AN EXTREMAL PROBLEM FOR FUNCTIONS WITH POSITIVE REAL PART

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Let $\mathfrak{P}$ be the class of regular analytic functions $P(z)$ in the unit circle $E = \{ z ! : | z | < 1\}$ with $P(0) = 1$ and Re $P(z) > 0$ in E. In the present paper complete solution of finding the extreme values of

$$\text{Re} \left\{ \mu P(z) \pm \frac{zP'(z)}{P(z) + \lambda} \right\}, \lambda > 0, -\infty < \mu < \infty, P(z) \in \mathfrak{P} \text{ has}$$

been given.

1. Introduction

Let E be the unit circle $\{z!: | z | < 1\}$ and P be the class of regular analytic functions $P(z)$ with $P(0) = 1$ and Re $P(z)>0$ in E. A large class of extremal problems in the class S of regular univalent functions in E and its subclasses lead to finding on $| z | = r$ the values of

$$\min_{P \in \mathfrak{P}} (\max \text{ Re } F(P(z), zP'(z)), \quad ...(1\cdot1)$$

where $F(u, v)$ is analytic in the half plane $Re\ u>0$ and the plane of v. Using variational methods Robertson (1963) proved the following:

Lemma A—If $F(u)$ is analytic in the half-plane $Re\ u>0$ and if $P(z) \in P$, then on $| z | = r<1$

$$\min_{P \in \mathfrak{P}} \text{Re } F(P(z)) = \min \text{ Re } F(P_0(z)), \quad ...(1\cdot2)$$

where

$$P_0(z) = \frac{1+z}{1-z} \cdot \quad ...(1\cdot3)$$

Lemma B—If $F(u, v)$ is analytic in the v-plane and in the half plane Re $u>0$, then on $| z | = r<1$

$$\min_{P \in \mathfrak{P}} \text{Re } F\{P(z), zP'(z)\} = \min_{\alpha, \theta, \phi} \text{Re } F\{p_0(z), zp_0'(z)\} \quad ...(1\cdot4)$$

where

$$p_0(z) = \frac{1+\alpha}{2} \frac{1+ze^{i\theta}}{1-ze^{i\theta}} + \frac{1-\alpha}{2} \frac{1+ze^{i\theta}}{1-ze^{-i\theta}}, \ z = re^{i\phi}, \quad ...(1\cdot5)$$

$$-1 \leqslant \alpha \leqslant 1, \ 0 \leqslant \theta \leqslant 2\pi \text{ and } 0 \leqslant \phi \leqslant 2\pi .$$

Using Lemma B Zomorovic (1965) obtained complete solution when $F(u, v)$ is given as follows

$$F\{P(z), zP'(z)\} = \pm \frac{1}{1+\lambda} P(z) + \frac{zP'(z)}{P(z)+\lambda}, \quad P \in \mathfrak{P}, \ \lambda > 0 \quad \ldots(1\cdot6)$$

Singh and Goel (1971) showed that extreme values of functionals of the form

$$\mathrm{Re}\left[\pm \mu P(z) + \frac{zP'(z)}{P(z)+\lambda}\right], \ \lambda > 0, \ 0 \leqslant \mu \leqslant 1, \ P(z) \in P, \quad \ldots(1\cdot7)$$

could be obtained by elementary methods which did not need the help of Lemma B.

In the present paper we obtain complete solution of finding the extreme values of

$$\mathrm{Re}\left[\mu P(z) \pm \frac{zP'(z)}{P(z)+\lambda}\right], \qquad \lambda > 0, \ -\infty < \mu < \infty, \ P \in \mathfrak{P}. \quad \ldots(1\cdot8)$$

The corresponding results when $\mu = 0$ are well known (Libera 1964, Zomorovic 1966) and have applications in the theory of univalent functions. The results obtained in the paper are directly useful in obtaining explicit results for a variety of extremal problems in the class S (Gupta 1973).

The main idea of our method is to replace $zP'(z)$ by using Lemma 1 (below) and obtain the extreme values of the resulting functional which is a function of Re $P(z)$ and $|P(z)|$ by using the variation formula for the class $\mathfrak{P}$ due to Sakaguchi (1964). This is an extension of the technique used by Gupta (1972). It should be noticed that the functional obtained after replacing $zP'(z)$ is not of the form needed in Lemma A and the paper has methodological interest in the sense that it systematically uses variational method to obtain explicit results for such a functional. We might also mention that Zomorovic (1965) had shown that Lemma 1 was true for functions of the form (1·5) but he seems to have overlooked the fact that it is true for the whole class $\mathfrak{P}$.

2. Preliminary

We shall, unless otherwise stated, adopt the following terminology:

$$|z| = r, \ a = \frac{1+r^2}{1-r^2}, \ \rho = \frac{2r}{1-r^2}, \ |P - a| = \rho_\cdot \leqslant \rho, \ P \in \mathfrak{P}$$

We shall denote by B the class of analytic functions $\phi(z)$, regular in E with $\phi(0) = 0$, $|\phi(z)| < 1$ for z in E.

With a view to subsequent application we reproduce below Sakaguchi's (1964) variational formula for P.

Lemma C—Let $p(z) \in \mathfrak{P}$. Then there exists a function $p^*(z) = p(z) + \delta p(z)$ belonging to $\mathfrak{P}$ with $\delta p(z)$ of the form

$$\frac{2}{\lambda}\delta p(z) = \varepsilon\left[p(z)\frac{1+\bar\beta z}{1-\bar\beta z} - \overline{p(\beta)}\left(p(z) - \frac{1+\bar\beta z}{1-\bar\beta z}\right) - 1\right] -$$
$$- \bar\varepsilon\left[p(z)\frac{\beta+z}{\beta-z} + p(\beta)\left(p(z) - \frac{\beta+z}{\beta-z}\right) - 1\right] + O(1) \quad \ldots(2\cdot1)$$

where β, ε are arbitrary complex numbers such that $|\beta| < 1$, $|\varepsilon| = 1$ and λ is a sufficiently small positive number.

We now proceed to prove the following:

Lemma 1—Let $P \in \mathcal{P}$. Then on $|z| = r$, $0 < r < 1$

$$\left| zP' - \frac{P^2 - 1}{2} \right| \leq \frac{\rho^2 - \rho_0^2}{2} \qquad \text{...(2·2}$$

where $|P - a| = \rho_0 \leq \rho$. This estimate is sharp.

PROOF: We first observe that for every $P \in \mathcal{P}$ there is a corresponding function $\phi(z) \in B$ such that

$$\phi(z) = \frac{P(z) - 1}{P(z) + 1}. \qquad \text{...(2·3)}$$

Moreover, the function

$$\psi(z) = \frac{\phi(z)}{z} \qquad \text{...(2·4)}$$

is likewise analytic in E and satisfies there $|\psi(z)| < 1$. Differentiating (2·4) and using the well known inequality (Nehari 1952, p. 18), we obtain

$$\left| \frac{\phi'(z)}{z} - \frac{\phi(z)}{z} \right| = |\psi'(z)| \leq \frac{1 - |\psi(z)|^2}{1 - r^2}. \qquad \text{...(2·5)}$$

On substituting for $\phi(z)$ and $\phi'(z)$ in terms of $P(z)$ and $P'(z)$ from (2·3) we obtain (2·2).

