SOME HIGHER OPERATIONAL TECHNIQUES IN THE THEORY OF SPECIAL FUNCTIONS

by Y. N. PRASAD *and* S. N. SINGH, *Applied Mathematics Section, Institute of Technology, Banaras Hindu University, Varanasi* 221005

(Communicated by R. S. Mishra, F.N.A.)

(*Received* 3 *April* 1975; *after revision* 20 *June* 1975)

In the present paper, an effort has been made to establish a formula on multiple transform which generalizes certain well-known results on Fox's *H*-function (1961) and Meijer's *G*-function due to Srivastava and Panda (1973), Srivastava and Singhal (1968) and Siddiqui and Prasad (1974), etc. We have defined the multiple transform as a linear operator and have applied it on various special functions. Various new results on generating functions have also been established.

1. INTRODUCTION

The single and double Euler transformations of the hypergeometric function ${}_pF_q$, given by Rainville (1965, p. 104) and Abdul-Halim and Al-Salam (1963) respectively and the double Laplace transformations, studied by subsequent workers (Jain 1965, Singh 1965) do indeed provide alternative, but only slightly more effective, techniques for augmenting parameters of the ${}_pF_q$ function. Recently Srivastava and Panda (1973) have discussed a double transformation involving Fox's *H*-function, viz.

$$\int_0^\infty\int_0^\infty x^{\alpha-1}\, y^{\beta-1}\, (x+y)^{\sigma}\, H^{m,\,n}_{p,\,q}\left[\lambda\,(x+y)\,\middle|\,\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p)\\(\beta_1, B_1), \ldots, (\beta_q, B_q)\end{matrix}\right]$$

$$\times\, H^{M,\,N}_{P,\,Q}\left[-t\, x^{s}\, y^{k}\, (x+y)^{r}\,\middle|\,\begin{matrix}(\gamma_1, C_1), \ldots, (\gamma_P, C_P)\\(\delta_1, D_1), \ldots, (\delta_Q, D_Q)\end{matrix}\right] dxdy$$

$$= \lambda^{-\alpha-\beta-\sigma}\, H^{M+n,\,N+m+2}_{P+q+2,\,Q+p+1}\left[-\frac{t}{\lambda^{\theta}}\,\middle|\,\begin{matrix}(1-\alpha,s),(1-\beta,k),(\lambda_1,\theta B_1),\ldots,(\lambda_m,\theta B_m),(\gamma_1,C_1),\\(\mu_1,\theta A_1),\ldots,(\mu_n,\theta A_n),(\delta_1,D_1),\ldots,(\delta_Q,D_Q),\end{matrix}\right.$$

$$\left.\begin{matrix}\ldots,(\gamma_P, C_P), (\lambda_{m+1}, \theta B_{m+1}), \ldots, (\lambda_q, \theta B_q)\\(1-\alpha-\beta, s+k), (\mu_{n+1}, \theta A_{n+1}), \ldots, (\mu_p, \theta A_p)\end{matrix}\right] \quad \ldots(1{\cdot}1)$$

where m, n, p, q and M, N, P, Q are integers such that $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$, $0 \leqslant M \leqslant Q$, $0 \leqslant N \leqslant P$; s, k, r and A_j, $1 \leqslant j \leqslant p$, B_j, $1 \leqslant j \leqslant q$, C_j, $1 \leqslant j \leqslant P$, D_j, $1 \leqslant j \leqslant Q$, are all positive and conditions given below hold appropriately:

$$|\arg Z| < \tfrac{1}{2}\pi\Delta, \text{ where } \Delta = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0,$$

$R(\alpha) > 0$, $R(\beta) > 0$

and $\xi_1 < R(\alpha + \beta + \sigma) < \xi_2$, with ξ_1, ξ_2 denoting, respectively, the first and last members of the inequality

$$-\min_{1 \leq j \leq m} R(\beta_j/B_j) < R(s) < \min_{1 \leq j \leq n} R\left(\frac{1-\alpha_j}{A_j}\right)$$ and, for the sake of brevity,

$$\theta = s + k + r$$

$$\lambda_j = 1 - \beta_j - (\alpha + \beta + \sigma) B_j, \; j = 1, \ldots, q$$

$$\mu_j = 1 - \alpha_j - (\alpha + \beta + \sigma) A_j, \; j = 1, \ldots, p.$$

Siddiqui and Prasad (1974) have introduced one more H-function in the integrand of the double transform and have obtained results similar to (1·1) due to Srivastava and Panda (1973). The present paper is a generalization of the double transform studied by Srivastava and Panda (1973) in a broader sense. Instead of double transform, we have introduced a multiple one.

We have applied the following result due to Edward (1954, p. 161) i.e.,

$$\int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{n} x_j^{\alpha_j - 1} f\left\{ \sum_{j=1}^{n} (x_j/a_j)^{p_j} \right\} \prod_{j=1}^{n} dx_j = \frac{\prod_{j=1}^{n} a_j^{\alpha_j} \prod_{j=1}^{n} \Gamma(a_j/p_j)}{\prod_{j=1}^{n} p_j \; \Gamma\left(\sum_{j=1}^{n} \frac{a_j}{p_j}\right)} \times \int_0^\infty Z^{\left(\sum_{j=1}^{n} \frac{a_j}{p_j}\right) - 1} f(z)\, dz \qquad \ldots(1\cdot2)$$

provided that $\sum_{j=1}^{n} (x_j/a_j)^{p_j}$ lies between 0 and ∞ and each $x_j > 0$ ($j = 1, 2, \ldots, n$).

