SOME OPERATION TRANSFORM FORMULAE FOR DISTRIBUTIONAL ONE SIDED LAPLACE TRANSFORMATION

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In this paper we obtain some operation transform formulae for one-sided Laplace transformable generalized functions.

1. INTRODUCTION

The conventional one-sided Laplace transformation is defined by

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \qquad \ldots(1)$$

where $f(t)$ is suitably restricted conventional function on $I : 0 < t < \infty$. Thus this transformation maps $f(t)$ into a function $F(s)$ of the complex variable s.

Let us assume that $a, b, c, d \in \mathbb{R}^1$ and $p, u, t \in I, s \in C^1$. Let $\mathcal{L}_{+,a}$ denote the space of all smooth functions $\phi(t)$ on I such that

$$\lambda_{a,l}(\phi) = \sup_{0 < t < \infty} | e^{at} D^l \phi(t) | < \infty, l = 0, 1, 2, \ldots, \qquad \ldots(2)$$

and its topology is generated by $\{\lambda_{a,l}\}_{l=0}^{\infty}$. $\mathcal{L}_{+,a}$ is a Hausdorff, locally convex, first countable linear space. It is complete and therefore Frechet space (Zemanian 1968, p. 90). Moreover $\mathcal{L}_{+,a}$ is a testing function space and $\mathcal{L}'_{+,a}$ is a generalized function space. $e^{-st} \in \mathcal{L}_{+,a}$ if and only if Re $s \geqslant a$. Let $\{a\nu\}_{\nu=1}^{\infty}$ be a monotonic sequence of real numbers such that $a\nu \longrightarrow w_+$, where w is a real number or $-\infty$. Then $\mathcal{L}_+(w) = \bigcup_{\nu=1}^{\infty} \mathcal{L}_{+,a\nu}$ is a countable union space. We call the generalized function f, $\mathcal{L}_+$ transformable if $f \in \mathcal{L}'_+(w)$ for some w, where $\mathcal{L}'_+(w)$ is the dual of $\mathcal{L}_+(w)$. Let σ be the infimum of all such w. The one-sided Laplace transform $\mathcal{L}_+(f)$ of f is defined by

$$F(s) \triangleq \mathcal{L}_+(f)(s) \triangleq \langle f(t), e^{-st} \rangle, s \in \Omega_f, \qquad \ldots(3)$$

where the symbol $\triangleq$ emphasizes that the equality is a definition and

$\Omega_f = \{s : \sigma < \text{Re}\, s < \infty\}$. Under these circumstances we call f a one-sided Laplace transformable generalized function and (3) a one-sided Laplace transform.

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The object of this paper is to obtain results similar to those of Buschman (1956) for Laplace transformable generalized functions.

Let A, k, g and $h=g^{-1}$ be single-valued analytic functions, real on $(0, \infty)$, and such that $g(0)=0$ and $g(\infty)=\infty$ (or $g(0)=\infty$ and $g(\infty)=0$). Let

$$\mathcal{L}_+[f(t)] \triangleq F(s), \qquad \mathcal{L}_+[A(t)\, f(t)] \triangleq F^*(s),$$

$$\mathcal{L}_+[\psi(s,u)] \triangleq \bar{\psi}(s,p), \qquad \mathcal{L}_+[\psi^*(s,u)] \triangleq \bar{\psi}^*(s,p),$$

where

$$\bar{\psi}(s,p) = e^{-sh(p)}\; k[h(p)]\, |h'(p)|$$

and $\bar{\psi}^*(s,p) = e^{-sh(p)}\; k[h(p)]\, |h'(p)|\, [A(p)]^{-1}$.

We prove the following results:

(*a*) Let $\phi(t) \in \mathcal{L}_{+,c}$. The mapping

$$\phi(t) \longrightarrow k[h(t)]\; \phi[h(t)]\, |h'(t)|$$

i.e. $\quad e^{-st} \longrightarrow k[h(t)]\; e^{-sh(t)}\, |h'(t)|$

is an isomorphism from $\mathcal{L}_{+,c}$ onto $\mathcal{L}_{+,a}$ where $a < c$. Consequently the mapping $f(t) \longrightarrow k(t) f[g(t)]$, which is defined by

$$< k(t) f[g(t)],\ \phi(t) > \; = \; < f(t),\ k[h(t)]\; \phi[h(t)]\, |h'(t)| >$$

is an isomorphism from $\mathcal{L}'_{+,a}$ onto $\mathcal{L}'_{+,c}$.

