THE DISTRIBUTION OF STRESS IN THE VICINITY OF AN EXTERNAL CRACK IN A SEMI-INFINITE SOLID

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The paper contains an analysis of the distribution of stress by the application of pressure to the faces of a plane crack covering the outside of a circle in a semi-infinite solid. Two problems are discussed. In the first problem it is assumed that the free end is stress free and in the second it is assumed to be rigidly clamped. By using the Hankel transforms and the theory of dual integral equations, each problem is reduced to the solution of a pair of simultaneous Fredholm integral equation of second kind. Expressions for various quantities of physical interest are derived for small value of its distance from the free boundary by finding iterative solution of these equations. When this distance is nearly unity, simultaneous Fredholm integral equations have been solved numerically.

1. INTRODUCTION

The strength of a material in the presence of cracks is a problem of interest in fracture as well as structural mechanics. Among quantities requiring theoretical prediction are the member rigidity and the shear centre location. In addition, knowledge of the elastic stress field is potentially useful for strength estimates based upon brittle fracture theory.

Recently several papers have appeared which treat stress distributions in an infinite solid due to the application of normal pressures to the faces of a flat external crack. The three-dimensional case, in which the crack covers outside of a circle, has been considered by Ufliand (1959) using toroidal co-ordinates and by Lowengrub and Sneddon (1963) from the Integral transform technique. Lowengrub (1966) has also solved the two dimensional plane strain problem for an external crack $y=0$, $|x|>1$ opened by normal pressures, using dual trigonometric-equations and by the author (Dhawan 1973) the case of crack in a thick plate when its free boundary is stress free.

In this paper we discuss a mixed boundary value problem in elasticity, which however, seem to have received scant attention so far in the scientific literature, even though they appear to be important for the design of various structures, which may not adequately be represented by a two-dimensional model. The problem arises in a natural way in the theoretical determination of stress inside a homogeneous elastic semi-infinite solid containing an exterior crack. The solid is supposed to be isotropic and two types of conditions are imposed on the free surface i.e. (i) free end is stress free and (ii) it is assumed to be rigidly clamped. The exterior crack is assumed to be in a plane normal to this axis occupying the region outside of a circle whose centre lies on the axis and whose radius is greater than the radius of the circle. The complementary problem of the penny-shaped crack has been considered by Srivastava and Singh (1969).

The technique employed in this paper is that of integral transforms and the theory of dual integral equations. The boundary conditions of mixed type lead to dual integral equations. These equations are then reduced to Fredholm integral equations of the second kind, which are amenable to numerical solutions. Using these solutions the quantities of physical interest may be calculated.

The analysis throughout this paper is purely formal and no attempt has been made to justify the interchange of various limiting processes. The numerical computation was carried out on the digital computer at the Computing Centre of the Tata Institute of Fundamental Research, Bombay. To solve, the integral equations, first the kernels were evaluated using Gauss-Laguerre quadrature and then approximate solutions were obtained by reducing integral equation to a system of linear algebraic equations using the method of Fox and Goodwin (1953).

2. Formulation of the Problem

In the problems that we shall consider here, we assume that there is symmetry about the z-axis. The position of a typical point of the solid may be expressed in terms of cylindrical coordinates (r, ϕ, z). For a symmetrical deformation of the solid the displacement vector (U) may be assumed to have components $(u, 0, w)$ and the only non-vanishing components of the stress-tensor will be σ_{rr}, $\sigma_{\phi\phi}$, σ_{zz} and τ_{rz}.

The crack is taken to lie on the line, $z = 0$, $1 \leq r < 0$. Let the solid be divided into two domains : (1) the layer $-h < z < 0$; and (2) the half-space $0 < z < \infty$.

We shall consider two problems.

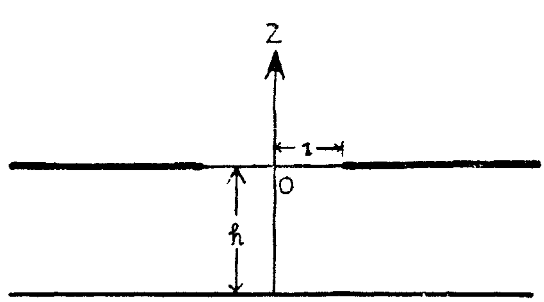

FIG. 1. Crack situated parallel to the free boundary.

Problem 1

The free boundary is assumed to be stress free and the stresses on the surface of the crack are prescribed. The boundary conditions can be written as

$$\sigma_{zz}(r, -h) = 0, \ \tau_{rz}(r, -h) = 0 \qquad \ldots(2\cdot1)$$

for all values of r, and

$$\left.\begin{aligned} \sigma_{zz}(r, 0^-) &= \sigma_1(r), \ \tau_{rz}(r, 0^-) = t_1(r) \\ \sigma_{zz}(r, 0^+) &= \sigma_2(r), \ \tau_{rz}(r, 0^+) = t_2(r) \end{aligned}\right\} r > 1. \qquad \ldots(2\cdot2)$$

For convenience we suppose that

$$\sigma_1(r) = \sigma_2(r) = -p(r),$$

$$t_1(r) = t_2(r) = -t(r).$$

Problem 2

It is assumed that the free edge is rigidly clamped and the stresses are prescribed on the surface of the crack. In this case in place of (2·1) we have

$$u(r, -h) = w(r, -h) = 0, \text{ for all values of } r, \qquad \ldots(2\cdot3)$$

while the conditions (1·2) remain unchanged.