2. The Main Result

We give our result on multiple transform as follows:

$$\int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{r} x_j^{\alpha_j - 1} \left\{ \sum_{j=1}^{r} (x_j/a_j)^{p_j} \right\}^{\sigma} \mathrm{H}_{u,v}^{f,g} \left[\lambda \left\{ \sum_{j=1}^{r} (x_j/a_j)^{p_j} \right\}^{\sigma_1} \middle| \begin{matrix} \{(A_u, \eta_u)\} \\ \{(B_v, \xi_v)\} \end{matrix} \right] H_{p,q}^{m,n} \left[t \prod_{j=1}^{r} x_j^{\beta_j} \left\{ \sum_{j=1}^{r} (x_j/a_j)^{p_j} \right\}^{\sigma_2} \middle| \begin{matrix} \{(c_p, \gamma_p)\} \\ \{(d_q, \delta_q)\} \end{matrix} \right] \prod_{j=1}^{r} dx_j$$

$$= \frac{\prod_{j=1}^{r} a_j^{\alpha_j}}{\prod_{j=1}^{r} p_j} \left(\frac{\lambda^{-K}}{\sigma_1} \right) H_{p+r+v,\, q+u+l}^{m+g,\, r+n+f} \left[t \prod_{j=1}^{r} a_j^{\beta_j} \lambda^{-K'} \middle| \begin{matrix} \left(1 - \frac{\alpha_1}{p_1}, \frac{\beta_1}{p_1}\right), \ldots, \left(1 - \frac{\alpha_r}{p_r}, \frac{\beta_r}{p_r}\right) \{(c_n, \gamma_n)\}, \{(1 - B_v - K\xi_v, K'\xi_v)\}, \\ \{(d_m, \delta_m)\}, \{(1 - A_u - K\eta_u, K'\eta_u)\}, (d_{m+1}, \delta_{m+1}), \ldots, (d_q, \delta_q), \end{matrix} \right.$$

$$\left. \begin{matrix} (c_{n+1}, \gamma_{n+1}), \ldots, (c_p, \gamma_p) \\ (1 + \sigma - \sigma_1 K, \sigma_1 K' - \sigma_2) \end{matrix} \right], \qquad \ldots(2\cdot1)$$

where $K = \frac{1}{\sigma_1}\left(\sigma + \sum_{j=1}^{r} \frac{\alpha_j}{p_j}\right)$; $K' = \frac{1}{\sigma_1}\left(\sigma_2 + \sum_{j=1}^{r} \frac{\beta_j}{p_j}\right)$ and the conditions given below are satisfied:

$\sigma_1, \sigma_2 > 0$; $R(\alpha_j), R(\beta_j) > 0$, $(j = 1, 2, \ldots, r)$; each $x_j > 0$, $(j = 1, 2, \ldots, r)$

$\sum_{j=1}^{r} (x_j/a_j)^{p_j} > 0$: $-\delta < R\left(\frac{\sigma}{\sigma_1} + \frac{1}{\sigma_1}\sum_{j=1}^{r}\frac{\alpha_j}{p_j}\right) < -\beta$; $\delta = \min R(B_j/\xi_j)$.

$(j = 1, 2, \ldots\ldots, f)$;

$\beta = \max R\left(\frac{A_i - 1}{\eta_i}\right)$, $(i = 1, 2, \ldots, g)$ and $|\arg \lambda| < \frac{1}{2}U\pi$, $U > 0$, where

$$U = \sum_{j=1}^{f} \xi_j - \sum_{j=f+1}^{r} \xi_j + \sum_{j=1}^{g} \eta_j - \sum_{j=g+1}^{u} \eta_j. \qquad \ldots(2\cdot2)$$

Proof: To evaluate (2·1), we express the second H-function as a Mellin-Barnes type contour integral and interchange the order of contour and multiple integrations which is justifiable under the given conditions. The left-hand side of (2·1) thus becomes

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + \beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j - \alpha_j s)}\, t^s\, ds \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{r} x_j^{\alpha_j + \beta_j s - 1}$$

$$\times \left\{\sum_{j=1}^{r}(x_j/a_j)^{p_j}\right\}^{\sigma + \sigma_2 s} H_{u,v}^{f,g}\left[\lambda\left\{\sum_{j=1}^{r}(x_j/a_j)^{p_j}\right\}^{\sigma_1}\middle|\begin{matrix}\{(A_u, \eta_u)\}\\ \{B_v, \xi_v)\}\end{matrix}\right]\prod_{j=1}^{r} dx_j. \ldots(2\cdot3)$$

With the help of (1·2), (2·3) reduces to

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + \beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j - \alpha_j s)}\, t^s\, ds$$

$$\times \frac{\prod_{j=1}^{r} a_j^{\alpha_j + \beta_j s - 1}\prod_{j=1}^{r}\Gamma\left(\frac{\alpha_j + \beta_j s}{p_j}\right)}{\prod_{j=1}^{r} p_j\, \Gamma\left(\sum_{j=1}^{r}\frac{\alpha_j + \beta_j s}{p_j}\right)}$$

$$\int_0^\infty Z^{\left(\sigma + \sigma_2 s + \sum_{j=1}^{r}\frac{\alpha_j + \beta_j s}{p_j}\right) - 1} H_{u,v}^{f,g}\left[\lambda Z^{\sigma_1}\middle|\begin{matrix}\{(A_u, \eta_u)\}\\ \{(B_v, \xi_v)\}\end{matrix}\right] dz, \qquad \ldots(2\cdot4)$$

provided that $\sigma_1, \sigma_2 > 0$, $R(\alpha_j) > 0$, $R(\beta_j) > 0$, $j = 1, \ldots, r$.