(*b*) Let $\phi(t) \in \mathcal{L}_{+,c}$. The mapping

$$\phi(t) \longrightarrow k[h(t)]\; \phi[h(t)]\, |h'(t)|\; [A(t)]^{-1}$$

i.e. $\quad e^{-st} \longrightarrow k[h(t)]\; e^{-sh(t)}\, |h'(t)|\; [A(t)]^{-1}$

is an isomorphism from $\mathcal{L}_{+,c}$ onto $\mathcal{L}_{+,a}$ where $a < c$. Consequently the mapping $A(t)\, f(t) \longrightarrow k(t) f[g(t)]$, which is defined by

$< k(t) f[g(t)],\ \phi(t) > \; = \; < A(t)\, f(t),\ k[h(t)]\; \phi[h(t)]\, |h'(t)|\, [A(t)]^{-1} >$ is an isomorphism from $\mathcal{L}'_{+,a}$ onto $\mathcal{L}'_{+,c}$.

For this purpose we prove the following theorem:

Theorem 1·1 — If k, g, h, $\psi(s,u)$, $\bar{\psi}(s,t)$, f and F are as above, then

$$< k(t) f[g(t)],\ e^{-st} > \; = \; < F(u),\ \psi(s,u) >.$$

Proof: Let $\phi(t)$ be an arbitrary member of $\mathcal{L}_{+,c}$. Let $k(t)$ be a single valued analytic function on I such that $[e^{(b-c)t}\, D^{l-\nu}\, k(t)]$ is bounded on $0 < t < \infty$, for an arbitrary real number b such that $b < c$. Then the operation $\phi(t) \longrightarrow k(t)\; \phi(t)$ is a continuous linear mapping of $\mathcal{L}_{+,c}$ into $\mathcal{L}_{+,b}$. To show this we write

$$e^{bt}\, D^l\, [k(t)\; \phi(t)] = \sum_{\nu=0}^{l} \binom{l}{\nu} [e^{(b-c)t}\, D^{l-\nu}\, k(t)]\, (e^{ct}\, D^{\nu}\; \phi(t)).$$

so that

$$\lambda_{b,l}\,[k(t)\,\phi\,(t)] \leqslant B_l \sum_{\nu=0}^{l} \binom{l}{\nu}\, \lambda_{c,\nu}\,[\phi(t)],$$

where B_l are constants.

Thus $k(t)\,\phi\,(t)$ is in $\mathcal{L}_{+,b}$ whenever $\phi\,(t)$ is in $\mathcal{L}_{+,c}$. The linearity of the mapping $\phi\,(t) \longrightarrow k(t)\,\phi\,(t)$ is obvious, and its continuity is implied by Lemma 1.10.1 of Zemanian (1968, p. 29). On the other hand the unique inverse mapping of $\phi\,(t) \longrightarrow k(t)\,\phi\,(t)$ by similar arguments is a continuous linear mapping of all of $\mathcal{L}_{+,b}$ onto $\mathcal{L}_{+,c}$. So truly $\phi\,(t) \longrightarrow k(t)\,\phi\,(t)$ is an isomorphism from $\mathcal{L}_{+,c}$ onto $\mathcal{L}_{+,b}$. In accordance with section 2·5 of Zemanian (1968, p. 43) it now follows that $f(t) \longrightarrow k(t)\,f(t)$ is an isomorphism from $\mathcal{L}'_{+,b}$ onto $\mathcal{L}'_{+,c}$ and we write

$$< k(t)\,f(t),\,\phi\,(t) > \;=\; < f(t),\,k(t)\,\phi\,(t) >.$$

Therefore if $\mathcal{L}_{+}(f) \triangleq F(s)$, $b < \text{Re}\,s < \infty$, the equation

$$< k(t)\,f(t),\; e^{-st} > \;=\; < f(t),\,k(t)\,e^{-st} >$$

has sense. Indeed, we have $f(t) \in \mathcal{L}'_{+,b}$, $k(t)\,e^{-st} \in \mathcal{L}_{+,b}$, $k(t)\,f(t), \in \mathcal{L}'_{+,c}$ and $e^{-st} \in \mathcal{L}_{+,c}$.