3. A Minimal Problem for $\mathcal{P}$

Theorem 1—Let $P \in \mathcal{P}$, then on $|z| = r$, $0 < r < 1$,

$$\text{Re}\left\{ \mu P + \frac{zP'}{P + \lambda} \right\}$$

$$\geq \begin{cases} -(\sqrt{\lambda(\mu + 1)} - \sqrt{a + \lambda})^2 \text{ if } \dfrac{\lambda(a + \lambda)}{(a - \rho + \lambda)^2} \geq \mu + 1 \geq \dfrac{\lambda(a + \lambda)}{(a + \rho + \lambda)^2} \\ (a - \rho)\left(\mu - \dfrac{\rho}{\lambda + a - \rho}\right) \text{ if } \mu + 1 > \dfrac{\lambda(a + \lambda)}{(a + \lambda - \rho)^2}; \\ (a + \rho)\left(\mu + \dfrac{\rho}{\lambda + a - \rho}\right) \text{ if } \mu + 1 < \dfrac{\lambda(a + \lambda)}{(a + \lambda + \rho)^2}; \end{cases} \qquad \text{...(3·1)}$$

where μ, λ are arbitrary but fixed real numbers, $-\infty < \mu < \infty$, $\lambda \geq 0$. The above estimates are sharp.

PROOF: Making use of (2·2) we obtain, on $|z| = r$

$$\text{Re}\left(\mu P + \frac{zP'}{P + \lambda}\right) \geq \text{Re}\left(\mu P + \tfrac{1}{2}\frac{P^2 - 1}{P + \lambda} - \tfrac{1}{2}\frac{\rho^2 - |P - a|^2}{|P + \lambda|}\right) \equiv \Psi_p. \qquad \text{...(3·2).}$$

We proceed to minimize the right side of (3·2). Since $P(ze^{i\theta}) \in \mathcal{P}$ wherever $P(z) \in \mathcal{P}$ for every real θ, we may assume that the minimum on $|z| = r$ occurs at $z = r$. Let p be the extremal function. Using Lemma C we obtain

$$\psi_p = \text{Re}\left(\mu p + \frac{1}{2}\frac{p^2-1}{p+\lambda} - \frac{1}{2}\frac{\rho^2 - |p-a|^2}{|p+\lambda|}\right) \qquad \ldots(3\cdot3)$$

$$\psi_p{}^* = \text{Re}\left(\mu(p+\delta p) + \frac{1}{2}\frac{(p+\delta p)^2-1}{p+\delta p+\lambda} - \frac{1}{2}\frac{\rho^2 - |p+\delta p-a|^2}{|p+\delta p+\lambda|}\right) \ldots(3\cdot4)$$

where δp is given by (3·1). We obtain, after some simplification

$$\psi_p{}^* - \psi_p = \text{Re}\,(w.\,\delta p) + o\,(1) \qquad \ldots(3\cdot5)$$

where

$$w = w(r) = \mu + \frac{1}{2} + \frac{1-\lambda^2}{2(p(r)+\lambda)^2} + \frac{\overline{p(r)}-a}{|p(r)+\lambda|} + \frac{1}{2}\frac{\rho^2 - |p(r)-a|^2}{(p(r)+\lambda)\,|p(r)+\lambda|} \qquad \ldots(3\cdot6)$$

Substituting the value of δp from (2·1) in (3·5), we get

$$\psi_p{}^*(r) - \psi_p(r) = \frac{\lambda}{2}\text{Re}\;\bar{\varepsilon}\Big[\left\{\overline{p(r)}\frac{1+\beta r}{1-\beta r} - p(\beta)\left(\overline{p(r)} - \frac{1+\beta r}{1-\beta r}\right) - 1\right\}\bar{w}$$

$$-\left\{p(r)\frac{\beta+r}{\beta-r} + p(\beta)\left(p(r) - \frac{\beta+r}{\beta+r}\right) - 1\right\}w\Big] + o\,(1). \ldots(3\cdot7)$$

Since p is extremal and ε is arbitrary with $|\varepsilon| = 1$ the expression within the square brackets in (3·7) must vanish. This gives, on changing β to z that

$$p(z) = \frac{A(z)}{B(z)} \qquad \ldots(3\cdot8)$$

where

$$A(z) = \left(\overline{p(r)}\frac{1+rz}{1-rz} - 1\right)\bar{w} - \left(p(r)\frac{z+r}{r-r} - 1\right)w$$

$$= -\frac{(\text{Re}\,(pw) + i\,\text{Im}\,w)}{(1-rz)(z-r)}A_1(z), \qquad \ldots(3\cdot9)$$

$$B(z) = \left(\overline{p(r)} - \frac{1+rz}{1-rz}\right)\bar{w} + \left(p(r) - \frac{z+r}{z-r}\right)w$$

$$= -\frac{(\text{Re}\,(pw) + i\,\text{Im}\,w)}{(1-rz)(z-r)}B_1(z), \qquad \ldots(3\cdot10)$$

$$A_1(z) = 1 + \frac{2i}{p}\left(\frac{\text{Im}\,(wp) - a\,\text{Im}\,w}{\text{Re}\,(wp) + i\,\text{Im}\,w}\right)z - \left(\frac{\text{Re}\,(wp) - i\,\text{Im}\,w}{\text{Re}\,(wp) + i\,\text{Im}\,w}\right)z^2,$$

$$B_1(z) = 1 - \frac{2}{p}\left(\frac{a\,\text{Re}\,(wp) - \text{Re}\,w}{\text{Re}\,(wp) + i\,\text{Im}\,w}\right)z + \left(\frac{\text{Re}\,(wp) - i\,\text{Im}\,w}{\text{Re}\,(wp) + i\,\text{Im}\,w}\right)z^2,$$

$w = w(r)$ and $p = p(r)$.

For the validity of (3·8) it is necessary that $B(z)$ does not vanish identically. However, $B(z) \equiv 0$ only if $w = 0$. Hence we first investigate the case $w = 0$ which is contained in the following:

Lemma 2—$w = 0$ is possible only if the following inequality is satisfied:

$$\frac{\lambda(a+\lambda)}{(a-\rho+\lambda)^2} \geqslant \mu + 1 \geqslant \frac{\lambda(a+\lambda)}{(a+\rho+\lambda)^2}.$$

Further in this case $\operatorname{Im} p = 0$, $p + \lambda = \sqrt{\dfrac{\lambda(a+\lambda)}{\mu+1}}$ and

$$\min \Psi_p = -\left[\sqrt{\lambda(\mu+1)} - \sqrt{a+\lambda}\right]^2.$$

PROOF: On putting $p + \lambda = Re^{i\phi}$ and equating the real and imaginary parts of w to zero we obtain the equations

$$\mu + \frac{1}{2} + \frac{(1-\lambda)^2}{2R^2}\cos 2\phi + \frac{R\cos\phi - \lambda - a}{R} + \frac{1}{2}\,\frac{\rho^2 - R^2 - (a+\lambda)^2 + 2R(a+\lambda)\cos\phi}{R^2} \times \cos\phi = 0. \qquad \ldots(3\cdot11)$$

$$\frac{1-\lambda^2}{2R^2}\sin 2\phi + \sin\phi + \frac{1}{2}\,\frac{\rho^2 - R^2 - (a+\lambda)^2 + 2R(a+\lambda)\cos\phi}{R^2}\sin\phi = 0 \quad \ldots(3\cdot12)$$

From (3·12) we obtain, either

$\sin\phi = 0.$

or

$$R^2 + 2(a+\lambda)R\cos\phi + 2(1-\lambda^2)\cos\phi - 1 - \lambda^2 - 2a\lambda = 0. \qquad \ldots(3\cdot13)$$

We show that (3·13) is not possible. Using the well-known inequality

$$|Re^{i\phi} - (a+\lambda)|^2 \leqslant \rho^2 \qquad \ldots(3\cdot14)$$

for functions $p \in \mathcal{P}$ in the form

$$R^2 + 1 + 2a\lambda + \lambda^2 \leqslant 2(\lambda + a)R\cos\phi, \qquad \ldots(3\cdot15)$$

we see that the expression on the left of (3·13)

$\geqslant 2(R^2 + (1-\lambda^2)\cos\phi) \geqslant 0$ if $\lambda \leqslant$ since $|\phi| < \pi/2$.