Now, evaluating the inner integral in (2·4) with the help of the following well-known result, viz.

$$\int_0^\infty x^{\rho-1} H_{p,q}^{m,n}\left[Z x^\sigma \middle| \begin{matrix}(a_p, \alpha_p)\\(b_q, \beta_q)\end{matrix}\right] dx$$

$$= \frac{1}{\sigma} Z^{-\rho/\sigma} \frac{\prod_{j=1}^{m} \Gamma\left(b_j + \frac{\rho}{\sigma}\beta_j\right) \prod_{j=1}^{n} \Gamma\left(1 - a_j - \frac{\rho}{\sigma}\alpha_j\right)}{\prod_{j=m+1}^{q} \Gamma\left(1 - b_j - \frac{\rho}{\sigma}\beta_j\right) \prod_{j=n+1}^{p} \Gamma\left(a_j + \frac{\rho}{\sigma}\alpha_j\right)} \qquad \ldots(2\cdot5)$$

provided that $\sigma > 0$, $\beta < R\left(\frac{\rho}{\sigma}\right) < \delta$, $|\arg z| < \frac{1}{2}\lambda\pi$, $\lambda > 0$,

$\delta = \min R\,(b_h/\beta_h)$, $(h = 1, 2, \ldots, m)$

$\beta = \max R\,(a_i/\alpha_i)$, $(i = 1, 2, \ldots, n)$

$$A = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j,\ \lambda = \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j + \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j$$

we get

$$\bar{\lambda}\frac{(K + K's)}{\sigma_1} = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=1}^{q} \Gamma(1 - b_j + \beta_j s) \prod_{j=1}^{s} \Gamma(a_j - \alpha_j s)} \times$$

$$t^s\, ds\ \frac{\prod_{j=1}^{r} a_j^{\alpha_j - \beta_j s} \prod_{j=1}^{r} \Gamma\left(\frac{\alpha_j + \beta_j s}{p_j}\right)}{\prod_{j=1}^{r} p_j\ \Gamma\left(\sum_{j=1}^{r} \frac{\alpha_j + \beta_j s}{p_j}\right)} \frac{\prod_{j=1}^{f} \Gamma(B_j + \xi_j \rho') \prod_{j=1}^{g} \Gamma(1 - A_j - \eta_j \rho')}{\prod_{j=f+1}^{v} \Gamma(1 - B_j - \xi_j \rho') \prod_{j=g+1}^{u} \Gamma(A_j + \eta_j \rho')} \qquad \ldots(2\cdot6)$$

where

$$K = \frac{1}{\sigma_1}\left(\sigma + \sum_{j=1}^{r} \frac{\alpha_j}{p_j}\right);\ K' = \frac{1}{\sigma_1}\left(\sigma_2 + \sum_{j=1}^{r} \frac{\beta_j}{p_j}\right);\ \rho' = K + K's$$

and the conditions given in (2·2) are satisfied. Finally interpreting the contour integral (2·6) with the definition of Fox's H-function, we arrive at the main result (2·1).

3. Particular Cases

(1) On putting $r = 2, p_1 = p_2 = 1 = a_1 = a_2 = \sigma_1$, we obtain the result recently obtained by Srivastava and Panda [1973, (3·6), p. 312].

(2) On taking $r = 2$, $p_1 = p_2 = 1 = a_1 = a_2 = \sigma_1$, $f = 2 = v$, $g = 0$, $u = 1$, $A_1 = \frac{1}{2}$, $\eta_1 = 1$, $B_1 = \nu$, $B_2 = -\nu$, $\xi_1 = \xi_2 = 1$, $m = 1$, $q = q + 1$, $n = p$, $d_1 = 0$, $\gamma_i = 1 = \delta_j$, $(i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q + 1)$ and replacing c_j, $(j = 1, 2, \ldots, p)$ and d_j, $(j = 2, 3, \ldots, q + 1)$ by $1 - c_j$, $(j = 1, 2, \ldots, p)$ and $1 - d_j$, $(j = 2, 3, \ldots, q + 1)$ respectively and using the well known result.

$$e^{-z} K_\nu (z) = \sqrt{\pi}\ G^{2,0}_{1\,2} \left(2z \middle| \begin{matrix} \frac{1}{2} \\ \nu, -\nu \end{matrix}\right),$$

we obtain the result obtained by Srivastava and Singhal (1968, p. 426).

(3) On taking $r = 2, p_1 = p_2 = 1 = a_1 = a_2, \sigma_1 = 2, f = 4 = v, g = 0, u = 2, \lambda = \lambda^2, A_1 = 0, A_2 = \frac{1}{2}, \eta_1 = \eta_2 = 1, B_j = \frac{1}{2}(\pm \mu \pm \nu), (j = 1, 2, 3, 4); \xi_j = 1, (j = 1, 2, 3, 4); m = 1, q = q + 1, n = p, d_1 = 0, \gamma_i = 1 = \delta_j, (i = 1, 2 \ldots, p; j = 1, 2, \ldots, q + 1)$ and replacing c_j, $(j = 1, 2, \ldots, p)$ and d_j, $(j = 2, 3, \ldots, q + 1)$ by $1 - c_j$, $(j = 1, 2, \ldots, p)$ and $1 - d_j$, $(j = 2, 3, \ldots, q + 1)$ respectively and using the known result

$$K_\mu [z] K_\nu [z] = \frac{\sqrt{\pi}}{2}\ G^{4,0}_{2,4} \left(Z^2 \middle| \begin{matrix} 0, \frac{1}{2} \\ \frac{1}{2}(\pm \mu \pm \nu) \end{matrix}\right);$$

we obtain the result due to Srivastava and Panda [1973, (1·7), p. 309].