In addition, for $z = 0$, to pass through the region unoccupied by the crack, the values of the components of displacement and stresses must be continuous. This requires the following additicnal boundary conditions

$$\left.\begin{aligned} \sigma_{zz}(r, 0^-) &= \sigma_{zz}(r, 0^+) \\ \tau_{rz}(r, 0^-) &= \tau_{rz}(r, 0^+) \\ u(r, 0^-) &= u(r, 0^+) \\ w(r, 0^-) &= w(r, 0^+) \end{aligned}\right\} \text{ for } r < 1. \qquad \ldots(2\cdot4)$$

Again the basic equations of elastic equilibrium in the axially symmetric case are

$$2(1-\eta)\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right] + (1-2\eta)\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial r \partial z} = 0 \qquad \ldots(2\cdot5)$$

$$(1-2\eta)\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right] + 2(1-\eta)\frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{u}{r}\right) = 0 \qquad \ldots(2\cdot6)$$

$$\sigma_{zz}(r, z) = \frac{2\mu}{1-\eta}\left[(1-\eta)\frac{\partial w}{\partial z} + \eta\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)\right] \qquad \ldots(2\cdot7)$$

$$\tau_{rz}(r, z) = \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right] \qquad \ldots(2\cdot8)$$

where $\mu = \dfrac{E}{2(1+\eta)}$, E is Young's modulus and η Poisson's ratio of the elastic medium.

For the solution of the partial differential equation (2·5) and (2·6) we introduce Hankel transform of u and w. We define

$$\bar{u}(\xi, z) = \mathfrak{H}_1[u(r, z), r \longrightarrow \xi] \qquad \text{...(2·9)}$$

$$\bar{w}(\xi, z) = \mathfrak{H}_0[w(r, z), r \longrightarrow \xi]. \qquad \text{...(2·10)}$$

Multiplying (2·5) by $\xi J_1(\xi r)$ and (2·6) by $\xi J_0(\xi r)$ and integrating with respect to r from 0 to ∞, we get

$$[(1-2\eta) D^2 - 2(1-\eta)\xi^2]\,\bar{u} - D\bar{w} = 0 \qquad \text{...(2·11)}$$

$$[2(1-\eta) D^2 - (1-2\eta)\xi^2]\,\bar{w} + D\bar{u} = 0 \qquad \text{...(2·12)}$$

where $D \equiv \left(\frac{d}{dz}\right)$. From (2·7) and (2·8), we have

$$\bar{\sigma}_{zz} = \mathfrak{H}_0[\sigma_{zz}(r, z), r \longrightarrow \xi] = \frac{2\mu}{1-\eta}\,(1-\eta) D\bar{w} + \eta\,\xi\,\bar{u}] \qquad \text{...(2·13)}$$

$$\bar{\tau}_{rz} = \mathfrak{H}_1[\tau_{rz}(r, z), r \longrightarrow \xi] = \mu\,[D\bar{u} - \xi\,\bar{w}]. \qquad \text{...(2·14)}$$

3. Solution for the Semi-Infinite Solid

In the case of semi-infinite solid $z >> 0$, assumed free from disturbances at infinity we are interested in the solutions of the equations (2·11) and (2·12) which tend to zero as $z \longrightarrow \infty$. The appropriate solutions are

$$\bar{u} = (A + B\,\xi\,z)\,e^{-\xi z}$$

$$\bar{w} = (A_1 + B\,\xi\,z)\,e^{-\xi z}$$

where

$$A_1 = A + (3 - 4\eta)\,B.$$

Hence the expressions for the components of displacement vector and stress tensor are

$$u(r, z) = \int_0^\infty \xi\,(A + B\,\xi\,z)\,e^{-\xi z} J_1(\xi r)\,d\xi, \qquad \text{...(3·1)}$$

$$w(r, z) = \int_0^\infty \xi\,(A_1 + B\,\xi\,z)\,e^{-\xi z} J_0(\xi r)\,d\xi, \qquad \text{...(3·2)}$$

$$\sigma_{zz}(r, z) = -2\mu \int_0^\infty \xi^2\,[A + 2(1-\eta)\,B + B\,\xi\,z]\,e^{-\xi z} J_0(\xi r)\,d\xi, \qquad \text{...(3·3)}$$

$$\tau_{rz}(r, z) = -2\mu \int_0^\infty \xi^2\,[A + (1-2\eta)\,B + B\,\xi\,z]\,e^{-\xi z} J_1(\xi r)\,d\xi. \qquad \text{...(3·4)}$$

4. Solution for the Layer $-h < z < 0$

The appropriate solutions in this case are

$$\bar{u}=[\{C+D\,(z+h)\xi\}\cosh(z+h)\xi+\{E+F(z+h)\xi\}\sinh(z+h)\xi]/\sinh\xi h \quad \ldots(4\cdot1)$$

$$\bar{w}=[\{C_1+D_1(z+h)\xi\}\cosh(z+h)\xi+\{E_1+F_1(z+h)\xi\}\sinh(z+h)\xi]/\sinh\xi h \quad \ldots(4\cdot2)$$

where

$$D=-F_1,\ D_1=-F,$$
$$C+E_1+(3-4\eta)\,D_1=0,\ C_1+E-(3-4\eta)D=0. \quad \ldots(4\cdot3)$$

Problem 1

We have to satisfy the condition (2·1). From (2·13) and (2·14) we have

$$\bar{\sigma}_{zz}=\frac{2\mu\xi}{\sinh\xi h}\,[\{(1-\eta)(C_1+D_1(z+h)\xi+F_1)+\eta(E+F(z+h)\xi)\}\times\sinh(z+h)\xi$$
$$+\{(1-\eta)(E_1+F_1(z+h)\xi+D_1)+\eta(C+D(z+h)\xi)\}\cosh(z+h)\xi] \quad \ldots(4\cdot4)$$

$$\bar{\tau}_{rz}=\frac{2\mu\xi}{\sinh\xi h}\,[\{C+D(z+h)\xi+F-E_1-F_1(z+h)\xi\}\sinh(z+h)\xi$$
$$+\{E+F(z+h)\xi+D-C_1-D_1(z+h)\xi\}\cosh(z+h)\xi]. \quad \ldots(4\cdot5)$$

