad \ldots(3\cdot3)$$

If $p(t)$ is constant (the trivial case $p(t) \equiv 0$ being excluded), then (3·2) and (3·3) are trivially satisfied. If not, then $\int_0^t |dp(u)|$ is not identically zero; hence, since it is non-decreasing, it is greater than or equal to some positive constant for all sufficiently large t. Hence (3·1) implies (3·2). Thus

$$p(0) = \text{const.} = o(|p_1(t)|),$$

so that, again using (3·1), we get

$$|p(t)| \leqslant |p(0)| + \int_0^t |dp(u)| = o(|p_1(t)|).$$

† The example suggested by Professor B. Kuttner and given at the end of Lemma 1 shows that the condition (3·1) is necessary.

Next we state some consequences of (3·3) of which we shall have to make use (cf. Choudhary 1970):

(i) given any constant $c > 0$, there is a $T = T(c)$ such that

$$| p_1(u) | \leqslant | p_1(t) | \, e^{c(u-t)} \text{ for } u > t \geqslant T \; ; \qquad \text{...(3·4)}$$

(ii) for any $c > 0$,

$$| p(u) | = O \, (e^{cu}) \qquad \text{...(3·5)}$$

We will write

$$P(s) = \beta + \int_0^\infty e^{-st} \, p(t) \, dt \; ; \qquad \text{...(3·6)}$$

it follows from (3·5), since $c > 0$ is arbitrary, that the integral in (3·6) converges absolutely for Re $s > 0$. We will use a similar notation with p replaced by $\breve{q}$.

Lemma 1—Let s_0 be a complex constant with Re $s_0 > 0$, and let (3·1) hold. Suppose that $P(s)$ has a zero of order at least $n + 1$ at $s = s_0$.

Then

$$t^n \, e^{s_0 t} \to O \, [N, \, p, \, \beta]_1.$$

PROOF: We express

$$t^n \, e^{s_0 t} = \eta + \int_0^t k(u) \, du$$

where

$$k(t) = n t^{n-1} \, e^{s_0 t} + s_0 t^n \, e^{s_0 t} \qquad \text{...(3·7)}$$

and

$$\eta = \begin{cases} 1 & \text{if } n = 0 \; ; \\ 0 & \text{if } n > 0. \end{cases} \qquad \text{...(3·8)}$$

In order to prove the assertion of the lemma, we have to show that

$$\int_0^t | \phi(u) | \, du = o \, (\, | p_1(t) | \,) \qquad \text{...(3·9)}$$

where

$$\phi(u) = \beta k(u) + (p * k)_u + \eta p(u). \qquad \text{...(3·10)}$$

Now

$$\phi\,(u) = \beta\, k(u) + n\int_0^\infty (u-v)^{n-1}\, e^{s_0(u-v)}\, p(v)\, dv$$

$$+\, s_0\int_0^\infty (u-v)^n\, e^{s_0(u-v)}\, p(v)\, dv$$

$$+\int_u^\infty \frac{d}{dv}\,[(u-v)^n\, e^{s_0(n-v)}]\, p(v)\, dv + \eta p(u). \qquad \ldots(3{\cdot}11)$$

Differentiating (3·6) n times, we have for $n \geqslant 1$

$$P^{(n)}\,(s) = \int_0^\infty (-u)^n\, e^{-su}\, p(u)\, du.$$

Hence

$$\int_0^\infty (u-v)^n\, e^{s_0(u-v)}\, p(v)\, dv$$

$$= e^{s_0 u} \sum_{j=0}^{n} \binom{n}{j} u^{n-j}\, P^{(j)}\,(s_0) - \beta\, e^{s_0 u}\, u^n.$$