In case $\lambda > 1$, the left side of (3·13) is

$\geqslant 2(R^2 + (1-\lambda^2))$

$\geqslant 2((a-\rho+\lambda)^2 + 1 - \lambda^2) = 2((a-\rho)^2 + 2(a-\rho)\lambda + 1) > 0$

where we have made use of the following consequence from (3·14)

$$a - \rho + \lambda \leqslant R \leqslant a + \rho + \lambda. \qquad \ldots(3\cdot16)$$

Therefore, we see that (3·13) is not possible for any $\lambda \geqslant 0$ and we must have $\sin\phi = 0$, that is, $\phi = 0$.

Putting $\phi = 0$ in (3·11) we obtain

$$R = R_0 = \sqrt{\frac{\lambda(a+\lambda)}{\mu+1}}. \qquad \ldots(3\cdot17)$$

In order that R_0 given by (3·17) be acceptable, we must have

$$R_0 \in [a - \rho + \lambda, a + \rho + \lambda].$$

This is possible only if

$$a - \rho + \lambda \leqslant \sqrt{\frac{\mu (a + \lambda)}{\mu + 1}} \leqslant a + \rho + \lambda,$$

or

$$\frac{\lambda (a + \lambda)}{(a - \rho + \lambda)^2} \geqslant \mu + 1 \geqslant \frac{\lambda (a + \lambda)}{(a + \rho + \lambda)^2}.$$

In this case the minimum of ψ_p is attained by

$$\rho + \lambda = R_0 = \sqrt{\frac{\lambda (a + \lambda)}{\mu + 1}},$$

and equals (on substitution in (3·3) and simplification)

$$-(\sqrt{\lambda (\mu + 1)} - \sqrt{a + \lambda})^2,$$

This completes the proof of Lemma 2.

Thus $W = 0$ is not possible if either

$$\mu + 1 > \frac{\lambda (a + \lambda)}{(a - \rho + \lambda)^2} \text{ or } \mu + 1 < \frac{\lambda (a + \lambda)}{(a + \rho + \lambda)^2}, \qquad \ldots(3\cdot18)$$

and $p(z)$ is now given by (3·8). Since on $|z| = 1$, $A(z)$ is pure imaginary and $B(z)$ is real, $p(z)$ is a rational function of z satisfying $\operatorname{Re} p(z) = 0$ on $|z| = 1$. Therefore $p(z)$ has no poles in $|z| \neq 1$ but has at least one pole on $|z| = 1$.

As $A(z)$ has no poles on $|z| = 1$ it further follows that $B(z)$ must have at least one zero on $|z| = 1$.

For $\delta > 0$ consider the variation

$$p^*(z) = \frac{p(z) + \delta \phi (z)}{1 + \delta} = p(z) + \delta (\phi(z) - p(z)) + o(\delta),$$

where $$\phi (z) = \frac{1 + \bar{\beta} z}{1 - \bar{\beta} z}, \; |\beta| < 1.$$

Then $p^*(z) \in \mathfrak{P}$ whenever $p(z) \in \mathfrak{P}$. For this variation we have

$$\psi_{p^*(r)} - \psi_{p(r)} = \frac{\delta}{2} \left[w \left(\frac{1 + \bar{\beta} r}{1 - \bar{\beta} r} - p(r) \right) + \bar{w} \left(\frac{1 + \beta r}{1 - \beta r} - \overline{p(r)} \right) \right] + o(\delta).$$

Since p gives the minimum and $\delta > 0$, we must have

$$\left(\overline{p(r)} - \frac{1 + \beta r}{1 - \beta r} \right) \bar{w} + \left(p(r) - \frac{1 + \bar{\beta} r}{1 - \bar{\beta} r} \right) w \leqslant 0.$$

Letting $|\beta| \to 1$, we obtain

$$B(\beta) \not< 0 \qquad \text{on } |\beta| = 1,$$

that is,

$$B(z) \not< 0 \qquad \text{on } |z| = 1. \qquad \ldots(3\cdot19)$$

From this we conclude that the zeros of $B(z)$ on $|z| = 1$ are of even order. Since $B(z) = \frac{B_1(z)}{(1-rz)(z-r)}$ where $B_1(z)$ is a quadratic in z, it follows that $B(z)$ has a single zero of order two on $|z| = 1$. As poles of $p(z)$ on $|z| = 1$ are necessarily simple it follows that $A(z)$ and $B(z)$ have a common zero. Accordingly $p(z)$ must have the form

$$p_\theta(z) = \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z}, \qquad 0 \leqslant \theta < 2\pi, \qquad \ldots(3\cdot20)$$

which may be written in the form

$$p_\theta(z) = \frac{A_1(z)}{B_1(z)} = \frac{1 - e^{2i\theta}z^2}{1 - 2e^{i\theta}z + e^{2i\theta}z^2}. \qquad \ldots(3\cdot21)$$

Comparing the coefficients of like powers of z in the numerator and denominator on both the sides of (3·21) we obtain the equations

$$\frac{\operatorname{Re}(wp) - i \operatorname{Im} w}{\operatorname{Re}(wp) + i \operatorname{Im} w} = e^{2i\theta}, \qquad \ldots(3\cdot22)$$

$$\frac{a \operatorname{Re}(wp) - \operatorname{Re} w}{\rho\,(\operatorname{Re}(wp) + i \operatorname{Im} w)} = e^{i\theta}, \qquad \ldots(3\cdot23)$$

$$a \operatorname{Im} w - \operatorname{Im}(wp) = 0. \qquad \ldots(3\cdot24)$$

From (3·23) we obtain

$$2re^{i\theta}(\operatorname{Re}(wp) + i \operatorname{Im} w) = (1 + r^2)\operatorname{Re}(wp) - (1-r^2)\operatorname{Re} w. \qquad \ldots(3\cdot25)$$

Again, since $\frac{z}{(1 - rz)(z - r)} = \frac{1}{(z - r)(\bar{z} - r)} = \frac{1}{|z - r^2|^2} > 0$ on $|z| = 1$, we obtain from (3·10), in view of (3·19)

$$(\operatorname{Re}(pw) + i \operatorname{Im} w)\frac{B_1(z)}{z} \geqslant 0, \quad \text{on } |z| = 1;$$

that is,

$$(\operatorname{Re}(pw) + i \operatorname{Im} w)\, e^{i\theta}\left(ze^{i\theta} + \frac{1}{z}e^{-i\theta} - 2\right) \geqslant 0, \quad \text{on } |z| = 1,$$

which is possible only if

$$(\operatorname{Re}(pw) + i \operatorname{Im} w)\, e^{i\theta} \leqslant 0. \qquad \ldots(3\cdot26)$$