Since $\sigma_{zz}=\tau_{rz}=0$ for $z=-h$ we have

$$\left.\begin{aligned}E+D-C_1&=0\\(1-\eta)(D_1+E_1)+\eta\,C&=0\end{aligned}\right\} \quad \ldots(4\cdot6)$$

From (4·3) and (4·6) we get

$$C=-2\,(1-\eta)\,D_1,\ E_1=-(1-2\eta)\,D_1,$$
$$C_1=2\,(1-\eta)\,D,\ E=(1-2\eta)\,D.$$

Hence the expressions for the components of displacement vectoı and stress tensor are

$$u\,(r,z)=\int_0^\infty[(\{D(z+h)\xi-2(1-\eta)D_1\}\cosh(z+h)\xi+\{(1-2\eta)D$$
$$-D_1(z+h)\xi\}\sinh(z+h)\xi)/\sinh\xi z]\xi J_1(\xi r)\,d\xi \quad \ldots(4\cdot7)$$

$$w\,(r,z)=\int_0^\infty[(\{2(1-\eta)D+D_1(z+h)\xi\}\cosh(z+h)\xi-\{D(z+h)\xi$$
$$+(1-2\eta)D_1\}\sinh(z+h)\xi)/\sinh\xi h]\,\xi J_0(\xi r)\,d\xi \quad \ldots(4\cdot8)$$

$$\sigma_{zz}\,(r,z)=2\mu\int_0^\infty[(\{D_1(z+h)\xi+D\}\sinh(z+h)\xi$$
$$-D(z+h)\xi\cosh(z+h)\xi)/\sinh\xi h]\,\xi J_0(\xi r)\,d\xi \quad \ldots(4\cdot9)$$

$$\tau_{rz}\,(r,z)=-2\mu\int_0^\infty\Big[(\{D_1-D(z+h)\xi\}\sinh(z+h)\xi+D_1(z+h)\xi$$
$$\times\cosh(z+h)\xi)\frac{\xi}{\sinh\xi h}\Big]\,\xi J_1(\xi r)\,d\xi. \quad \ldots(4\cdot10)$$

Problem 2

The conditions (2·3) imply that

$C = C_1 = 0;\ E_1 = -(3-4\eta)D_1;\ E = (3-4\eta)\ D.$

The expressions for the components of displacement vector and stress tensor are

$$u(r,z) = \int_0^\infty [(D(z+h)\xi \cosh(z+h)\ \xi + \{(3-4\eta)\ D - D_1(z+h)\xi\} \times \sinh(z+h)\ \xi)/\sinh \xi h]\ \xi J_1(\xi r)\ d\xi \quad \ldots(4·11)$$

$$w(r,z) = \int_0^\infty [(D_1(z+h)\ \xi \cosh(z+h)\ \xi - \{(3-4\eta)\ D_1 + D(z+h)\ \xi\} \times \sinh(z+h)\xi)/\sinh \xi h]\ \xi J_0(\xi r)\ d\xi. \quad \ldots(4·12)$$

$$\sigma_{zz}(r,z) = -2\mu \int_0^\infty \left[\frac{\xi}{\sinh \xi h}(\{D(z+h)\ \xi + 2(1-\eta)\ D_1\} \cosh(z+h)\xi + \{D(1-2\eta) - D_1(z+h)\ \xi\} \sinh(z+h)\ \xi)\right] \xi J_0(\xi r)\ d\xi \quad \ldots(4·13)$$

$$\tau_{rz}(r,z) = -2\mu \int_0^\infty \left[\frac{\xi}{\sinh \xi h}(\{D_1(z+h)\ \xi - 2(1-\eta)\ D\} \cosh(z+h)\xi - \{D_1(1-2\eta) + D(z+h)\ \xi\} \sinh(z+h)\ \xi)\right] \xi J_1(\xi r)\ d\xi \quad \ldots(4·14)$$

5. Reduction of the Problem to a System of Simultaneous Dual Integral Equations

Problem 1

We still have to satisfy the boundary conditions for $z = 0$. The conditions (2.2) imply that for $r > 1$, we have

$$\int_0^\infty [D(1-y\coth y) + D_1 y]\ \xi^2 J_0(\xi r)\ d\xi = -\int_0^\infty [A + 2(1-\eta)B]\xi^2 J_0(\xi r) d\xi = \frac{-p(r)}{2\mu} \quad \ldots(5·1)$$

$$\int_0^\infty [D_1(1+y\coth y) - Dy]\ \xi^2 J_1(\xi r)\ d\xi = \int_0^\infty [A + (1-2\eta)B]\ \xi^2 J_1(\xi r) d\xi = \frac{-t(r)}{2\mu}. \quad \ldots(5.2)$$

The conditions (2·4) imply that for $r < 1$, we have

$$\int_0^\infty [D(1 - y\coth y) + D_1 y + A + 2(1-\eta)B]\ \xi^2 J_0(\xi r)\ d\xi = 0 \quad \ldots(5·3)$$

$$\int_0^\infty [D(1 + y \cosh y) - Dy - A - (1 - 2\eta)B] \, \xi^2 J_1(\xi r) d\xi = 0 \qquad \ldots(5{\cdot}4)$$

$$\int_0^\infty [D(y \coth y + 1 - 2\eta) - D_1(y + 2(1 - \eta) \coth y - A)] \, \xi J_1(\xi r) \, d\xi = 0 \qquad \ldots(5{\cdot}5)$$

$$\int_0^\infty [D \, \{2(1 - \eta) \coth y - y\} + D_1 \, \{y \coth y - 1 + 2\eta\} \,] \, \xi J_0(\xi r) \, d\xi = 0 \qquad \ldots(5{\cdot}6)$$

where $y = \xi h$.