Thus (3·11) becomes

$$\phi\,(u) = e^{s_0 u}\left\{n \sum_{j=0}^{n-1} \binom{n-1}{j}\; u^{n-j-1}\, P^{(j)}\,(s_0)\right.$$

$$\left. +\, s_0 \sum_{j=0}^{n} \binom{n}{j} u^{n-j}\, P^{(j)}\,(s_0)\right\}$$

$$+ \Big[(u-v)^n\, e^{s_0(u-v)}\, p(v)\Big]_u^\infty$$

$$- \int_u^\infty (u-v)^n\, e^{s_0(u-v)}\, dp(v) + \eta\, p(u).$$

It can be verified that the contribution of $[\;\int_u^\infty\;]$ is $-\eta p(u)$; and thus

$$\phi(u) = e^{s_0 u}\left[n \sum_{j=0}^{n-1} \binom{n-1}{j} u^{n-j-1} p^{(j)}(s_0) + s_0 \sum_{j=0}^{n} \binom{n}{j} u^{n-j} P^{(j)}(s_0) \right] - \int_0^\infty (-w)^n e^{-s_0 w} d_w p(u+w)$$

$$= -\int_0^\infty (-w)^n e^{-s_0 w} d_w p(u+w) \qquad \ldots(3\cdot12)$$

since, by hypothesis,

$$P(s_0) = P'(s_0) = \ldots = P^{(n)}(s_0) = 0.$$

Hence, for (3·9) it is enough to show that

$$\int_0^t \left| \int_0^\infty (-w)^n e^{-s_0 w} d_w p(u+w) \right| du = o(|p_1(t)|) \qquad \ldots(3\cdot13)$$

But the left side of (3·13) is

$$< \int_0^\infty |(-w)^n e^{-s_0 w}| dw \int_0^t |d_w p(u+w)|$$

$$\leqslant \int_0^\infty |(-w)^n e^{-s_0 w}| dw \int_0^{t+w} |dp(v)| . \qquad \ldots(3\cdot14)$$

Now, given $\varepsilon > 0$, there is a T_0 such that

$$\int_0^{t+w} |dp(v)| < \varepsilon\, |p_1(t+w)| \qquad \ldots(3\cdot15)$$

for all $t + w \geqslant T_0$ and hence for all positive w, if $t \geqslant T_0$. Choose c such that $0 < c < \operatorname{Re} s_0$, and suppose that $T_0 \geqslant T$ (c and T as in (3·4)). Then, by (3·15), the right side of (3·14) does not exceed

$$\varepsilon \int_0^\infty |p_1(t+w)|\, w^n e^{-\gamma_0 w} dw \quad (\gamma_0 = \operatorname{Re} s_0)$$

$$\leqslant \varepsilon \mid p_1(t) \mid \int_0^\infty w^n e^{(\varepsilon-\gamma_0)w} dw \quad (\text{by } (3\cdot4))$$

$$= \varepsilon \mid p_1(t) \mid \text{const.},$$

this proves (3·13), and hence the lemma.

Remark: If we do not assume (3·1), then the following example shows that the conclusion of Lemma 1 is not valid even in simple cases.

Example—Take

$$p(t) = a - \sin t \qquad (a > 1).$$

(We must have $a > 1$ in order that (2·1) should hold). Considering the case $\beta = 0$, we find

$$P(s) = \frac{as^2 - s + a}{s(s^2 + 1)} .$$

Thus $P(s)$ has a zero of order 1 at $s = s_0$, where

$$s_0 = \frac{1}{2a}(1 \pm \surd(4a^2 - 1)\, i).$$

We have to consider the case $f(t) = e^{s_0 t}$, where s_0 is given by either of the above values. With the notation of (2·3), we have

$$Y(t) = s_0 e^{s_0 t} \int_0^t e^{-s_0 u} (a - \sin u)\, du$$

$$= s_0 e^{s_0 t} \left[e^{-s_0 u} \left(- \frac{a}{s_0} + \frac{\cos u + s_0 \sin u}{s_0^2 + 1} \right) \right]_0^t$$

$$= - a + \frac{s_0 (\cos t + s_0 \sin t)}{s_0^2 + 1} .$$

Since $s = 0$, $f(0) = 1$, (2·2) (with $\lambda = 1$) would give

$$\frac{1}{(s_0^2 + 1)} \int_0^t \mid s_0 \cos u - \sin u \mid du = o(t).$$

But, for large t, the expression on the left is asymptotically equivalent to $\dfrac{2t}{\pi \surd(s_0^2 + 1)}$, so the result is false.