In view of (3·26) we may write (3·25) in the form

$$-2r\,|\operatorname{Re}(wp) + i \operatorname{Im} w| = (1 + r^2)\operatorname{Re}(wp) - (1 - r^2)\operatorname{Re} w. \qquad \ldots(3\cdot27)$$

Again, from (3·6) we easily obtain

$$\operatorname{Im} w = -\operatorname{Im}(p + \lambda)\left[\frac{1 - \lambda^2}{|p + \lambda|^4}\operatorname{Re}(p + \lambda) + \frac{1}{|p + \lambda|}\,\frac{1}{2}\,\frac{\rho^2 - |p - a|^2}{|p + \lambda|^3}\right]$$

and $\ldots(3\cdot28)$

$$\operatorname{Im}\{w(p + \lambda)\} = \operatorname{Im}(p + \lambda)\left[\mu + \frac{1}{2}\,\frac{1 - \lambda^2}{2|p + \lambda|^2} - \frac{a + \lambda}{|p + \lambda|}\right]. \qquad \ldots(3\cdot29)$$

Substituting from (3·28) and (3·29) for Im w and Im w $\{w(p+\lambda)\}$ in (3·24) we obtain

$$\mathrm{Im}\,(p+\lambda)\left[\left\{\frac{(1-\lambda^2)\,\mathrm{Re}\,(p+\lambda)}{|p+\lambda|^4}+\frac{1}{|p+\lambda|}+\frac{1}{2}\,\frac{\rho^2-|p-a|^2}{|p+\lambda|^3}\right\}\right.$$

$$\left.\times(a+\lambda)+\mu+\frac{1}{2}-\frac{1-\lambda^2}{2\,|p+\lambda|^2}-\frac{a+\lambda}{|p+\lambda|}\right]=0. \qquad \ldots(3\cdot30)$$

Hence either (3·31)

$$\mathrm{Im}\,(p+\lambda)=0, \qquad \ldots(3\cdot31)$$

or

$$\left\{\frac{(1-\lambda^2)\,\mathrm{Re}\,(p+\lambda)}{|p+\lambda|^4}+\frac{1}{|p+\lambda|}+\frac{1}{2}\,\frac{\rho^2-|p-a|^2}{|p+\lambda|^3}\right\}(a+\lambda)+\mu+\frac{1}{2}$$

$$-\frac{1-\lambda^2}{2\,|p+\lambda|^2}-\frac{a+\lambda}{|p+\lambda|}=0. \qquad \ldots(3\cdot32)$$

(3·31) implies that $(p+\lambda)$ is real and in view of (3·28) that w is real. However w is now given by

$$w=1+\mu-\frac{\lambda(a+\lambda)}{(p+\lambda)^2}. \qquad \ldots(3\cdot33)$$

From (3·27) we now have

$$\frac{-\,2rp\,|w|}{w}=(1+r^2)\,p-(1-r^2). \qquad \ldots(3\cdot34)$$

If $\mu+1>\dfrac{\lambda(a+\lambda)}{(a+\lambda-\rho)^2}$ then from (3·33) $w>0$ and (3·34) gives

$$p(r)=\frac{1-r}{1+r}, \qquad \ldots(3\cdot35)$$

and (3·20) reduces to $p(z)=\dfrac{1-z}{1+z}$.

The minimum of ψ_p is obtained from (3·3) by putting $p=\dfrac{1-r}{1+r}=a-$ and equals $(a-p)\left(\mu-\dfrac{\rho}{(\lambda+a-\rho)}\right)$.

If $\mu+1<\dfrac{\lambda(a+\lambda)}{(a+\lambda+\rho)^2}$ then from (3·33) $w<0$ and (3·34) yields $p(r)=\dfrac{1+r}{1-r}$, ...(3·36)

and (3·20) reduces to $p(z)=\dfrac{1+z}{1-z}$. The minimum of ψ_p now equals $(a+\rho)\left(\mu+\dfrac{\rho}{\lambda+a+\rho}\right)$.

We have yet to investigate whether (3·32) holds. We proceed to show that this is not possible.

For functions of the form (3·20) we have the well-known equality

$$|p + \lambda - (a + \lambda)|^2 = \rho^2, \qquad \ldots(3\cdot37)$$

on $|z| = r$ which may also be written as

$$|p + \lambda|^2 + 1 + \lambda + 2a\lambda = 2(a + \lambda)\,\mathrm{Re}\,(p + \lambda). \qquad \ldots(3\cdot38)$$

Making use of (3·38) we see that (3·32) can hold if and only if

$$|p + \lambda|^4 = \frac{(\lambda^2 - 1)(a + \lambda - \rho)(a + \lambda + \rho)}{2\mu + 1}. \qquad \ldots(3\cdot39)$$

This can be valid only if $\lambda^2 - 1$ and $2\mu + 1$ have the same sign, that is, only if either

(*a*) $\lambda \geqslant 1$ and $2\mu + 1 \geqslant 0$.

or

(*b*) $0 \leqslant \lambda \leqslant 1$ and $2\mu + 1 \leqslant 0$.

Since $|p + \lambda|$ is to satisfy (3·16), (3·39) is acceptable only if

$$(a - \rho + \lambda)^2 \leqslant \sqrt{\frac{(\lambda^2 - 1)(a + \lambda - \rho)(a + \lambda + \rho)}{2\mu + 1}} \leqslant (a + \rho + \lambda)^2;$$

that is, if

Case (*a*)

$$\frac{(\lambda^2 - 1)(a + \lambda - \rho)}{(a + \lambda + \rho)^3} \leqslant 2\mu + 1 \leqslant \frac{(\lambda^2 - 1)(a + \lambda + \rho)}{(a + \lambda - \rho)^3}, \text{ and}$$

$$\lambda \geqslant 1,\ 2\mu + 1 \geqslant 0. \qquad \ldots(3\cdot40)$$

Or

Case (*b*)

$$\frac{(1 - \lambda^2)(a + \lambda - \rho)}{(a + \lambda + \rho)^3} \leqslant -(2\mu + 1) \leqslant \frac{(1 - \lambda^2)(a + \lambda + \rho)}{(a + \lambda - \rho)^3}, \text{ and}$$

$$0 \leqslant \lambda \leqslant 1,\ -(2\mu + 1) \geqslant 0. \qquad \ldots(3\cdot41)$$

Case (*a*)—In this case $\lambda \geqslant 1$ and (3·40) hold. The two possibilities (3·18) give rise to the following cases:

(Ia) $0 \leqslant 2\mu + 1 < \dfrac{2\lambda(a + \lambda)}{(a + \rho + \lambda)^2} - 1$, and (3·40).

(IIa) $2\mu + 1 > \dfrac{2\lambda(a + \lambda)}{(a - \rho + \lambda)^2} - 1$, and (3·40).