From equations (5·1) to (5·4) we have

$$X = D(1 - y \coth y) + D_1 y = - A - 2(1 - \eta) B \qquad \ldots(5{\cdot}7)$$

$$Y = - Dy + D_1 (1 + y \coth y) = A + (1 - 2\eta) B. \qquad \ldots(5{\cdot}8)$$

Let us suppose

$$M = D(y \coth y + 1 - 2\eta) - D_1 (y + 2 (1 - \eta) \coth y) - A \qquad \ldots(5{\cdot}9)$$

$$N = D\{2 (1-\eta) \coth y - y\} + D_1\{y \coth y - 1 + 2\eta\} - A - (3-4\eta)B \qquad \ldots(5{\cdot}10)$$

From these equations, we have

$$4(1 - \eta) X = N - I(y)N - J(y)M \qquad \ldots(5{\cdot}11)$$

$$- 4(1 - \eta) Y = M - K(y)M - L(y)N. \qquad \ldots(5{\cdot}12)$$

where

$$I(y) = (1 + 2y + 2y^2) \, e^{-2y}; \; k(y) = (1 - 2y + 2y^2) \, e^{-2y}; \; L(y) = J(y) = 2y^2 \, e^{-2y}.$$

Equations (5·1), (5·2), (5·5) and (5·6) lead to the system of simultaneous dual integral equations.

$$\left.\begin{aligned} &\int_0^\infty [N(\xi) - I(\xi h) \, N(\xi) - J(\xi h) \, M(\xi)] \, \xi^2 \, J_0 \, (\xi r) \, d\xi = - P(r) \qquad \ldots(5{\cdot}13) \\ &\int_0^\infty [M(\xi) - K(\xi h) \, M(\xi) - L(\xi h) \, N(\xi)] \, \xi^2 \, J_1 \, (\xi r) \, d\xi = T(r) \qquad \ldots(5{\cdot}14) \end{aligned}\right\} \; r > 1$$

$$\left.\begin{aligned} &\int_0^\infty \xi \, N(\xi) \, J_0 \, (\xi r) \, d\xi = 0 \qquad \ldots(5{\cdot}15) \\ &\int_0^\infty \xi \, M(\xi) \, J_1 \, (\xi r) \, d\xi = 0 \qquad \ldots(5{\cdot}16) \end{aligned}\right\} \; 0 < r < 1$$

where

$$\left.\begin{aligned} P(r) &= \frac{2\,p(r)\,(1-\eta)}{\mu} \\ T(r) &= \frac{2\,t(r)\,(1-\eta)}{\mu}. \end{aligned}\right\} \qquad \ldots(5{\cdot}17)$$

Problem 2

The conditions (2·2) and (2·4) and an exactly similar procedure as above leads to the set of simultaneous dual integral equation (5·13)—(5·17) with the values of I, J, K and L as

$$I(y) = \frac{-2}{3-4\eta}\,(a+y+y^2)\,e^{-2y}$$

$$K(y) = \frac{-2}{3-4\eta}\,(b-y+y^2)\,e^{-2y}$$

$$J(y) = L(y) = -\,2(y^2+b)\,e^{-2y}/3-4\eta$$

$$a = \tfrac{1}{2}\,(5-12\eta+8\eta^2)$$

$$b = 2\,(1-\eta)\,(1-2\eta).$$

6. Solution of the Simultaneous Dual Integral Equations

We have to solve the integral equations

$$\int_0^\infty \xi\, N(\xi)\, J_0\,(\xi r)\, d\xi = 0, \quad r < 1 \qquad \ldots(6{\cdot}1)$$

$$\int_0^\infty \xi\, M(\xi)\, J_1\,(\xi r)\, d\xi = 0, \quad r < 1 \qquad \ldots(6{\cdot}2)$$

$$\left.\begin{aligned} &\int_0^\infty [N(\xi) - I(\xi h)\,N(\xi) - J(\xi h)\,M(\xi)]\,\xi^2\,J_0\,(\xi r)\,d\xi = -\,P(r) \qquad \ldots(6{\cdot}3)\\ &\int_0^\infty [M(\xi) - K(\xi h)\,M(\xi) - L(\xi h)\,N(\xi)]\,\xi^2\,J_1\,(\xi r)\,d\xi = T(r) \qquad \ldots(6{\cdot}4) \end{aligned}\right\}\ r > 1$$

where I, J, K, L, P and T are known functions and M, N are the unknown functions to be determined. We shall presently show that these equations can be reduced to simultaneous Fredholm integral equations of the second kind which are best solved by numerical methods. However in the case where $h >> 1$ and the integrals

$$\int_0^\infty u^n\, I(u)\, du, \quad n = 0, 1, 2, \ldots$$

and similar integrals for J, K, L are convergent, with slight modification of Copson *et al.* (see Sneddon 1966) can be used for solving the above set of simultaneous dual integral equations. Let the trivial solution be.

$$\xi N(\xi) = \int_1^\infty n(t) \cos \xi t \, dt, \qquad \ldots(6{\cdot}5)$$

$$\xi M(\xi) = \int_1^\infty m(t) \sin \xi t \, dt, \qquad \ldots(6{\cdot}6)$$

with

$$\lim_{t \to \infty} n(t) = 0 \; ; \; \lim_{t \to \infty} m(t) = 0. \qquad (6{\cdot}7)$$