§ 4. Now define

$$q(t) = \alpha\, p(t) + \beta\, r(t) + \int_0^t p(t-u)\, r(u) du \quad (\alpha \neq 0) \qquad \ldots(4{\cdot}1)$$

$$R(s) = \alpha + \int_0^\infty e^{-st}\, r(t)\, dt \quad \text{with} \int_0^\infty |\, r(t)\, |\, dt < \infty. \qquad \ldots(4{\cdot}2)$$

Write, for Re $s > 0$,

$$P(s) = \beta + \int_0^\infty e^{-st}\, p(t)\, dt, \qquad \ldots(4{\cdot}3)$$

$$Q(s) = \alpha\, \beta + \int_0^\infty e^{-st}\, q(t)\, dt. \qquad \ldots(4{\cdot}4)$$

It is clear from (4·2) that $R(s)$ has only a finite number of zeros in Re $s \geqslant 0$. Further, by (3·5), the integral in (4·3) is absolutely convergent, and therefore, by (4·1) and (4·2),

$$Q(s) = P(s)\, R(s). \qquad \ldots(4{\cdot}5)$$

Lemma 2—Define $q(t)$, $R(s)$, $P(s)$ and $Q(s)$ as defined in (4·1), (4·2), (4·3) and (4·4). Suppose that $r(t) \to 0$ as $t \to \infty$ and $R(0) \neq 0$. Suppose also that $p(t)$ satisfies (3·3) and that

$$\int_0^t |\, p(u)\, |\, du = O\, (\, |\, p_1(t)\, |\,). \qquad \ldots(4{\cdot}6)$$

Then $q(t)$ satisfies (3·3) and (4·6); and

$$q_1\, (t) = (R(0) + o\, (1)\,)\, p_1(t). \qquad \ldots(4{\cdot}7)$$

Further suppose that $\int_0^\infty |\, dr(t)\, | < \infty$. Then, if $p(t)$ satisfies (3·1), $q(t)$ satisfies

$$\int_0^t |\, dq(u)\, | = o\, (\, |\, q_1(t)\, |\,). \qquad \ldots(4{\cdot}8)$$

PROOF: The first part of the lemma is contained in Lemma 7 of Choudhary (1970). For the second part, with the aid of (3·1), from (4·1) we have

$$\int_0^t |\, dq(u) \,| \leqslant o(\,|\, p_1(t) \,|\,) + \int_0^t |\, d(p^*r)_u \,|$$

since $\int_0^\infty |\, dr(t) \,| < \infty$ and $|\, p_1(t) \,| \to \infty$ as $t \to \infty$. Now

$$\int_0^t |\, d\,(p^*r)_u \,| \leqslant |\, p(0) \,| \int_0^t |\, r(u) \,|\, du + \int_0^t |\, r(v) \,|\, dv \int_0^{t-v} |\, dp(u) \,|$$

$$= \int_0^t |\, r(v) \,|\, \rho(t - v)\, dv$$

where $\rho(t) = |\, p(0) \,| + \int_0^t |\, dp(u) \,| = o\,(\,|\, p_1(t) \,|\,)$.

Since $\int_0^\infty |\, r(t) \,|\, dt < \infty$, $r(t) \to 0$ as $t \to \infty$ and $\rho(t) = o\,(\,|\, p_1\,(t) \,|\,)$,

then $$\int_0^t |\, d(p^*r)_u \,| = o\,(\,|\, p_1(t) \,|\,).$$

Thus $$\int_0^t |\, dq(u) \,| = o\,(\,|\, p_1(t) \,|\,)$$

$$= o\,(\,|\, q_1(t) \,|\,) \qquad \text{(by (4·7))}.$$

Lemma 3—Let s_0 be a complex constant, Re $s_0 > 0$. Assume that $P(s)$ is regular for Re $s > 0$. Define

$$\breve{q}(t) = p(t) - (s_0 + 1)\,\{\beta e^{-t} + (e^{-t} * p)_t\}. \qquad \text{...(4·9)}$$

If $\breve{q}(t)$ satisfies

$$\int_0^t |\, d\breve{q}(u) \,| = o\,(\,|\, \breve{q}_1(t) \,|\,), \qquad \text{...(4·10)}$$

then $p(t)$ satisfies (3·1).