Similar results on double transforms due to Abdul-Halim and Al-Salam (1963) Jain (1965), Singh (1965), etc. can be deduced from our results by specializing the parameters.

4. Applications

In terms of the linear operator, let us define

$$\Omega^{(\alpha_r), (a_r), (p_r), \sigma}_{f, g, u, v, \lambda, \sigma_1} \{ f(x_1, x_2, \ldots\ldots, x_r) \} = \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^r x_j^{\alpha_j - 1} \left(\sum_{j=1}^r \left(\frac{x_j}{a_j}\right)^{p_j} \right)^\sigma$$

$$\times H^{f, g}_{u, v} \left[\lambda \left(\sum_{j=1}^r \left(\frac{x_j}{a_j}\right)^{p_j} \right)^{\sigma_1} \middle| \begin{matrix} \{ (A_u, \eta_u) \} \\ \{ (B_v, \xi_v) \} \end{matrix} \right] \{ f(x_1, \ldots, x_r) \} \prod_{j=1}^r dx_j, \quad \ldots(4·1)$$

where (a_r) stands for $a_1, a_2, \ldots\ldots, a_r$ and $\{f(x_1, \ldots\ldots, x_r)\}$ is such that the right-hand side of (4·1) exists.

(i) On taking $\{f(x_1, \ldots\ldots, x_r)\} = 1$, and using the results (1·2) and (2·5) in (4·1), we obtain

$$\Omega^{(\alpha_r), (a_r), (p_r), \sigma}_{f, g, u, v, \lambda, \sigma_1} \{1\} = \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^r x_j^{\alpha_j - 1} \left(\sum_{j=1}^r \left(\frac{x_j}{a_j}\right)^{p_j} \right)^\sigma$$

$$\times H^{f, g}_{u, v} \left[\lambda \left(\sum_{j=1}^r \left(\frac{x_j}{a_j}\right)^{p_j} \right)^{\sigma_1} \middle| \begin{matrix} \{ (A_u, \eta_u) \} \\ \{ (B_v, \xi_v) \} \end{matrix} \right] \prod_{j=1}^r dx_j$$

$$= \frac{\prod_{j=1}^r a_j^{\alpha_j} \prod_{j=1}^r \Gamma\left(\frac{\alpha_j}{p_j}\right)}{\prod_{j=1}^r p_j\ \Gamma\left(\sum_{j=1}^r \frac{\alpha_j}{p_j}\right)} \left(\frac{\lambda^{-K}}{\sigma_1}\right) \frac{\prod_{j=1}^f \Gamma(B_j + K\xi_j) \prod_{j=1}^g \Gamma(1 - A_j - K\eta_j)}{\prod_{j=f+1}^v \Gamma(1 - B_j - K\xi_j) \prod_{j=g+1}^u \Gamma(A_j + K\eta_j)}, \quad \ldots(4·2)$$

provided that $K = \frac{1}{\sigma_1}\left(\sigma + \sum_{j=1}^{r} \frac{\alpha_j}{p_j}\right)$, $\sigma_1 > 0$, $R(\alpha_j) > 0$, $(j = 1, 2, \ldots, r)$;

$-\delta < R(K) < -\beta$, $|\arg \lambda| < \frac{1}{2}U\pi$, $U > 0$ and δ, β, U are as usual.

(ii) On taking $\{f(x_1,\ldots,x_r)\} = \prod_{j=1}^{r} x_j^{\beta_j} \left\{ \sum_{j=1}^{r} \left(\frac{x_j}{a_j}\right)^{p_j} \right\}^{\sigma_2}$

we obtain

$$\Omega^{(\alpha_r), (a_r), (p_r), \sigma}_{f, g, u, v, \lambda, \sigma_1} \left\{ \prod_{j=1}^{r} x_j^{\beta_j} \left(\sum_{j=1}^{r} \left(\frac{x_j}{a_j}\right)^{p_j} \right)^{\sigma_2} \right\}$$

$$= \frac{\lambda^{-K_1}}{\sigma_1} \frac{\prod_{j=1}^{r} a_j^{\alpha_j} \prod_{j=1}^{r} \Gamma\left(\frac{\alpha_j + \beta_j}{p_j}\right) \prod_{j=1}^{f} \Gamma\left(B_j + K_1 \xi_j\right) \prod_{j=1}^{g} \Gamma\left(1 - A_j - K_1 \eta_j\right)}{\prod_{j=1}^{r} p_j \, \Gamma\left(\sum_{j=1}^{r} \frac{\alpha_j + \beta_j}{p_j}\right) \prod_{j=f+1}^{v} \Gamma\left(1 - B_j - K_1 \xi_j\right) \prod_{j=g+1}^{u} \Gamma\left(A_j + K_1 \eta_j\right)} \qquad \ldots(4\cdot3)$$

where $K_1 = K + \frac{\sigma_2}{\sigma_1}$, $\sigma > 0$, $R(a_j + \beta_j) > 0$ $(j = 1, 1, 2, \ldots, r)$, $-\delta < R(K_1) < -\beta$,

$|\arg\lambda| < \frac{1}{2} U\pi$, $U > 0$ and δ, β, U are as usual.