With these choice of M and N, we see that (6·1) and (6·2) are satisfied while (6·3) and (6·4) give

$$-\int_r^\infty \frac{n(t)}{\sqrt{t^2 - r^2}} \, dt + \frac{1}{\pi} \int_r^\infty \frac{dt}{\sqrt{t^2 - r^2}} \int_1^\infty n\,(x) \left[\frac{d}{dx} \int_0^\infty - 2\, I(\xi h) \cos \xi x \cos \xi t \, d\xi \, dx \right]$$

$$+ \frac{1}{\pi} \int_r^\infty \frac{dt}{\sqrt{t^2 - r^2}} \int_1^\infty m\,(x) \left[\frac{d}{dx} \int_0^\infty 2\, J(\xi h) \sin \xi x \cos \xi t \, d\xi \right] dx = -P(r) \qquad \ldots(6{\cdot}7)$$

$$-\frac{d}{dr} \int_r^\infty \frac{m(t)}{\sqrt{t^2 - r^2}} \, dt + \frac{1}{\pi} \frac{d}{dr} \int_r^\infty \frac{dt}{\sqrt{t^2 - r^2}} \int_1^\infty m\,(x) \left[\int_0^\infty 2K(\xi h) \sin \xi x \,.\, \sin \xi t \, d\xi \right] dx$$

$$+ \frac{1}{\pi} \frac{d}{dr} \int_r^\infty \frac{dt}{\sqrt{t^2 - r^2}} \int_1^\infty n\,(x) \left[\int_0^\infty 2\, L(\xi h) \cos \xi x \sin \xi t \, d\xi \right] dx = T(r) \qquad \ldots(6{\cdot}8)$$

From the above equations we have

$$n\,(t) - \int_1^\infty [n\,(x)\, K_1\,(t, x) + m\,(x)\, k_2\,(t, x)] \, dx = \frac{-2}{\pi} \int_t^\infty \frac{r\, P\,(r)\, d\xi}{\sqrt{r^2 - t2}}, \qquad \ldots(6{\cdot}9)$$

$$m\,(t) - \int_1^\infty [m\,(x)\, k_3\,(t, x) + n\,(x)\, k_4\,(t, x)] \, dx = \frac{-2\,t}{\pi} \int_t^\infty \frac{T\,(r)\, d\xi}{\sqrt{r^2 - t^2}}. \qquad \ldots(6{\cdot}10)$$

where

$$K_1 = \frac{2}{\pi h} \int_0^\infty I\,(\omega) \cos \frac{\omega x}{h} \cos \frac{\omega t}{h} \, d\omega$$

$$= \frac{2}{\pi} \left[\frac{I_0}{h} + \frac{I_1}{h^3} (x^2 + t^2) + \frac{I_2}{h^5} (x^4 + 6x^2 t^2 + t^4) + o\,(h)^{-6} \right]$$

$$K_2 = \frac{2}{\pi h} \int_0^\infty J(\omega) \sin \frac{\omega x}{h} \cos \frac{\omega t}{h} \, d\omega$$

$$= \frac{2}{\pi} \left[\frac{J_0 x}{h^2} + \frac{J_1 x}{h^4}(x^2+3t^2) + \frac{J_2 x}{h^6}(x^4+10x^2t^2+5t^4) + o\,(h^{-8}) \right]$$

$$K_3 = \frac{2}{\pi h} \int_0^\infty K(\omega) \sin \frac{\omega x}{h} \sin \frac{\omega t}{h} \, d\omega = \frac{2}{\pi} \left[\frac{K_0\, xt}{h^3} + \frac{K_1\, xt}{h^5}(x^2+t^2) + o\,(h^{-7}) \right]$$

$$K_4 = \frac{2}{\pi h} \int_0^\infty L(\omega) \cos \frac{\omega x}{h} \sin \frac{\omega t}{h} \, d\omega =$$

$$\frac{2}{\pi} \left[\frac{L_0\, t}{h^2} + \frac{L_1\, t}{h^4}(3x^2+t^2) + \frac{L_2\, t}{h^6}(5x^4+10x^2t^2+t^4) + o\,(h^{-8}) \right].$$

The above expressions have been obtained by substitution $\xi h = \omega$, writing the expan sions of sine and cosine function in power series and integrating term by term. The values of the constants are

$$I_0 = \int_0^\infty I(\omega)\, d\omega\,; \qquad K_0 = \int_0^\infty \omega^2 K(\omega)\, d\omega\,;$$

$$I_1 = \frac{-1}{2!} \int_0^\infty \omega^2 I(\omega)\, d\omega\,; \qquad K_1 = \frac{-1}{3!} \int_0^\infty \omega^4 K(\omega)\, d\omega\,;$$

$$I_2 = \frac{1}{4!} \int_0^\infty \omega^4 I(\omega)\, d\omega\,; \qquad L_0 = \int_0^\infty \omega\, L(\omega)\, d\omega\,;$$

$$J_0 = \int_0^\infty \omega\, J(\omega)\, d\omega\,; \qquad L_1 = \frac{-1}{3!} \int_0^\infty \omega^3 L(\omega)\, d\omega\,;$$

$$J_1 = \frac{-1}{3!} \int_0^\infty \omega^3 J(\omega)\, d\omega\,; \qquad L_2 = \frac{1}{5!} \int_0^\infty \omega^5 L(\omega)\, d\omega.$$

$$J_2 = \frac{1}{5!} \int_0^\infty \omega^5 J(\omega)\, d\omega\,;$$

The constants L are obtained from expressions for the constants J by replacing the function $J(x)$ by $L(x)$. By a simple extension of the classical theory of Fredholm integral equations (Tricomi 1957) we can obtain the solutions of (6·9) and (6·10) as a power series in $1/h$ provided h is sufficiently large. Suppose that $n(t)$ and $m(t)$ can be written as

$$m(t) = m_0(t) + \frac{m_1(t)}{h} + \frac{m_2(t)}{h^2} + \ldots\ldots + \frac{m_6(t)}{h^6} + o\,(h^{-7}) \qquad \ldots(6{\cdot}11)$$

$$n(t) = n_0(t) + \frac{n_1(t)}{h} + \frac{n_2(t)}{h^2} + \ldots\ldots + \frac{n_6(t)}{h^6} + o\,(h^{-7}). \qquad \ldots(6{\cdot}12)$$