To show that (Ia) is not possible it is enough to show that

$$\frac{2\lambda(a + \lambda)}{(a + \rho + \lambda)^2} - 1 \leqslant \frac{(\lambda^2 - 1)(a + \lambda - \rho)}{(a + \lambda + \rho)^3},$$

which on simplification yields

$$(a+\rho)^3-(a-\rho)+2\lambda\left((a+\rho)^2-1\right)\geqslant 0.$$

which is always true. Thus (Ia) is not possible.

We proceed to show that (IIa) is not possible. We first write the inequality

$$2\mu+1>\frac{2\lambda\,(a+\lambda)}{(a-\rho+\lambda)^2}-1,$$

in the equivalent form

$$C=\mu\lambda^2+\lambda\left\{2(\mu+1)(a-\rho)-a\right\}+(\mu+1)(a-\rho)^2>0. \qquad \ldots(3{\cdot}42)$$

We shall show that in this case

$$2\mu+1>\frac{(\lambda^2-1)(a+\lambda+\rho)}{(a+\lambda-\rho)^3} \qquad \ldots(3{\cdot}43)$$

demonstrating that (IIa) is not possible. To establish (3·43) is equivalent to showing that

$$D=2\mu\lambda^3+\left\{3(2\mu+1)(a-\rho)-(a+\rho)\right\}\lambda^2+\left\{3(2\mu+1)(a-\rho)^2+1\right\}\lambda$$
$$+(2\mu+1)(a-\rho)^3+(a-\rho)>0.$$

But it is readily verified that

$$D=2(\lambda+a-\rho)\,C+\lambda\left\{-3(a-\rho)^2+1+2a(a-\rho)\right\}-(a-\rho)^3+(o+\rho)$$

and this is positive since $C>0$ and $a-\rho=\dfrac{1-r}{1+r}>1$.

Case (*b*)—We show that in this case (3·39) yields the maximum and not the minimum of ψ_p. Indeed, since p is of the form (3·20) we obtain from (3·3)

$$\psi_p=\operatorname{Re}\left[\mu p+\frac{1}{2}\,\frac{p^2-1}{p+\lambda}\right],$$

where we have made use of (3·37). Writing $p+\lambda=Re^{i\phi}$ and making use of (3·38) this may be rewritten as

$$\psi_p=\psi=\frac{1}{4(a+\lambda)}\left[(2\mu+1)R^2-\frac{(1-\lambda^2)(1+2a\lambda+\lambda^2)}{R^2}\right.$$
$$\left.+(2\mu+1)(1+2a\lambda+\lambda^2)-(1-\lambda^2)-2(a+\lambda)(\mu+1)\lambda\right],$$

from which it easily follows that

$$\frac{d\psi}{dR}=0 \text{ if } R^4=\frac{(\lambda^2-1)(1+2a\lambda+\lambda)^2}{2\mu+1}$$

which coincides with (3·39) and that $\dfrac{d^2\psi}{dR^2}<0$ for this value of R since $2\mu+1\leqslant 0$ in case (*b*). Consequently (3·39) yields the maximum of ψ.

To complete the proof of the theorem we must show that the estimates in (3·1) are sharp. It is readily verified that, given μ, λ and r, if

$$\frac{\lambda\,(a+\lambda)}{(a-\rho+\lambda)^2}\geqslant\mu+1\geqslant\frac{\lambda\,(a+\lambda)}{(a+\rho+\lambda)^2},$$

the function $\dfrac{1 - z^2}{1 - 2z \cos \theta + z^2}$,

where $\cos \theta$ is to be determined from the equation

$$\frac{1 - r^2}{1 - 2r \cos \theta + r^2} + \lambda = \sqrt{\frac{\lambda\,(a + \lambda)}{\mu + 1}}\,,$$

belongs to $\mathfrak{P}$ and yields equality in the first inequality of (3·1).

If $\mu + 1 > \dfrac{\lambda\,(a + \lambda)}{(a + \lambda - \rho)^2}$ the function $\dfrac{1 - z}{1 + z}$ yields equality in the second inequality of (3·1). If $\mu + 1 < \dfrac{\lambda\,(a + \lambda)}{(a + \lambda - \rho)^2}$ the function $\dfrac{1 + z}{1 - z}$ yields equality in the last inequality of (3·1).

More generally, extremal functions are of the type $p(ze^{i\gamma})$ for arbitrary real γ. This completes the proof of the theorem.

Corollary—Let $P \in \mathfrak{P}$, then on $|z| = r$, $0 < r < 1$,

$$\operatorname{Re}\left(\mu P - \frac{zP'}{P + \lambda}\right) \leq$$

$$\leq \begin{cases} \left(\sqrt{\lambda\,(1 - \mu)} - \sqrt{a + \lambda}\right)^2 & \text{if } \dfrac{\lambda\,(a + \lambda)}{(a - \rho + \lambda)^2} \geq 1 - \mu \geq \dfrac{\lambda\,(a + \lambda)}{(a + \rho + \lambda)^2} \\ (a - \rho)\left(\mu + \dfrac{\rho}{\lambda + a - \rho}\right) & \text{if } 1 - \mu > \dfrac{\lambda\,(a + \lambda)}{(a + \lambda - \rho)^2}\,; \\ (a + \rho)\left(\mu - \dfrac{\rho}{\lambda + a + \rho}\right) & \text{if } 1 - \mu < \dfrac{\lambda\,(a + \lambda)}{(a + \lambda + \rho)^2}\,; \end{cases}$$

where μ, λ are given numbers, $- < \mu < \infty$, $\lambda \geq 0$. The above estimates are sharp.

PROOF: The above results follow immediately from (3·1) by changing μ to $-\mu$.

4. A Maximal Problem for $\mathfrak{P}$

Theorem 2—Let $P \in \mathfrak{P}$, then on $|z| = r$, $0 < r < 1$

$$\operatorname{Re}\left\{\mu P + \frac{zP'}{P + \lambda}\right\}$$

$$\leq \begin{cases} a - \mu\lambda - 2\sqrt{-\mu\,(1 + a\lambda)} & \text{if } \dfrac{1 + a\lambda}{(a + \rho + \lambda)^2} \leq -\mu \leq \dfrac{1 + a\lambda}{(a - \rho + \lambda)^2}\,; \\ (a + \rho)\left(\mu + \dfrac{\rho}{a + \rho + \lambda}\right) & \text{if } -\mu < \dfrac{1 + a\lambda}{(a + \rho + \lambda)^2}\,; \\ \dfrac{\mu\,(1 + a\lambda) - \sqrt{(2\mu + 1)(\lambda^2 - 1)(1 + 2a\lambda + \lambda^2)}}{2(a + \lambda)} & \end{cases}$$

(*Continued next page*)

(from per page)

$$\left| \begin{array}{l} \text{if } -\mu > \dfrac{1 + a\lambda}{(a - \rho + \lambda)^2} \text{ and } (e)\ 0 \leq \lambda < 1,\ 2\mu + 1 < 0 \\ \text{and } (a - \rho)(\lambda - \rho) + \lambda^2 \leq 0; \\ (a - \rho)\left(\mu - \dfrac{\rho}{a - \rho + \lambda}\right) \text{ if } -\mu > \dfrac{1 + a\lambda}{(a - \rho + \lambda)^2} \end{array} \right.$$

and (e) does not hold; ...(4·1)

where μ, λ are arbitrary but fixed real numbers, $-\infty < \mu < \infty$, $\lambda \geqslant 0$. The above estimates are sharp.