PROOF: From (4·9), we have

$$\widehat{Q}(s) = \left(\frac{s-s_0}{s+1}\right) P(s) \qquad \ldots(4\cdot11)$$

which gives

$$p(t) = \widehat{q}(t) + (s_0+1)\{\beta e^{s_0 t} + (e^{s_0 t} * \widehat{q})_t\}. \qquad \ldots(4\cdot12)$$

Since $P(s)$ is regular for Re $s > 0$, hence it follows from (4·11) that $\widehat{Q}(s_0) = 0$. Thus (4·12) becomes

$$p(t) = \widehat{q}(t) - (s_0+1)\int_0^\infty e^{-s_0 v}\widehat{q}(t+v)\,dv. \qquad \ldots(4\cdot13)$$

Hence, by (4·10), we have

$$\int_0^t |dp(u)| \leqslant o(|\widehat{q_1}(t)|) + |(s_0+1)| \int_0^t \left| d_u \int_0^\infty e^{-s_0 v}\widehat{q}(u+v)\,dv \right|. \qquad \ldots(4\cdot14)$$

By the same argument used in proving (3·13), we find that the second term on the right side of (4·14) is $o(|\widehat{q_1}(t)|)$ and thus

$$\int_0^t |dp(u)| = o(|\widehat{q_1}(t)|)$$

which gives (3·1) provided $o(|\widehat{q_1}(t)|)$ is the same as $o(|p_1(t)|)$.

Now, from (4·13), we have

$$p_1(t) = \widehat{q_1}(t) - (s_0+1)\int_0^\infty e^{-s_0 v}\,dv \int_v^{t+v} \widehat{q}(u)\,du,$$

i.e., $$\frac{p_1(t)}{\widehat{q_1}(t)} = 1 - (s_0+1)\frac{e^{s_0 t}}{\widehat{q_1}(t)} \int_t^\infty e^{-s_0 w}\widehat{q_1}(w)\,dw$$

$$+ (s_0+1)\frac{1}{\widehat{q_1}(t)}\int_0^\infty e^{s_0 v}\,\widehat{q_1}(v)\,dv$$

$$= 1 - (s_0+1) I_1 + (s_0+1) I_2$$

say. Integrating I_1 by parts and using Lemma 1 of Chaudhary (1970) (with $m=0$ and p replaced by $\widehat{q}$), we obtain

$$I_1 = \frac{1}{s_0} + o(1).$$

Next, since $|\breve{q}_1(t)| \to \infty$ as $t \to \infty$, we see that $I_2 = o(1)$. Thus

$$p_1(t) = (-\frac{1}{s_0} + o(1))\,\breve{q}_1(t)$$

which shows that $o(|p_1(t)|)$ is the same as $o(|\breve{q}_1(t)|)$.

Lemma 4—Let s_0 be a given complex number with Re $s_0 > 0$. Suppose that $P(s)$ has a zero of order n at $s=s_0$. Define $\breve{q}(t)$ by (4·9). Then, if $p(t)$ satisfies (3·3), $\breve{q}(t)$ also satisfies (3·3); and

$$\breve{q}_1(t) = (-s_0 + o(1))\,p_1(t). \qquad \ldots (4{\cdot}15)$$

Further suppose that (3·1) holds. Then $f(t) \to o\,[N, \breve{q}, \beta]_1$ if and only if

$$f(t) = g(t) + Ct^n e^{s_0 t} \qquad \ldots (4{\cdot}16)$$

where $g(t) \to o\,[N, p, \beta]_1$ and C is a constant.

PROOF: The first part of the lemma is given by (Choudhary 1970, Lemma 4). For the second part, suppose first that $f(t) \to O\,[N, \breve{q}, \beta]_1$. Write

$$\gamma(t) = \beta a(t) + (\breve{q}*a)_t + f(0)\,\breve{q}(t),$$

thus we are given that

$$\int_0^t |\gamma(u)|\,du = o(|\breve{q}_1(t)|)$$

$$= o(|p_1(t)|) \text{ (by (4·15))}. \qquad \ldots (4{\cdot}17)$$

Define $g(t)$ by (4·16) (where C is to be chosen later), *i.e.*,

$$g(t) = g(0) + \int_0^t \breve{b}(u)\,du$$

where

(i) $\breve{b}(t) = a(t) - Ck(t)$ and $k(t)$ is given by (3·7) ...(4·18)

and

(ii) $g(0) = f(0) - C\eta$ and η is given by (3·8).