We can determine the values of $m_i(t)$ and $n_i(t)$, $i = 0, 1, \ldots\ldots$ by substitution in (6·9) and (6·10).

$$n_0(t) = -\frac{2}{\pi}\int_t^\infty \frac{rP(r)}{\sqrt{r^2-t^2}}\; dr.$$

$$n_1(t) = \frac{2}{\pi} I_0 \int_1^\infty n_0(x)\; dx$$

$$n_2(t) = \frac{2}{\pi}\int_1^\infty [\, I_0\, n_1(x) + J_0 x\; m_0(x)\,]\; dx$$

$$n_3(t) = \frac{2}{\pi}\int_1^\infty [\, I_1\,(x^2+t^2)\, n_0(x) + I_0 n_2(x) + J_0 x\; m_1(x)]\; dx$$

$$n_4(t) = \frac{2}{\pi}\int_1^\infty [\, I_1\,(x^2+t^2) n_1(x) + I_0 n_3(x) + J_0 x m_2(x) + J_1 x\,(x^2+3t^2)\; m_1(x)\,]\; dx$$

$$n_5(t) = \frac{2}{\pi}\int_1^\infty [\, I_2\,(x^4+6x^2t^2+t^4)\; n_1(x) + I_1\,(x^2+t^2)\; n_2\,(x) + I_0 n_4\,(x)$$

$$+ J_1 x\,(x^2+3t^2)\; m_1(x) + J_0 x\; m_3(x)\,]\; dx$$

$$n_6(t) = \frac{2}{\pi}\int_1^\infty [\, I_2\,(x^4+6x^2t^2+t^4)\; n_1(x) + I_1\,(x^2+t^2)\; n_3(x)$$

$$+ J_2 x\,(x^4+10x^2t^2+5t^4)\; m_0(x) + J_1 x\,(x^2+3t^2)\; m_2(x) + J_0 x m_4(x)\,]\; dx.$$

$$m_0(t) = \frac{2t}{\pi}\int_t^\infty \frac{T(r)dr}{\sqrt{r^2-t^2}}$$

$$m_1(t) = 0$$

$$m_2(t) = \frac{2}{\pi} L_0 t\int_1^\infty n_0(x)\; dx$$

$$m_3(t) = \frac{2}{\pi}\int_1^\infty [\, K_0 x t\; m_0(x) + L_0 t n_1(x)\,]\; dx$$

$$m_4(t) = \frac{2}{\pi}\int_1^\infty [\, K_0 x t\; m_1(x) + L_1 t\,(3x^2+t^2)\; n_0(x) + L_1 t n_2(x)\,]\; dx$$

$$m_5(t) = \frac{2}{\pi} \int_1^\infty [K_1 xt\,(x^2+t^2) m_0(x) + K_0 xt m_2(x) + L^1 t(3x^2+t^2) n_1(x) + L_0 t n_3(x)]\, dx$$

$$m_6(t) = \frac{2}{\pi} \int_1^\infty [K_1 xt\,(x^2+t^2)\, m_1(x) + K_0 xt m_3(x) + L_2 t\,(5x^4 + 10x^2t^2 + t^4)\, n_0(x)$$

$$+ L_1 t(3x^2+t^2)\, n_2(x) + L_0\, t\, n_4(x)]\, dx.$$

7. Solution for Particular Type of Loading and Quantities of Physical Interest

In this section we solve the integral equations (6·9) and (6·10) for large values of h, by giving a particular type of loading which is important from the physical point of view.

Let $p(r) = p\ H(a-r),\ a > 1$ and $t(r) = 0$ ...(7·1)

where $H(t)$ is the Heaviside unit function, then

$$n_0(t) = \frac{-2}{\pi} \int_t^\infty \frac{rP(n)\, dr}{\sqrt{r^2-t^2}}$$

$$= \begin{cases} -K\sqrt{a^2-t^2}, & t \leq a \\ 0, & t > a \end{cases} \quad \text{where } K = \frac{4(1-\eta)p}{\pi\mu} \qquad \text{...(7·2)}$$

and

$$m_0(t) = 0. \qquad \text{...(7·3)}$$

Substituting these values in (6·9) and (6·10), we see that for $t < a$ the integral equation becomes:

$$n(t) - \int_1^\infty n(x)\, K_1(t,x) + m(x)\, K_2(t,x)]\, dx = \frac{-2}{\pi}\sqrt{a^2-t^2}, \qquad \text{...(7.4)}$$

$$m(t) - \int_1^\infty [m(x)\, K_3(t,x) + n(x)\, K_4(t,x)]\, dx = 0 \qquad \text{...(7·5)}$$

which give

$$n_0(t) = -K\sqrt{a^2-t^2}, \text{ where } K = \frac{4(1-\eta)p}{\pi\mu}$$

$$n_1(t) = -\frac{I_0 K}{2\pi}\left[\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2-1}\right]$$

$$n_2(t) = -\frac{1}{\pi^2} I_0^2 K(a-1)\left[\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - \sqrt{a^2-1}\right]$$