PROOF: Making use of (2·2) we obtain, on $|z| = r$

$$\text{Re}\left(\mu P + \frac{zP'}{P + \lambda}\right) \geqslant \text{Re}\left(\mu P + \frac{1}{2}\frac{P^2 - 1}{P + \lambda} + \frac{1}{2}\frac{\rho^2 - |P - a|^2}{|P + \lambda|}\right) \equiv \psi_p \quad ...(4·2)$$

We maximize the right side. Assume that the maximum occurs at $z = r$ and let p be the extremal function. Using Lemma C we obtain

$$\psi_p = Re\left(\mu p + \frac{1}{2}\frac{p^2 - 1}{p + \lambda} + \frac{1}{2}\frac{p^2 - |p - a|^2}{|p + \lambda|}\right), \quad (4·3)$$

$$\psi_{p_*} = Re\left(\mu(p + \delta p) + \frac{1}{2}\frac{(p + \delta p)^2 - 1}{(p + \delta p + \lambda)} + \frac{1}{2}\frac{\rho^2 - |p + \delta p - a|^2}{|p + \delta p + \lambda|}\right) \quad ...(4·4)$$

where $p^* = p + \delta p$ and δp is given by (2·1). We obtain, after simplification

$$\psi_{p*} - \psi_p = \text{Re}\,(w.\,\delta p) + o\,(1),$$

where

$$w = \mu + \frac{1}{2} + \frac{1 - \lambda^2}{2(p + \lambda)^2} - \frac{\bar{p} - a}{|p + \lambda|} - \frac{1}{2}\frac{\rho^2 - |p - a|^2}{(p + \lambda)|p + \lambda|}. \quad ...(4·5)$$

Proceeding as in Theorem 1 we find that the extremal function is given by

$$p(z) = \frac{A(z)}{B(z)} \quad ...(4·5a)$$

where $A(z)$ and $B(z)$ are given by (3·9), (3·10) and w is given by (4·5), provided $B(z) \not\equiv 0$. But $B(z) \equiv 0$ only if $w = 0$. We now prove the following:

Lemma 3—For the extremal function $p(z)$, $w = 0$ only if $\text{Im}\, p(r) = 0$.

PROOF: Let $w = 0$. Since $p + \lambda \neq 0$, this implies that $(p + \lambda)\,w = 0$. Equating the real and imaginary parts of $(p + \lambda)\,w$ to zero we obtain the equations.

$$\{(2\mu + 1)R^2 + 2(\lambda + a)R - (1 - \lambda^2)\}\sin\phi = 0 \quad ...(4·6)$$

$$\{(2\mu + 1)R^2 + (1 - \lambda^2)\}\cos\phi + 1 + 2a\lambda + \lambda^2 - R^2 = 0 \quad ...(4·7)$$

where $p + \lambda = Re^{i\phi}$. From (4.6), (4.7) we obtain, either

$$\phi = 0 \text{ and } R^2 = \frac{1 + a\lambda}{-\mu},$$

or

$$(2\mu + 1)R^2 + 2(\lambda + a)R - (1 - \lambda^2) = 0 \quad ...(4·8)$$

and (4·7) holds. We proceed to show that the solutions of eqns. (4·7) and (4·8) do not yield the maximum of ψ_p. Indeed, $w = 0$ corresponds to

$$\frac{\partial \psi_p}{\partial \phi} = -\frac{1}{2}\left[(2\mu + 1) R + \frac{\lambda^2 - 1}{R} + 2(a + \lambda)\right] \sin \phi = 0 \qquad \ldots(4{\cdot}9)$$

$$\frac{\partial \psi_p}{\partial R} = \frac{1}{2}\left[\left((2\mu + 1) - \frac{\lambda^2 - 1}{R^2}\right) \cos \phi - 1 + \frac{1 + 2a\lambda + \lambda^2}{R^2}\right] = 0 \qquad \ldots(4{\cdot}10)$$

which are the same as (4·6) and (4·7). Thus the equations w=0 contain the solutions for the maximum and minimum of ψ_p. In order to distinguish which solutions give the maximum, we need to go to the sufficient conditions which involve the second partial derivatives. Hence to show that (4·7) and (4.8) do not yield the maximum of ψ_p it is sufficient to show that

$$\left(\frac{\partial^2 \psi_p}{\partial \phi\, \partial R}\right)^2 - \frac{\partial^2 \psi_p}{\partial \phi^2} \cdot \frac{\partial^2 \psi_p}{\partial R^2} > 0 \qquad \ldots(4{\cdot}11)$$

when ϕ, R are given by (4·7) and (4·8). We obtain from (4·9) and (4·10).

$$\frac{\partial^2 \psi_p}{\partial \phi^2} = -\frac{1}{2}\left[(2\mu + 1) R + \frac{\lambda^2 - 1}{R} + 2(a + \lambda)\right] \cos \phi \qquad \ldots(4{\cdot}12)$$

$$\frac{\partial^2 \psi_p}{\partial \phi\, \partial R} = -\frac{1}{2}\left[2\mu + 1 + \frac{1 - \lambda^2}{R^2}\right] \sin \phi. \qquad \ldots(4{\cdot}13)$$

Because of (4·8), (4·12) yields

$$\frac{\partial^2 \psi}{\partial \phi^2} = 0. \qquad \ldots(4{\cdot}14)$$

We show that

$$\frac{\partial^2 \psi}{\partial \phi \partial R} \neq 0. \qquad \ldots(4{\cdot}15)$$

If $\dfrac{\partial^2 \psi}{\partial \phi \partial R} = 0$, we obtain from (4·13)

$$(2\mu + 1) R^2 + (1 - \lambda^2) = 0.$$

This combined with (4·8) yields

$$R = \frac{1 - \lambda^2}{a + \lambda}, \qquad \ldots(4{\cdot}16)$$

and because $R > 0$ we must have $0 \nless \lambda \leqslant 1$.

Again, from (4·7), (4·8) we obtain

$$2\,[(1 - \lambda^2) - (\lambda + a) R] \cos \phi + 1 + 2a\lambda + \lambda^2 - R^2 = 0.$$

Substituting the value of R from (4·16) this yields

$$(1 + 2a\lambda + \lambda^2)(a + \lambda)^2 - (1 - \lambda^2)^2 = 0$$

which is not possible since the first term is greater than 1 whereas the second term does not exceed 1. Thus (4·15) holds. Because of (4·14) and (4·15), (4·11) holds. It follows that w=0 can only lead to ϕ=0 and proof of the lemma is complete.

We continue with the proof of the theorem. From Lemma 3 we see that $w = 0$ implies $\phi = 0$. Putting $\phi = 0$ in (4·7) we see that the maximum of ψ_p is attained by $p + \lambda = R_0$ where $R_0 = \sqrt{\dfrac{1+a}{-\mu}}$.

Two cases now arise:

Case I – $R_0 \in [a - \rho + \lambda, a + \rho + \lambda]$.