In order to prove $g(t) \to O\,[N, p, \beta]_1$, we have to show that

$$\int_0^t |\sigma(u)|\,du = o(|p_1(t)|) \qquad \ldots (4{\cdot}19)$$

where

$$\sigma(u) = \beta\,\breve{b}(u) + (p*\breve{b})_u + (f(0) - C\eta)\,p(u).$$

With the aid of (4·12) and (4·18) we find that

$$\sigma(u) = \gamma(u) + (s_0+1)(e^{s_0 u} * \gamma)_u + \beta f(0)(s_0+1) e^{s_0 u} - C\phi(u)$$

where $\phi(u)$ is given by (3·10). Next, since $P(s)$ has a zero of order n at $s = s_0$, it follows from (3·12) that

$$\phi(u) = s_0 P^{(n)}(s_0)\, e^{s_0 u} - \int_0^\infty (-w)^n e^{-s_0 w} d_w\, p(u+w).$$

Thus, writing

$$A = \int_0^\infty e^{-s_0 v} \gamma(v)\, dv$$

(so that A is a complex constant), we obtain

$$\sigma(u) = \gamma(u) + (s_0+1)(A + \beta f(0)) e^{s_0 u} - (s_0+1) \int_u^\infty e^{s_0(u-v)} \gamma(v)\, dv$$

$$- C s_0 P^{(n)}(s_0)\, e^{s_0 u} + C \int_0^\infty (-w)^n e^{-s_0 w} d_w p(u+w).$$

We now choose

$$C = \frac{(s_0+1)(A+\beta f(0))}{s_0 P^{(n)}(s_0)}$$

so that

$$\sigma(u) = \gamma(u) - (s_0+1) \int_u^\infty e^{s_0(u-v)} \gamma(v)\, dv + C \int_0^\infty (-w)^n e^{-s_0 w} d_w p(u+w).$$

Thus, by (4·17) and (3·13), we get

$$\int_0^t |\sigma(u)|\, du \leqslant o(|p_1(t)|) + |(s_0+1)| \int_0^t \left| \int_0^\infty e^{-s_0 v} \gamma(u+v)\, dv \right| du. \qquad \ldots(4{\cdot}20)$$

By the same argument used in proving (3·13), it can be shown that the second term on the right side of (4·20) is $o(|p_1(t)|)$. This proves (4·19), and hence the necessity part of the lemma is proved.

To prove the sufficiency part, suppose that $g(t) \rightarrow O\,[N, p, \beta]_1$, (where $g(t) = g(0) + \int_0^t b(u)\, du$). Thus we are given that

$$\int_0^t | \psi(u) | \, du = o(| p_1(t) |) \qquad \ldots(4\cdot21)$$

where $\psi(t) = \beta\, b(t) + (p^*b)_t + g(0)\, p(t)$. Now

$$\int_0^t | \beta b(u) + (\breve{q}^*b)_u + g(0)\, \breve{q}(u) | \, du$$

$$= \int_0^t | \psi(u) - (s_0 + 1)(e)^{-u*}\psi)_t - \beta (s_0 + 1)\, g(0) e^{-u} | \, du$$

$$\leqslant o(| p_1(t) |) + | (s_0 + 1) | \int_0^t | (e^{-u*}\psi)_u | \, du$$

$$+ | \beta (s_0 + 1)\, g(0) | \int_0^t e^{-u} \, du \qquad \ldots(4\cdot22)$$

by (4·21). It can be easily verified that the last two terms on the right side of (4·22) are $o(| p_1(t) |)$. Thus

$$\int_0^t | \beta\, b(u) + (\breve{q}^*b)_u + g(0)\, \breve{q}(u)\, du = o(| p_1(t) |)$$

$$= o(| \breve{q}_1(t) |).$$

Hence $g(t) \to o\,[N, \breve{q}, \beta]_1$.

Finally, by Lemma 2, $\breve{q}(t)$ satisfies (4·8), and so, by Lemma 1, $Ct^n e^{s_0 t} \to O\,[N, \breve{q}, \beta]_1$; and hence $f(t) \to O\,[N, \breve{q}, \beta]_1$. This completes the proof of the lemma.

Lemma 5—Assume that (3·2), (3·3) and (4·6) hold. Define $q(t)$, $R(s)$, $P(s)$ and $Q(s)$ as defined in (4·1), (4·2), (4·3) and (4·4). Suppose that $R(s)$ has no zeros in Re $s \geqslant 0$ and that $r(t) \to 0$ as $t \to \infty$. Then $[N, p, \beta]_1$ and $[N, q, \alpha\beta]_1$ are equivalent.