(iii) On taking $m = 2 = q$, $n = 0$, $p = 1$, $c_1 = \frac{1}{2}$, $\gamma_1 = 1$, $d_1 = \nu$ $d_2 = -\nu$, $\delta_1 = \delta_2 = 1$ in the result (2·1); we obtain

$$\Omega^{(\alpha_r), (a_r), (p_r), \sigma}_{f, g, u, v, \lambda, \sigma_1} \left\{ \exp\left[-\tfrac{1}{2} t \prod_{j=1}^{r} x_j^{\beta_j} \left(\sum_{j=1}^{r} \left(\frac{x_j}{a_j}\right)^{p_j} \right)^{\sigma_2} \right] \times K_\nu \left[\tfrac{1}{2} t \prod_{j=1}^{r} x_j^{\beta_j} \left(\sum_{j=1}^{r} \left(\frac{x_j}{a_j}\right)^{p_j} \right)^{\sigma_2} \right] \right\}$$

$$= \frac{\lambda^{-K}}{\sqrt{\pi}\,\sigma_1} \frac{\prod_{j=1}^{r} a_j^{\alpha_j}}{\prod_{j=1}^{r} p_j} H^{2+g,\, r+f}_{1+r+v,\, u+3} \left[t \prod_{j=1}^{r} a_j^{\beta_j} \lambda^{-K'} \middle| \begin{matrix} \left(1 - \frac{\alpha_1}{p_1}, \frac{\beta_1}{p_1}\right), \ldots, \left(1 - \frac{\alpha_r}{p_r}, \frac{\beta_r}{p_r}\right), \{(1 - B_v - K\xi_v, K'\xi_v)\}, (\frac{1}{2}, 1) \\ (\nu, 1), (-\nu, 1), \{(1 - A_u - K\eta_u, K'\eta_u)\}, (1 + \sigma - \sigma_1 K, \sigma_1 K' - \sigma_2) \end{matrix} \right], \qquad \ldots(4\cdot4)$$

provided that $\sigma_1, \sigma_2 > 0$, $R(\alpha_j), R(\beta_j) > 0$, $(j = 1, 2, \ldots\ldots, r)$, $-\delta < R(K) < -\beta$ $\arg \lambda| < \frac{1}{2} U\pi$, $U > 0$ and K, K', δ, β, U are as usual.

(iv) On taking $m = 4 = q$, $n = 0$, $p = 2$, $c_1 = 0$, $c_2 = \frac{1}{2}$, $\gamma_1 = \gamma_2 = 1$, $d_j = \frac{1}{2}(\pm \mu \pm \nu)$, $(j = 1, 2, 3, 4)$ in the result (2·1), we obtain

$$\Omega^{(\alpha_r), (a_r), (p_r), \sigma}_{f, g, u, v, \lambda, \sigma_1} \left\{ K_\mu \left[t \prod_{j=1}^{r} x_j^{\beta_j} \left(\sum_{j=1}^{r} \left(\frac{x_j}{a_j}\right)^{p_j} \right)^{\sigma_2} \right] \times K_\nu \left[t \prod_{j=1}^{r} x_j^{\beta_j} \left(\sum_{j=1}^{r} \left(\frac{x_j}{a_j}\right)^{p_j} \right)^{\sigma_2} \right] \right\} =$$

(equation contd. on p. 1304)

$$= \frac{2}{\sqrt{\pi}} \frac{\prod_{j=1}^{r} a_j^{\alpha_j}}{\prod_{j=1}^{r} p_j} \frac{\lambda^{-K}}{\sigma_1} H^{4+g,\, r+f}_{2+r+v,\, u+5}\left[t^2 \prod_{j=1}^{r} a_j^{2\beta_j} \lambda^{-2K'} \middle| \begin{matrix} \left(1-\frac{\alpha_1}{p_1}, \frac{2\beta_1}{p_1}\right), \ldots \\ (\pm \frac{1}{2}\mu \pm \frac{1}{2}\nu, 1), \end{matrix} \right.$$

$$\left. \begin{matrix} \ldots, \left(1-\frac{\alpha_r}{p_r}, \frac{2\beta_r}{p_r}\right), \{(1-B_v-K\xi_v, 2K'\xi_v)\}, (0, 1), (\frac{1}{2}, 1) \\ \{(1-A_u-K\eta_u, 2K'\eta_u)\}, (1+\sigma-\sigma_1 K, 2\sigma_1 K'-\sigma_2) \end{matrix} \right] \quad \ldots(4{\cdot}5)$$

provided that $\sigma_1, \sigma_2 > 0$, $R(\alpha_j)$. $R(\beta_j) > 0$, $(j = 1, 2, \ldots, r)$. $-\delta < R(K) < -\beta$, $|\arg \lambda| < \frac{1}{2} U\pi$, $U > 0$ and K, K', δ, β, U are usual.