In this case $a - \rho + \lambda \leqslant \sqrt{\dfrac{1 + a\lambda}{-\mu}} \leqslant a + \rho + \lambda$,

that is,

$$\frac{1 + a\lambda}{(a + \rho + \lambda)^2} \leqslant -\mu \leqslant \frac{1 + a\lambda}{(a - \rho + \lambda)^2}.$$

The maximum of ψ_p is obtained by putting $p + \lambda = R_0 = \sqrt{\dfrac{1 + a\lambda}{-\mu}}$ in (4·3) and equals $a - \mu\lambda - 2\sqrt{-\mu(1 + a\lambda)}$.

Case II—$R_0\, [a - \rho + \lambda, a + \rho + \lambda]$.

In this case either

$$-\mu < \frac{1 + a\lambda}{(a + \rho + \lambda)^2} \text{ or } -\mu > \frac{1 + a\lambda}{(a - \rho + \lambda)^2}. \qquad \ldots(4\cdot17)$$

Moreover, $w \neq 0$ and so the extremal function is given by (4·5).

As in Theorem 1 we may prove that on $|z| = 1$

$$B(z) > 0 \qquad \ldots(4\cdot18)$$

and that the extremal function is given by

$$p_\theta(z) = \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z} \qquad \ldots(4\cdot19)$$

which may be written in the form

$$p_\theta(z) = \frac{A_1(z)}{B_1(z)} = \frac{1 - e^{2i\theta}}{1 - 2e^{i\theta}z + e^{2i\theta}z^2} \qquad \ldots(4\cdot20)$$

where $A_1(z)$, $B_1(z)$ are given by (3·9), (3·10). We again obtain the equations

$$\frac{\text{Re}(wp) - i\,\text{Im}\,w}{\text{Re}(wp) + i\,\text{Im}\,w} = e^{2i\theta} \qquad \ldots(4\cdot21)$$

$$\frac{a\,\text{Re}(wp) - \text{Re}\,w}{\rho\,\text{Re}(wp) + i\,\text{Im}\,w)} = e^{i\theta} \qquad \ldots(4\cdot22)$$

$$a\,\text{Im}\,w - \text{Im}(wp) = 0. \qquad \ldots(4\cdot23)$$

From (4·22) we obtain

$$2re^{i\theta}(\text{Re}(wp) + i\,\text{Im}\,w) = (1 + r^2)\,\text{Re}(wp) - (1 - r^2)\,\text{Re}\,w. \qquad \ldots(4\cdot24)$$

Because of (4·18), however, (3·26) now becomes

$$(\text{Re}(pw) + i\,\text{Im}\,w)\,e^{i\theta} > 0. \qquad \ldots(4\cdot25)$$

in view of (4·25), (4·24) may be written in the form

$$2r \mid \operatorname{Re}(wp) + i \operatorname{Im} w \mid = (1 + r^2) \operatorname{Re}(pw) - (1 - r^2) \operatorname{Re} w. \qquad \ldots(4\cdot26)$$

Again, from (4·5) we easily obtain

$$\operatorname{Im} w = - \operatorname{Im}(p + \lambda) \left[\frac{(1 - \lambda^2) \operatorname{Re}(p + \lambda)}{|p + \lambda|^4} - \frac{1}{|p + \lambda|} - \frac{1}{2} \frac{\rho^2 - |p - a|^2}{|p + \lambda|^3} \right], \qquad \ldots(4\cdot27)$$

$$Im\, w\,(p + \lambda) = Im\,(p + \lambda) \left[\mu + \frac{1}{2} - \frac{1 - \lambda^2}{2 |p + \lambda|^2} + \frac{a + \lambda}{|p + \lambda|} \right] \ldots(4\cdot28)$$

Substituting these values in (4·23) we obtain

$$\operatorname{Im}(p + \lambda) \left[\left\{ \frac{(1 - \lambda^2) \operatorname{Re}(p + \lambda)}{|p + \lambda|^4} - \frac{1}{|p + \lambda|} - \frac{1}{2} \frac{\rho^2 - |p - a|^2}{|p + \lambda|^3} \right\} (a + \lambda) + \mu + \frac{1}{2} - \frac{1 - \lambda^2}{2 |p + \lambda|^2} + \frac{a + \lambda}{|p + \lambda|} \right] = 0. \qquad \ldots(4\cdot29)$$

Hence either $\operatorname{Im}(p + \lambda) = 0$ or

$$\left\{ \frac{(1 - \lambda^2) \operatorname{Re}(p + \lambda)}{|p + \lambda|^4} - \frac{1}{|p + \lambda|} - \frac{1}{2} \frac{\rho^2 - |p - a|^2}{|p + \lambda|^3} \right\} (a + \lambda) + \mu + \frac{1}{2} - \frac{1 - \lambda^2}{2 |p + \lambda|^2} + \frac{a + \lambda}{|p + \lambda|} = 0 \qquad \ldots(4\cdot30)$$

If $\operatorname{Im}(p + \lambda) = 0$, then $(p + \lambda)$ is real and in view of (4·27) w is real. Moreover, w is now given by

$$w = \mu + \frac{1 + a\lambda}{(p + \lambda)^2}. \qquad \ldots(4\cdot31)$$

From (4·26) we now have

$$\frac{2rp \mid w \mid}{w} = (1 + r^2)p - (1 - r^2). \qquad \ldots(4\cdot32)$$

If $\dfrac{1 + a\lambda}{(a + \rho + \lambda)^2} > - \mu$ then from (4·31) $w > 0$ and (4·32) gives

$$p(r) = \frac{1 + r}{1 - r} = a + \rho,$$

and (4·19) reduces to

$$p(z) = \frac{1 + z}{1 - z}.$$

The maximum of ψ_p is obtained from (4·3) by putting $p(r) = a + \rho$ and equals $(a + \rho)\left(\mu + \dfrac{\rho}{a + \rho + \lambda} \right)$.

If $\dfrac{1 + a\lambda}{(a - \rho + \lambda)^2} < - \mu$, then $w < 0$ and (4·32) similarly gives

$$p(r) = \frac{1-r}{1+r} = a - \rho,$$

and (4·19) reduces to

$$p(z) = \frac{1-z}{1+z}.$$

The maximum of ψ_p now equals $(a-\rho)\left(\mu - \frac{\rho}{a-\rho+\lambda}\right)$.

We now investigate whether (4·30) can hold. We observe that for functions of the form (4·19), (4·30) is identical with (3·72) and conclude, as in Theorem 1, that (4·30) is possible only if

$$|p+\lambda|^4 = \frac{(\lambda^2-1)(1+2a\lambda+\lambda^2)}{2\mu+1} \qquad \ldots(4\cdot33)$$

and

$$(a)\ \frac{(\lambda^2-1)(a+\lambda-\rho)}{(a+\lambda+\rho)^3} \leqslant 2\mu+1 \leqslant \frac{(\lambda^2-1)(a+\lambda+\rho)}{(a+\lambda-\rho)^3},\ \lambda \geqslant 1,\ 2\mu+1 \geqslant 0,$$

or

$$(b)\ \frac{(1-\lambda^2)(a+\lambda-\rho)}{(a+\lambda+\rho)^3} \leqslant -(2\mu+1) \leqslant \frac{(1-\lambda^2)(a+\lambda+\rho)}{(a+\lambda-\rho)^3},$$

$$0 \leqslant \lambda \leqslant 1,\ -(2\mu+1) > 0, \qquad \ldots(4\cdot34)$$

hold. Further, since $2\mu+1>0$ yields a minimum of ψ_p (see the proof of case *b*, Theorem 1) we have only to investigate case (*b*) above.