PROOF : Suppose first that $f(t) \to s\,[N, p, \beta]_1$. Thus

$$\int_0^t | \phi(u) | \, du = o(| p_1(t) |).$$

where $\qquad \phi(t) = \beta a(t) + (p^*a)_t - (s - f(0))\, p(t).$

Now

$$\int_0^t | \alpha\beta a(u) + (q^*a)_u - (s-f(0))\, q(u) |\, du$$

$$= \int_0^t | \alpha\phi(u) + (r^*\phi)_u - \beta\,(s-f(0))\, r(u) |\, du$$

$$\leqslant o(|p_1(t)|) + \int_0^t | r(t-v) | \{\int_0^t |\phi(w)|\, dw\}\, dv + |\beta\,(s-f(0))| \int_0^t | r(u) |\, du$$

$$= o(|p_1(t)|) + I_1 + |\beta\,(s-f(0))|\, I_2 \text{ (say)}.$$

Since

$$\int_0^t |\phi(u)|\, du = o(|p_1(t)|), \int_0^\infty | r(t) |\, dt < \infty \text{ and } r(t) \to 0 \text{ as } t \to \infty,$$

therefore $I_1 = o(|p_1(t)|)$. Also, by (3·2), $I_2 = o(|p_1(t)|)$. Hence

$$\int_0^t | \alpha\beta\, a(u) + (q^*a)_u - (s-f(0))\, q(u) |\, du = o(|p_1(t)|)$$
$$= o(|q_1(t)|);$$

i.e., $[N, p, \beta]_1 \subseteq [N, q, \alpha\beta]_1$. ...(4·23)

Also, $P(s) = Q(s)/R(s)$ where $1/R(s)$ satisfies the same conditions as does $R(s)$ (see Lemma 6 of Choudhary 1970). We thus obtain

$$[N, q, \alpha\beta]_1 \subseteq [N, p, \beta]_1. \quad \text{...(4·24)}$$

Thus, from (4·23) and (4·24), the result follows.

5. Main Result

Theorem 1—Assume that (3·1) and (4·6) hold. Define $q(t)$, $R(s)$, $P(s)$ and $Q(s)$ as defined in (4·1), (4·2), (4·3) and (4·4).

Suppose that $\int_0^\infty | dr(t) | < \infty$ and that $r(t) \to 0$ as $t \to \infty$. Suppose also that no zeros of $R(s)$ occur on $\text{Re}\, s=0$, and let the zeros of $R(s)$ in Re $s>0$ occur at $s=s_1, s_2, \ldots, s_k$ (all different), with multiplicities $\gamma_1, \gamma_2, \ldots\ldots \gamma_k\ (>0)$, and suppose that the zeros of $R(s)$ in Re $s>0$ are zeros of $P(s)$ with non-negative multiplicities $\lambda_1, \lambda_2, \ldots, \lambda_k$.

Then $f(t) \to o\,[N, q, \alpha\beta]_1$ if and only if

$$f(t) = g(t) + \sum_{i=1}^{k} e^{s_i t} \sum_{j=\lambda_i}^{\lambda_i+\gamma_i-1} C_{ij}\, t^j \tag{5·1}$$

where $g(t) \to O[N, p, \beta]_1$ and $C_{ij} = \text{const.}$

PROOF: Consider the functions

$$P^*(s) = \prod_{i=1}^{k} \left(\frac{s-s_i}{s+1}\right)^{\gamma_i} P(s)$$

$$R^*(s) = \prod_{i=1}^{k} \left(\frac{s-s_i}{s+1}\right)^{-\gamma_i} R(s)$$

so that

$$Q(s) = P^*(s)\; R^*(s).$$

Here it may be observed that $R^*(s)$ satisfies the same conditions as does $R(s)$ (cf. the proof of Theorem 1 of Choudhary 1970).

It follows by repeated applications of Lemma 4 that $f(t) \to O\,[N, p^*, \beta]_1$ if and only if (5·1) holds with $g(t) \to O[N, p, \beta]_1$. But, by Lemma 5 with P^*, Q in place of P, Q

$f(t) \to o\,[N, q, \alpha\beta]_1$ if and only if $f(t) \to O\,[N, p^*, \beta]_1$, and the theorem follows.

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