(*v*) On taking $m = 2 = q$, $n = 0$, $p = 1$. $c_1 = 1 - \mu$, $\gamma_1 = 1$, $d_j = \frac{1}{2} \pm \nu$ $(j = 1, 2)$, $\delta_1 = \delta_2 = 1$ and using the known result

$$e^{-z/2} W_{\mu,\nu}(z) = G^{2,0}_{1,2}\left(z \middle| \begin{matrix} 1-\mu \\ \frac{1}{2} \pm \nu \end{matrix}\right).$$

we obtain

$$\Omega \left\{ \begin{matrix} (\alpha_r)\,(a_r), (p_r), \sigma \\ f, g, u, v, \lambda, \sigma_1 \end{matrix} \right| \exp\left[-\tfrac{1}{2} t \prod_{j=1}^{r} x_j^{\beta_j} \left(\sum_{j=1}^{r} \left(\frac{x_j}{a_j}\right)^{p_j} \right)^{\sigma_2} \right]$$

$$\left. \times\, W_{\mu,\nu}\left[t \prod_{j=1}^{r} x_j^{\beta_j} \left(\sum_{j=1}^{r} \left(\frac{x_j}{a_j}\right)^{p_j} \right)^{\sigma_2} \right] \right\}$$

$$= \frac{\prod_{j=1}^{r} a_j^{\alpha_j}}{\prod_{j=1}^{r} p_j} \frac{\lambda^{-K}}{\sigma_1} H^{2+g,\, r+f}_{1+r+v,\, u+3}\left[t \prod_{j=1}^{r} a_j^{\beta_j} \lambda^{-K'} \middle| \begin{matrix} \left(1-\frac{\alpha_1}{p_1}, \frac{\beta_1}{p_1}\right), \ldots \\ (\frac{1}{2}+\nu, 1), \end{matrix} \right.$$

$$\left. \begin{matrix} \ldots, \left(1-\frac{\alpha_r}{p_r}, \frac{\beta_r}{p_r}\right), \{(1-B_v-K\xi_v, K'\xi_v)\}, (1-\mu, 1) \\ \{(1-A_u-K\eta_u, K'\eta_u)\}, (1+\sigma-\sigma_1 K, \sigma_1 K'-\sigma_2) \end{matrix} \right] \quad \ldots(4{\cdot}6)$$

provided that $\sigma_1, \sigma_2 > 0$, $R(\alpha_j)$, $R(\beta_j) > 0$, $(j = 1, 2, \ldots, r)$, $-\delta < R(K) < -\beta$, $|\arg \lambda| < \frac{1}{2} U\pi$, $U > 0$, and K, K', δ, β, U being as usual.

(vi) On taking $r = 2$, $a_1 = a_2 = 1 = p_1 = p_2$, $f = 4 = v$, $g = 0$, $u = 2$, $A_1 = 0$, $A_2 = \frac{1}{2}$, $\eta_1 = \eta_2 = 1$, $B_j = \pm \frac{1}{2}\mu \pm \frac{1}{2}\nu$, $(j = 1, 2, 3, 4)$, $\xi_j = 1$, $(j = 1, 2, 3, 4)$, $\sigma_1 = 2$ in (4·5), we arrive at the result given by Srivastava and Panda [1973, (4·4) p. 314].

(vii) On taking $r = 2$, $a_1 = a_2 = 1 = p_1 = p_2 = \sigma_1 = \sigma_2 = f$, $n = p$, $B_1 = 0$, $\eta_1 = \eta_2 = \ldots = \eta_u = 1$, $\xi_1 = \xi_2 = \ldots = \xi_{v+1} = 1$ and replacing A_j, $(j = 1, 2, \ldots, u)$ and B_j, $(j = 2, \ldots, v+1)$ by $1 - A_j$ and $1 - B_j$ respectively in (4·5); we arrive at the result obtained by Srivastava and Panda [1973, (4·2), p. 314].

5. Generating Functions

In Bateman's generating function for the classical Jacobi polynomials (Rainville 1965, p. 256), viz.

$$ {}_0F_1[-\mu;1+a;\tfrac{1}{2}(x-1)t]\,{}_0F_1[-\nu;1+b;\tfrac{1}{2}(x+1)t]=\sum_{n=0}^{\infty}\frac{t^n}{(1+a)_n(1+b)_n}P_n^{(a,b)}(x) \quad\ldots(5\cdot1)$$

if we replace t by $t\prod_{j=1}^{r}x_j^{\beta_j}\left(\sum_{j=1}^{r}\left(\frac{x_j}{a_j}\right)^{p_j}\right)^{\sigma_2}$ and operate upon both sides by the operator

$$\Omega^{(\alpha_r),(a_r),(p_r),\sigma}_{f,g,u,v,\lambda,\sigma_1}$$

then in view of (4·3), we shall get

$$\sum_{n=0}^{\infty}\frac{t^n}{(1+a)_n(1+b)_n}P_n^{(a,b)}(x)\frac{\lambda^{K_1}}{\sigma_1}\frac{\prod_{j=1}^{r}a_j^{\alpha_j+n\beta_j}\prod_{j=1}^{r}\Gamma\left(\frac{\alpha_j+n\beta_j}{p_j}\right)}{\prod_{j=1}^{r}p_j\,\Gamma\left(\sum_{j=1}^{r}\frac{\alpha_j+n\beta_j}{p_j}\right)}$$

$$\times\frac{\prod_{j=1}^{f}\Gamma(B_j+K_1\xi_j)\prod_{j=1}^{g}\Gamma(1-A_j-K_1\eta_j)}{\prod_{j=f+1}^{v}\Gamma(1-B_j-K_1\xi_j)\prod_{j=g+1}^{u}\Gamma(A_j+K_1\eta_j)}$$

$$=\sum_{N,h=0}^{\infty}\frac{\left(\frac{x-1}{2}\right)^N\left(\frac{x+1}{2}\right)^h t^{N+h}}{\lfloor N\lfloor h(1+a)_N(1+b)_h}\frac{\lambda^{-K_2}}{\sigma_1}\frac{\prod_{j=1}^{r}a^{(\alpha_j+\overline{N+h}\beta_j)}\prod_{j=1}^{r}\Gamma\left(\frac{\alpha_j+\overline{N+h}\beta_j}{p_j}\right)}{\prod_{j=1}^{r}p_j\,\Gamma\left(\sum_{j=1}^{r}\frac{\alpha_j+\overline{N+h}\beta_j}{p_j}\right)}$$