If $\chi\,(t) = f[\,g(t)\,] \in \mathcal{L}'_{+,b}$ then $\chi\,(t) \longrightarrow k(t)\;\;\chi(t)$ is an isomorphism from $\mathcal{L}'_{+,b}$, onto $\mathcal{L}'_{+,c}$, and we can write

$$< k(t)\,f\,[\,g(t)\,],\; e^{-st} > \;=\; < f\,[\,g(t)],\,k(t)\,e^{-st}> \qquad \ldots(4)$$

Here $f\,[\,g(t)\,] \in \mathcal{L}'_{+,b}$, $k(t)\,e^{-st} \in \mathcal{L}_{+,b}$, $k(t)\,f\,[g(t)\,] \in \mathcal{L}'_{+,c}$ and $e^{-st} \in \mathcal{L}_{+,c}$.

Let $k(t)\phi\,(t) = \eta\,(t)$ be an arbitrary member of $\mathcal{L}_{+,b}$. Choose a real number a, $a < c$, such that $\eta\,[\,h(t)\,]\;\;|\,h'(t)\,| \in \mathcal{L}_{+,a}$. Then the mapping

$$\eta\,(t) \longrightarrow \eta\,[\,h(t)\,]\,|\,h'(t)\,|$$

is a continuous linear mapping of $\mathcal{L}_{+,b}$ into $\mathcal{L}_{+,a}$. The unique inverse mapping is

$$\eta\,(t) \longrightarrow \eta\,[\,g(t)\,]$$

and it maps all of $\mathcal{L}_{+,a}$ into $\mathcal{L}_{+,b}$. Hence

$$\eta\,(t) \longrightarrow \eta\,[\,h(t)\,]\,|\,h'(t)\,|$$

is an isomorphism from $\mathcal{L}_{+,b}$ onto $\mathcal{L}_{+,a}$. We denote the adjoint of the mapping $\eta\,(t) \longrightarrow \eta\,[h\,(t)\,]\,|\,h'(t)\,|$ by $f\,(t) \longrightarrow f\,[\,g\,(t)\,]$, since this is what we would have if f were a conventional function, and we write

$$< f\,[\,g(t)\,],\,\eta\,(t)> \;=\; < f\,(t),\;\eta\,[\,h(t)\,]\,|\,h'(t)\,| >.$$

By theorem 1·10·2 of Zemanian (1968, p. 29), $f\,(t) \longrightarrow f\,[g\,(t)\,]$ is an isomorphism from $\mathcal{L}'_{+,a}$ onto $\mathcal{L}'_{+,b}$. Therefore, if $\mathcal{L}_{+}\;(f) \triangleq F(s)$, $a < \text{Re}\,s < \infty$, the equation

$$< f[\,g(t)\,],\, k(t)\, e^{-st} > \; = \; < f(t),\, k\,[h(t)\,]\, e^{-sh(t)}\,|\,h'(t)\,|\, > \qquad \ldots(5)$$

has sense. Indeed, we have $f(t) \in \mathcal{L}'_{+,a}$, $k\,[h(t)\,]\, e^{-sh(t)}\,|\,h'(t)\,| \in \mathcal{L}_{+,a}$; $f[g(t)\,] \in \mathcal{L}'_{+,b}$ and $k(t)\,e^{-st} \in \mathcal{L}_{+,b}$.

From (4) and (5) we conclude that

$$f(t) \longrightarrow k(t)\, f[g(t)\,]$$

is an isomorphism from $\mathcal{L}'_{+,a}$ onto $\mathcal{L}'_{+,c}$ where $a < c$, and we write

$$< k(t)\, f[g(t)\,],\, e^{-st} > \; = f(t),\, k\,[h(t)\,]\, e^{-sh(t)}\,|\,h'(t)\,|\, > \qquad \ldots(6)$$

where $f(t) \in \mathcal{L}'_{+,a}$, $k\,[h(t)\,]\, e^{-sh(t)}\,|\,h'(t)\,| \in \mathcal{L}_{+,a}$, $k(t)\, f\,[\,g(t)\,] \in \mathcal{L}'_{+c}$, and $e^{-st} \in \mathcal{L}_{+,c}$.

The equation (6) further can be written as

$$
\begin{aligned}
< k(t)\, f[\,g(t)\,],\, e^{-st} > &= < f(t),\, k\,[h(t)\,]\, e^{-sh(t)}\,|\,h'(t)\,|\, > \\
&= < f(t),\, \bar{\Psi}(s, t) > \\
&= < f(t),\, < \psi(s, u),\, e^{-tu} >> \\
&= << f(t),\, e^{-ut} >,\, \psi(s, u) >.
\end{aligned}
$$

because of Lemma 3·5·1 (Zemanian 1968, p. 64)

$$= < F(u),\, \psi(s, u) >$$

i.e. $\mathcal{L}_{+}\,[k(t) f[\,g(t)]\,] \triangleq < F(u),\, \psi(s, u) >$ *Q.E.D.*

It is worth noting at this point that four special cases of $k(t)$ which are discussed by Buschman (1956) will also hold.