We must see whether (4·34 *b*) is compatible with (4·17), that is, whether (4·34*b*) is compatible with (4·17)

(i) $0 < -(2\mu+1) < -1 + \dfrac{2(1+a\lambda)}{(a+\rho+\lambda)^2}$,

and

(ii) $-(2\mu+1) \geqslant \dfrac{2(1+a\lambda)}{(a-\rho+\lambda)^2} - 1.$

We first show that (4·34*b*) is incompatible with (i) above. Now (i) is equivalent to

$$C = \mu\lambda^2 + \{(2\mu+1)(a+\rho) - \rho\}\lambda + \mu(a+\rho)^2 + 1 > 0.$$

We show that in this case

$$-(2\mu+1) < \frac{(1-\lambda^2)(a+\lambda-\rho)}{(a+\lambda+\rho)^3},$$

This is equivalent to showing that

$$D = (1-\lambda^2)(a+\lambda-\rho) + (2\mu+1)(a+\lambda+\rho)^3 > 0.$$

However, D may be written as

$$D = 2(\lambda+a+\rho)C + 4\rho\lambda^2 + \{(a+\rho)^2 + 2\rho(a+\rho) - 1\}\lambda + (a+\rho)^3 - 3\rho - a,$$

and this is clearly positive. Thus (4·34*b*) is incompatible with (i).

We now investigate whether (4·34*b*) and (ii) are possible. In the first place we verify that

$$\frac{2(1+a\lambda)}{(a-\rho+\lambda)^2}-1>\frac{(1-\lambda^2)(a+\lambda-\rho)}{(a+\lambda+\rho)^3}.$$

In order to verify this we must show that

$$2(1+a\lambda)(a+\lambda+\rho)^3-(a-\rho+\lambda)^2(a+\lambda+\rho)^3-(1-\lambda^2)(a-\rho+\lambda)^3>0.$$

However, the left side

$$>2(1+a\lambda)(a+\lambda+\rho)^3-(1+\lambda)^2(a+\lambda+\rho)^3-(1-\lambda^2)(a+\rho+\lambda)^3$$
$$=2(a+\lambda+\rho)^3\lambda(a-1)>0.$$

Therefore, the left side >0. Next, we must see whether

$$\frac{2(1+a\lambda)}{(a-\rho+\lambda)^2}-1\leqslant\frac{(1-\lambda^2)(a+\lambda+\rho)}{(a+\lambda-\rho)^3}. \qquad \ldots(4\cdot35)$$

After some calculations we see that (4·35) will hold provided that

$$(a-\rho)(\lambda-\rho)+\lambda^2\leqslant 0. \qquad \ldots(4\cdot36)$$

As shown in case (*b*) of Theorem 1, the maximum in this case is given by (4·33).

To sum up, we obtain in Case II

(i) If $-\mu<\dfrac{1+a\lambda}{(a+\rho+\lambda)^2}$, max. ψ_p is given by $p(r)=a+\rho$ at $z=r$ and equals $(a+\rho)\left(\mu+\dfrac{\rho}{a+\rho+\lambda}\right)$;

(ii) If $-\mu>\dfrac{1+a\lambda}{(a-\rho+\lambda)^2}$,

and $0\leqslant\lambda<1$, $2\mu+1<0$, and $(a-\rho)(\lambda-\rho)+\lambda^2\leqslant 0$, (A)

the max. ψ_x is attained by (4·33) and equals

$$\mu(1+a\lambda)-\sqrt{\frac{(2\mu+1)(\lambda^2-1)(1+2a\lambda+\lambda^2)}{2(a+\lambda)}};$$

(iii) If $-\mu>\dfrac{1+a\lambda}{(a-\rho+\lambda)^2}$ and (A) does not hold, the max. ψ_p is attained by $p(r)=a-\rho$ at $z=r$ and equals $(a-\rho)\left(\mu-\dfrac{\rho}{a-\rho+\lambda}\right)$.

To complete the proof of the theorem we must show that the estimates in (4·1) are sharp. It is readily verified that, given μ, λ and r, if

$$\frac{1+a\lambda}{(a+\rho+\lambda)^2}\leqslant-\mu\leqslant\frac{1+a\lambda}{(a-\rho+\lambda)^2},$$

the function

$$\frac{1-2mz+z^2}{1-z^2},$$

where m is to be determined from the equation

$$\frac{1 - 2mr + r^2}{1 - r^2} + \lambda = \left(\frac{1 + a\lambda}{-\mu}\right)^{\frac{1}{2}},$$

belongs to $\mathcal{P}$ and yields equality in the first inequality of (4·1). If $-\mu < \frac{1+a\lambda}{(a+\rho+\lambda)^2}$, the function $\frac{1+z}{1-z}$ yields equality in the second inequality of (4·1).

If $-\mu > \frac{1+a\lambda}{(a-\rho+\lambda)^2}$ and (A) holds then the function

$$p_\alpha(z) = \frac{1 + ze^{i\alpha}}{1 - ze^{i\alpha}},$$

where a is to be determined from the equation

$$\left| \frac{1 + re^{i\alpha}}{1 - re^{i\alpha}} + \lambda \right|^4 = \frac{(\lambda^2 - 1)(1 + 2a\lambda + \lambda^2)}{2\mu + 1},$$

yields equality in the third inequality of (4·1). If $- > \mu \frac{1+a\lambda}{(a-\rho+\lambda)^2}$ and (A) does not hold then the function $\frac{1-z}{1+z}$ yields equality in the last inequality of (4·1).

This completes the proof of Theorem 2.

Corollary—Let $P \in \mathcal{P}$, then on $|z| = r$, $0 < r < 1$,

$$\text{Re}\left(P\mu - \frac{zP'}{\rho + \lambda}\right)$$

$$\geqslant \begin{cases} -a - \mu\lambda + 2\sqrt{\mu(1 + a\lambda)} & \text{if } \frac{1 + a\lambda}{(a + \rho + \lambda)^2} \leqslant \mu \leqslant \frac{1 + a\lambda}{(a - \rho + \lambda)^2}; \\ (a + \rho)\left(\mu - \frac{\rho}{a + \rho + \lambda}\right) & \text{if } \mu < \frac{1 + a\lambda}{(a + \rho + \lambda)^2}; \\ \frac{\mu(1 + a\lambda) + \sqrt{(1 - 2\mu)(\lambda^2 - 1)(1 + 2a\lambda + \lambda^2)}}{2(a + \lambda)} & \text{if } \mu > \frac{1 + a\lambda}{(a - \rho + \lambda)^2} \\ \text{and} & \\ (b)\ 0 \leqslant \lambda < 1,\ 1 - 2\mu < 0,\ (a - \rho)(\lambda - \rho) + \lambda^2 \leqslant 0; & \\ (a - \rho)\left(\mu + \frac{\rho}{a - \rho + \lambda}\right) & \text{if } \mu > \frac{1 + a\lambda}{(a - \rho + \lambda)^2} \text{ and } (b) \text{ does not hold;} \end{cases}$$

where μ, λ are given numbers, $-\infty < \mu < \infty$, $\lambda \geqslant 0$. These estimates are sharp.

Proof: The above results follow immediately from (4·1) by changing μ to $-\mu$.

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