$$\times\frac{\prod_{j=1}^{f}\Gamma(B_j+K_2\xi_j)\prod_{j=1}^{g}\Gamma(1-A_j-K_2\eta_j)}{\prod_{j=f+1}^{v}\Gamma(1-B_j-K_2\xi_j)\prod_{j=g+1}^{u}\Gamma(A_j-K_2\eta_j)} \quad\ldots(5\cdot2)$$

where $K_1=\frac{1}{\sigma_1}\left[\sigma+n\,\sigma_2+\sum_{j=1}^{r}\frac{a_j+n\,\beta_j}{p_j}\right]$,

$$K_2=\frac{1}{\sigma_1}\left[\sigma+(N+h)\,\sigma_2+\sum_{j=1}^{r}\frac{a_j+\overline{N+h}\beta_j}{p_j}\right], \quad\ldots(5\cdot3)$$

$\sigma_1,\sigma_2>0,\ a_j+n\,\beta_j>0,\ (j=1,2,\ldots,r)$;

$-\delta<R(K_1)<-\beta;\ |\arg\lambda|<\frac{1}{2}U\pi,\ U>0$

Further on putting $r=2, p_1=p_2=1=a_1=a_2, f=4=v, g=0, u=2, \lambda=4, \sigma_1=2,$ $A_1=0, A_2=\frac{1}{2}, B_j=(\pm\mu\pm\nu), (j=1,2,3,4), \sigma_2=0, \beta_1=\beta_2=1, \eta_1=\eta_2=1.$ $\xi_j=1, (j=1,2,3,4)$ and $\sigma=2c-a_1-a_2$ in (5·2), we obtain

$$\sum_{n=0}^{\infty} \frac{(t/16)^n}{(1+a)_n\,(1+b)_n} P_n^{(a,b)}(x) \frac{(a_1)_n\,(a_2)_n\,(c\pm\mu\pm\nu)_n}{\left(\frac{a_1+a_2}{2}\right)_n \left(\frac{a_1+a_2+1}{2}\right)_n (c)_n\,(c+\frac{1}{2})_n}$$

$$= \sum_{N,\,h=0}^{\infty} \frac{\left(\frac{x-1}{2}\right)^N \left(\frac{x+1}{2}\right)^h (t/16)^{N+h}}{\lfloor N \lfloor h\,(1+a)_N\,(1+b)_h}$$

$$\times \frac{(a_1)_{N+h}\,(a_2)_{N+h}\,(c\pm\mu\pm\nu)_{N+h}}{\left(\frac{a_1+a_2}{2}\right)_{N+h}\left(\frac{a_1+a_2+1}{2}\right)_{N+h}(c)_{N+h}\,(c+\frac{1}{2})_{N+h}}$$

which with the replacement of $t/16$ by t, yields the result recently obtained by Srivastava and Panda [1973, (4·9), p. 314].

A similar application of the operator $\Omega\,{}^{(\alpha_r),\,(a_r),\,(p_r),\,\sigma}_{f,\,g,\,u,\,v,\,\lambda,\,\sigma_1}$ to the generating relation (Srivastava 1972)

$$\sum_{n=0}^{\infty} \frac{(\lambda_1)_n}{(-a-b)_n} H_n^{(a-n,b-n)}[\rho,\,\varepsilon,\,x]\,t^n = F_2[\lambda_1;\,-a,\rho;\,-a-b,\,\varepsilon;\,-t,\,xt]$$

$$= \sum_{N,\,h=0}^{\infty} \frac{(\lambda_1)_{N+h}\,(-a)_N(\rho)_h\,(-t)^N\,(xt)^h}{\lfloor N\,\lfloor h\,(-a-b)_N\,(\varepsilon)_h}$$

involving the generalized Rice polynomials

$$H_n^{(a,\,b)}[\xi;\,p,\,v] = \binom{a+n}{n}\,{}_3F_2\left[\begin{matrix}-n,\,a+b+n+1,\,\xi;\\ a+1,\,p;\end{matrix}\;v\right], \qquad n \geqslant 0,$$

with replacement of t by $t \prod_{j=1}^{r} x_j^{\beta_j} \left(\sum_{j=1}^{r}\left(\frac{x_j}{a_j}\right)^{p_j}\right)^{\sigma_2}$ would result in the formula

$$\sum_{n=0}^{\infty} \frac{(\lambda_1)_n}{(-a-b)_n} H_n^{(a-n,\,b-n)}[\rho,\,\varepsilon,\,x]\,t^n$$

$$\frac{\lambda^{-K_1}}{\sigma_1} \cdot \frac{\prod_{j=1}^{r} a_j^{\alpha_j+n\beta_j} \prod_{j=1}^{r}\Gamma\left(\frac{\alpha_j+n\beta_j}{p_j}\right) \prod_{j=1}^{f}\Gamma(B_j+K_1\xi_j) \prod_{j=1}^{r}\Gamma(1-A_j-K_1\eta_j)}{\prod_{j=1}^{r} p_j\,\Gamma\left(\sum_{j=1}^{r}\frac{\alpha_j+n\beta_j}{p_j}\right) \prod_{j=f+1}\Gamma(1-B_j-K_1\xi_j) \prod_{j=g+1}^{u}\Gamma(A_j+K_1\eta_j)}$$