Theorem 1·2—Let k, g, h, $\psi^*(s, u)$, $\bar{\Psi}^*(s, t)$, F^* are as above, then

$$< k(t)\, f[\,g(t)\,],\, e^{-st} > \; = \; < F^*(u),\, \psi^*(s, u) >.$$

Proof of this theorem can be carried through in the same manner as that for Theorem 1·1.

It is noted that if $f(t) = t^{\nu}$, Re $\nu > -1$, then the conclusion of the Theorem 1·1 reads

$$< k(t)\,[\,g(t)\,]^{\nu}\, e^{-st} > \; = \; < \Gamma(\nu+1)\, u^{-(\nu+1)},\, \psi(s, u) >$$

2. Theorem

The next substitution theorem involves the representation of $\mathcal{L}_{+}^{-1}\,[\,k(s)\, F[g(s)\,]$ in terms of $\mathcal{L}_{+}^{-1}\,[\,F(s)\,]$, where

$\mathcal{L}_{+} f = F(s)$, $a <$ Re $s < \infty$, and k, g are analytic functions. Let $\mathcal{L}_{+}\,[\,\theta(t, u)\,] = k(s)\, e^{-g(s)u} = k(s)\,\Phi\,[\,g(s), u\,]$.

Theorem 2·1—If k, g, f, F and θ [(t, u)] are as above, then

$$\mathcal{L}_+^{-1} [k(s) \; F[g(s)]] = \langle f(u), \theta \; (t, u) \rangle.$$

PROOF: Now

$$F(p) = \langle f(u), e^{-pu} \rangle = \langle f(u), \Phi \; (p, u) \rangle$$

$$F [g(s)] = \langle f(u), \Phi [g(s), u] \rangle$$

$$k(s) \; F [g(s)] = \langle f(u), k(s) \; \Phi [g(s), u] \rangle$$

$$= \langle f(u), \langle \theta \; (t, u), e^{-st} \rangle \rangle$$

$$= \langle \langle f(u), \theta \; (t, u) \rangle, e^{-st} \rangle$$

because of Lemma 3·5·1 (Zemanian 1968, p. 64)

$$= \mathcal{L}_+ [\langle f(u), \theta \; (t, u) \rangle]$$

$\therefore \; \mathcal{L}_+^{-1} [k(s) \; F [g(s)]] = \langle f(u), \theta \; (t, u) \rangle$ *Q.E.D.*

Note that we can prove similarly some operation transform formulae for two-sided Laplace transformation as below

Suppose that A, k, g, h, $\psi \; (s, u)$, $\psi^* \; (s, u)$, $\overline{\Psi} \; (s, t)$, $\overline{\Psi}^* \; (s, t)$, f, F, and F^* are as defined above.

(a') Let $\phi(t) \in \mathcal{L}_{c,d}$. The mapping $\phi(t) \longrightarrow k \; [h(t)] \; \phi[h(t)] \; | \; h'(t) \; |$

i.e. $\quad e^{-st} \longrightarrow k \; [h(t)] \; e^{\; sh(t)} \; | \; h'(t) \; |$

is an isomorphism from $\mathcal{L}_{c,d}$ onto $\mathcal{L}_{a,b}$ where a $< c$ and $d < b$. Consequently the mapping $f(t) \longrightarrow k(t) f [g(t)]$, which is defined by

$$\langle k(t) f [g(t),], \; \phi(t) \rangle = \langle k(t), k \; [h(t)] \; \phi[h(t)] \; | \; h'(t) \; | \rangle$$

is an isomorphism from $\mathcal{L}'_{a,b}$ onto $\mathcal{L}'_{c,d}$

(b') Let $\phi \; (t) \in \mathcal{L}_{c,d}$. The mapping

$$\phi \; (t) \longrightarrow k \; [h(t)] \; \phi \; [h(t)] \; | \; h'(t) \; | \; [A(t)]^{-1}$$

i.e. $\quad e^{-st} \longrightarrow K [h(t)] \; e^{-sh(t)} \; | \; h'(t) \; | \; [A(t)]^{-1}$

is an isomorphism from $\mathcal{L}_{c,d}$ onto $\mathcal{L}_{a,b}$.