$$n_3(t) = -\frac{2}{\pi} I_1K\left[\frac{1}{16}\left(3\pi a^4 - 2a^4 \sin^{-1}\left(\frac{1}{a}\right) - 2(2 - a^2)\sqrt{a^2 - 1}\right)\right.$$
$$\left. + \frac{t^2}{4}\left(\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2-1}\right)\right]$$
$$- \left(\frac{2}{\pi}\right)^3 I_0^3 (a-1)^2 \left[\frac{1}{4}\left(\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2 - 1}\right)\right]$$

$$n_4(t) = -\frac{4I_0I_1K}{3\pi^2}(a^3 - 1)\left[\frac{1}{4}\left(\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2 - 1}\right)\right]$$
$$- \frac{4I_0I_1K}{\pi^2} t^2 (a-1)\left[\frac{1}{4}\left(\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2 - 1}\right)\right]$$
$$- \frac{4I_0I_1K}{\pi^2}(a-1)\left[\frac{1}{16}\left(3\pi a^4 - 2a^4 \sin^{-1}\left(\frac{1}{a}\right) - 2(2 - a^2)\sqrt{a^2 - 1}\right)\right]$$
$$- \frac{4I_0I_1K}{3\pi^2}(a^3 - 1)\left[\frac{1}{16}\left(3\pi a^4 - 2a^4 \sin^{-1}\left(\frac{1}{a}\right) - 2(2 - a^2)\sqrt{a^2 - 1}\right)\right]$$
$$- \left(\frac{2}{\pi}\right)^3 I_0^4K(a^3 - 1)\left[\frac{1}{4}\left(\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2 - 1}\right)\right]$$
$$- \frac{4}{3\pi^2} L_0J_0K(a^3 - 1)\left[\frac{1}{4}\left(\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2 - 1}\right)\right]$$

$$n_5(t) = -\frac{2}{\pi} I_2K\left[\frac{1}{96}\left(3\pi a^6 - 2a^6 \sin^{-1}\left(\frac{1}{a}\right) - 2(8 - 50a^2 + 69a^4)\sqrt{a^2 - 1}\right)\right.$$
$$+ \frac{6t^2}{14}\left(3\pi a^4 - 2a^4 \sin^{-1}\left(\frac{1}{a}\right) - 2(2 - a^2)\sqrt{a^2 - 1}\right)$$
$$\left. + \frac{1}{4} t^4\left(\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2 - 1}\right)\right]$$
$$- \left(\frac{2}{\pi}\right)^3 I_0^2I_1K(a-1)\left\{\frac{a^3 - 1}{3} + t^2(a-1)\right\}$$
$$\times\left\{\frac{1}{4}\left(\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2 - 1}\right)\right\}$$
$$- \frac{2I_0(a-1)}{\pi} n_4(t) - \left(\frac{2}{\pi}\right)^3 L_0J_0KI_1 \frac{(a-1)(a^3 - 1)}{12}$$
$$\times\left(\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2 - 1}\right)$$

$$m_0(t) = 0$$
$$m_1(t) = 0$$

$$m_2(t) = -\frac{2}{\pi} L_0Kt \left[\frac{1}{4}\left(\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2-1} \right)\right]$$

$$m_3(t) = -\left(\frac{2}{\pi}\right)^2 I_0L_0Kt\,(a-1)\left[\frac{1}{4}\left(\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2-1} \right)\right]$$

$$m_4(t) = -\frac{1}{\pi} L_1K \left[\frac{1}{16}\left(3\pi a^4 - 2a^4 \sin^{-1}\left(\frac{1}{a}\right) - 2\,(2-a^2)\sqrt{a^2-1} \right)\right]$$

$$-\frac{2L_1t^3K}{\pi}\left[\frac{1}{4}\left(3\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2-1} \right)\right]$$

$$-\left(\frac{2}{\pi}\right)^3 L_0I_0^2Kt\,(a-1)^2\left[\frac{1}{4}\left(3\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2-1} \right)\right]$$

$$m_5(t) = -\left(\frac{2}{\pi}\right)^2 K_0KtL_0\,\frac{(a^3-1)}{3}\left[\frac{1}{4}\left(3\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2-1} \right)\right]$$

$$-\left(\frac{2}{\pi}\right)^2 I_0KL_1t\left\{ a^3 - 1 + t^2\,(a-1) \right\}$$

$$\left\{ \frac{1}{4}\left(3\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2-1} \right) \right\}$$

$$-\left(\frac{2}{\pi}\right)^2 L_0I_1tK\Bigg[(a-1)\left\{\frac{1}{16}\left(3\pi a^4 - 2a^4 \sin^{-1}\left(\frac{1}{a}\right) - 2(2-a^2)\sqrt{a^2-1} \right)\right\}$$

$$+\left(\frac{a^3-1}{3}\right)\left\{ \frac{1}{4}\left(3\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2-1} \right) \right\}$$

$$-\left(\frac{2}{\pi}\right)^4 I_0^3K\,(a-1)^3\left[\frac{1}{4}\left(3\pi a^2 - 2a^2 \sin^{-1}\left(\frac{1}{a}\right) - 2\sqrt{a^2-1} \right)\right]$$

Thus the iterative solutions for $m(t)$ and $n(t)$ are

$$n(t) = \frac{2}{\pi} K\left[\sqrt{a^2-t^2} + a_0 + a_1t^2 + a_2t^4\right] + o(h^{-6}) \qquad \ldots(7{\cdot}6)$$

$$m(t) = \frac{2}{\pi} K\left[b_0t + b_1t^3 + b_2t^5\right] + o\,(h^{-6}) \qquad \ldots(7{\cdot}7)$$

where

$$a_0 = \left(\frac{2}{\pi h}\right)I_0A_1\sum_{n=0}^{4}\left[\left(\frac{2I_0}{\pi h}\right)(a-1)\right]^n$$

$$-\left(\frac{2I_0I_1}{3\pi^2h^4}\right)\left[1 + \left(\frac{2I_0}{\pi h}\right)(a-1)\right](a^3-1)\,A_1 -$$