$$= \sum_{N,\,h=0}^{\infty} \frac{(\lambda_1)_{N+h}\,(-a)_N\,(\rho)_h\,(-t)^N\,(xt)^h}{(-a-b)_N\,(\varepsilon)_h \lfloor N \lfloor h}$$

$$\frac{\lambda^{-K_2}}{\sigma_1} \cdot \frac{\prod_{j=1}^{r} a_j^{\alpha_j+(N+h)\beta_j} \prod_{j=1}^{r}\Gamma\left(\frac{\alpha_j+(N+h)\beta_j}{p_j}\right) \prod_{j=1}^{f}\Gamma(B_j+K_2\xi_j) \prod_{j=1}^{g}\Gamma(1-A_j-K_2\eta_j)}{\prod_{j=1}^{r} p_j\,\Gamma\left(\prod_{j=1}^{r}\frac{\alpha_j+(N+h)\beta_j}{p_j}\right) \prod_{j=f+1}^{r}\Gamma(1-B_j-K_2\xi_j) \prod_{j=g+1}^{u}\Gamma(A_j+K_2\eta_j)}$$

...(5.4)

provided that the conditions given in (5·3) are satisfied.

On further taking $r = 2$, $p_1 = p_2 = 1 = a_1 = a_2$, $f = 4 = v$, $g = 0$, $u = 2$, $\lambda = 4$, $\sigma_1 = 2$, $A_1 = 0$, $A_2 = \frac{1}{2}$, $B_j = \frac{1}{2}\ (\pm 2\mu \pm 2\nu)$, $(j=1, 2, 3, 4)$, $\sigma_2 = 0$: $\beta_1 = \beta_2 = 1$, $\eta_1 = \eta_2 = 1$, $\xi_j = 1$, $(j = 1, 2, 3, 4)$ and $\sigma = 2c - \alpha_1 - \alpha_2$, in (5·4), we obtain

$$\sum_{n=0}^{\infty} \frac{(\lambda_1)_n\ H_n^{(a-n,\, b-n)}\ [\rho, \varepsilon, x]\ (t/16)^n\ (\alpha_1)_n\ (\alpha_2)_n\ (c \pm \mu \pm \nu)_n}{(-a-b)_n \left(\frac{\alpha_1+\alpha_2}{2}\right)_n \left(\frac{\alpha_1+\alpha_2+1}{2}\right)_n (c)_n\ (c+\frac{1}{2})_n}$$

$$= \sum_{N,h=0}^{\infty} \frac{(\lambda_1)_{N+h}\, (-a)_N\, (\rho)_h\, (-t/16)^N \left(\frac{xt}{16}\right)^h (\alpha_1)_{N+h}\, (\alpha_2)_{N+h}\, (c \pm \mu \pm \nu)_{N+h}}{\lfloor N \lfloor h\ (-a-b)_N\ (\varepsilon)_h \left(\frac{\alpha_1+\alpha_2}{2}\right)_{N+h} \left(\frac{\alpha_1+\alpha_2+1}{2}\right)_{N+h} (c)_{N+h}\ (c+\frac{1}{2})_{N+h}}$$

Finally replacing $t/16$ by t in the above result, we arrive at the result due to Srivastava and Panda [1973, (4·13), p. 315].

References

Abdul-Halim, N., and Al-Salam, W. A. (1963). Double Euler transformations of certain hypergeometric functions. *Duke math. J.*, **30**, 51-62.

Burchnall, J. L., and Chaundy, T. W. (1941). Expansions of Appell's double hypergeometric functions (II). *Q. Jl Math., Oxford Ser.*, **12**, 112-28.

Chundy, T. W. (1943). An extension of hypergemetric functions (I). *Q. Jl Math., Oxford Ser.*, **14** ,55-78.

Edward, J. (1954). A Treatise on the Integral Calculus, Vol. II. Chelsea Publishing Co., New York.

Erdelyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G. (1953). Higher Transcendental Functions, Vol. I. McGraw-Hill Book Co., Inc., New York.

——————(1954). Tables of Integral Transforms, Vol. I, McGraw-Hill Book Co., Inc., New York.

Fox, C. (1961). The *G*- and *H*-functions as symmetrical Fourier-kernel. *Trans. Am. math. Soc.*, **98**, 395-429.

Jain, R. N. (1965). Some double integral transformations of certain hypergeometric functions. *Math. Japon.*, **10**, 17-26.

Meijer, C. S. (1946). On the *G*-function I-VIII. *Nederl. Akad. Wetensch. Proc. Ser. A.*, **49**, 227-37, 344-56, 457-69, 632-41, 765-72, 936-43, 1063-72, 1165-75.

Rainville, E. D. (1965). Special Functions. The Macmillan Co., New York.

Singh, R. P. (1965). A note on double transformations of certain hypergeometric functions. *Proc. Edinb. math. Soc.* (2), **14**, 221-27.

Srivastava, H. M., and Panda, Rekha (1973). Some operational techniques in the theory of Special functions. *Nederl. Akad. Wetensch Proc. Ser. A*, **76**, 308-18.

Srivastava, H. M., and Singhal, J. P. (1968). Double Meijer transformations of certain hypergeometric functions. *Proc. Camb. phil. Soc.*, **64**, 425-30.

Srivastava, H. M. (1972). Certain formulas associated with generalized Rice polynomials, II. *Ann. Polon. Math.*, **27**, 73-83.

Siddiqui, A., and Prasad, Y. N. (1974). On some operational techniques in the theory of special functions. *Indian J. pure appl. Math.* (accepted for publication).