Consequently the mapping $A(t) \, f(t) \longrightarrow k \; (t) \, f [\, g \; (t) \,]$, which is defined by

$$\langle k(t) f [g(t)], \phi \; (t) \rangle = \langle A(t) \, f(t), \; k \; [h(t)] \; \phi \; [h(t)] \; | \; h'(t) \; | \; [A(t)]^{-1} \rangle$$

is an isomorphism from $\mathcal{L}'_{a,b}$ onto $\mathcal{L}'_{c,d}$

EXAMPLES

Example 3·1—Let $\mathfrak{L}_+ f \triangleq F(s),\ a < \text{Re } s < \infty$.

Let $g(t) = t^{-1},\ k(t) = t^{\nu-1}$, Re $\nu > -1$.

We have

$$< t^{\nu-1} f(t^{-1}), e^{-st} > \; = \; < F(u),\ u^{\frac{1}{2}\nu} s^{-\frac{1}{2}\nu} J_\nu [2\sqrt{su}] >$$ [see Erdelyi 1954, 4·1(25)].

Example 3·2—Let $\mathfrak{L}_+ f \triangle F(s),\ \ a < \text{Re } s < \infty$.

Let $g(t) = \mathcal{J} t$, with $\mathcal{J}$ positive real number and

Let $k(t) = 1$.

We have

$$< f(\mathcal{J} t), e^{-st} > \; = \mathcal{J}^{-1} F(s/J).$$

Example 3·3—Let $\mathfrak{L}_+ f \triangleq F(s),\ a < \text{Re } s < \infty$.

Let $g(t) = t^2 + 2t,\ k(t) = t + 1$.

We have

$$< (t+1) f(t^2 + 2t),\ e^{-st} >$$

$$= < F(u), \frac{1}{\sqrt{2\pi}} \exp ((-s^2/8u + u + s)(2u)^{-1} D_1 (s/\sqrt{2u}) >$$

where $D_1(z)$ is the parabolic cylinder function [see Erdelyi 1954, 5·6(6)].

Example 3·4—Let $\mathfrak{L}_+ f = F(s),\ a < \text{Re } s < \infty$.

Let $g(t) = e^{\alpha t} - 1,\ k(t) = e^{\alpha t} - 1$ with α positive.

We have

$$< (e^{\alpha t} - 1) f(e^{\alpha t} - 1), e^{-st} >$$

$$= < F(u), \left[\alpha \, \Gamma \left(\frac{s}{2a} \right) \right]^{-1} u^{(s-2\alpha)\,2\alpha}\, e^{-u/2}\, M^{(u)}_{(s+2\alpha)\,2\alpha,\ (s-2\alpha)\,2\alpha} >$$

where $M_{K,\mu}(Z)$ is the Whittaker's function [see Erdelyi 1954, 5·6(6)].

Example 3·5—Let $\mathfrak{L}_+ f \triangleq F(s),\ a < \text{Re } s < \infty$.

Let $k(s) = s^{-\frac{1}{2}},\ g(s) = s^{\frac{1}{2}}$.

$$\therefore\ \mathfrak{L}_+^{-1} [s^{-\frac{1}{2}} F(s^{\frac{1}{2}})] = < f(u), (\pi t)^{-\frac{1}{2}} e^{-u^2/4t} >.$$

Example 3·6—Let $\mathfrak{L}_+ f \triangleq F(s),\ a < Re\ s < \infty$.

Let $k(s) = 1,\ g(s) = s^{\frac{1}{2}}$.

$$\therefore\ \mathfrak{L}_+^{-1} [F(s^{\frac{1}{2}})] = < f(u), (4\pi t^3)^{-\frac{1}{2}}\, u\, e^{-u^2/4t} >.$$

Example 3·7—Let $\mathfrak{L}_+ f \triangleq F(s),\ a < \text{Re } s < \infty$.

Let $g(s) = \sin h^{-1}(s),\ k(s) = s^{-1}$.

$$\therefore\ \mathfrak{L}_+^{-1} [s^{-1} F[\sin h^{-1}(s)]] = < f(u), J_u(t) >$$ [see Erdelyi 1954, 5·4 (23)].

Example 3·8—Let $\mathcal{L}_+ f \triangleq F(s)$, $a < \text{Re } s < \infty$.

Let $g(s) = s^{-1}$, $k(s) = s^{-d}$.

$$\therefore \ \mathcal{L}_+^{-1} [s^{-d} F(s^{-1})] = < f(u), (\sqrt{t}/u)^{d-1} J_{d-1} (2\sqrt{ut}) >.$$

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