(equation contd. next page)

$$-\left(\frac{2I_1}{\pi h^3}\right)\left[1+\left(\frac{2I_0}{\pi h}\right)(a-1)+\left(\frac{2I_0}{\pi h}\right)^2(a-1)^2\right]A_2$$

$$+\left(\frac{2I_2}{\pi h^5}\right)A_3-\left(\frac{2}{\pi}\right)^2\frac{L_0J_0A_1}{3h^4}\left\{1+2\frac{I_0}{\pi h}(a-1)\right\}(a^3-1)+O(h^{-6})$$

$$a_1=\left(\frac{2}{\pi h^3}\right)\left[I_1\left\{1-\left(\frac{2I_0}{\pi h}\right)\left(1-\frac{2I_0}{\pi h}\right)(a-1)\right\}A_1-\frac{6I_2}{h^2}A_2\right]+O(h^{-6})$$

$$a_2=\left(\frac{2I_2}{\pi h^5}\right)A_1+O(h^{-6})$$

$$b_0=\left(\frac{2}{\pi h}\right)L_0A_1-\left(\frac{2}{\pi h^2}\right)I_0A_1\sum_{n=0}^{2}\left[\left(\frac{2I_0}{\pi h}\right)(a-1)\right]^n-\frac{3}{\pi h^4}L_1A_2-\frac{5}{\pi h^6}L_2A_3$$

$$-\left(\frac{2}{\pi}\right)^2\frac{A_1K_0}{3h^5}\left[\left\{\frac{I_0L_1}{\pi h}-\frac{I_0L_0}{\pi h}-\frac{I_0^2L_1}{\pi h}+\frac{L_0J_0}{3\pi h}\right\}(a-1)\right](a^3-1)$$

$$\left(\frac{2}{\pi}\right)^2A_1\frac{(a^3-1)}{h^5}\left[I_0L_1-\frac{L_0I_1}{3h}\right]+O(h^{-7})$$

$$b_1=\frac{2}{\pi h^4}A_1-\frac{10L_2A_2}{\pi h^6}-\left(\frac{2}{\pi}\right)^2\frac{I_0L_1A_1}{h^5}\left[1+\frac{(a-1)}{\pi h}+\frac{I_1(a-1)}{\pi h}\right](a-1)-O(h^{-7})$$

$$b_2=-\left(\frac{2L_2}{\pi h^6}\right)A_1-O(h^{-7})$$

with

$$A_1=\left(\frac{1}{4}\right)\left[\pi a^2-2a^2\sin^{-1}\left(\frac{1}{a}\right)-2\sqrt{a^2-1}\right] \quad \ldots(7\cdot8)$$

$$A_2=\left(\frac{1}{16}\right)\left[3\pi a^4-2a^4\sin^{-1}\left(\frac{1}{a}\right)-2(2-a^2)\sqrt{a^2-1}\right] \quad \ldots(7\cdot9)$$

$$A_3=\left(\frac{1}{96}\right)\left[3\pi a^6-2a^6\sin^{-1}\left(\frac{1}{a}\right)-2(8-50a^2+69a^4)\sqrt{a^2-1}\right]. \quad \ldots(7\cdot10)$$

The values of the constants for the Problem I are:

$I_0=1\cdot500000$	$J_0=0\cdot750000$	$K_0=0\cdot750000$
$I_1=-1\cdot250000$	$J_1=-0\cdot625000$	$K_1=-0\cdot625000$
$I_2=0\cdot656250$	$J_2=0\cdot328125$	

(I) *Normal component of stress*

The left-hand side of (6·3) is the value of $\sigma_{zz}(r, 0)$. Making use of (6·3) and (6·6), we get

$$\left[\sigma_{zz}(r,0)\right]_{0<r<1} = \frac{-n(1)}{\sqrt{1-r^2}} - \int_1^\infty \frac{n(t)}{\sqrt{t^2-r^2}}\,dt$$

$$+\frac{2}{\pi h^2}\int_r^\infty \frac{dt}{\sqrt{t^2-r^2}}\int_0^\infty n(x)\,w\,I(w)\,\sin\frac{wx}{h}\cos\frac{wt}{h}\,dw\,dx$$

$$+\frac{2}{\pi h^2}\int_r^\infty \frac{dt}{\sqrt{t^2-r^2}}\int_0^\infty m(x)w\,J(w)\sin\frac{wt}{h}\sin\frac{wx}{h}\,dw\,dx.$$

Putting the values of $n(t)$ and $m(t)$ and after simplification we get

$$\frac{\pi}{2K}\left[\sigma_z(r,0)\right]_{0<r<1} = -\frac{a_0+a_1+a_2+\sqrt{a^2-1}}{\sqrt{1-r^2}} + \sin^{-1}\left(\frac{a^2-1}{a^2-r^2}\right)^{\frac{1}{2}}$$

$$+\frac{1}{h^3}\left[I_1 D + \frac{2I_2}{h^2}\left\{\frac{2}{3}(4-r^2)D + 6E\right\}\right]\sqrt{1-r^2}$$

$$+\frac{1}{h^4}\left[3J_1F + \frac{10}{h^2}J_2\left\{G + \frac{1}{3}F(4-r^2)\right\}\right]\sqrt{1-r^2} + O(h^{-6}) \ldots (7.11$$

where

$$D = A_1 + (a-1)a_0 + \frac{1}{3}(a^3-1)a_1 + \frac{1}{5}(a^5-1)a_2$$

$$E = A_2 + \frac{1}{3}(a^3-1)a_0 + \frac{1}{5}(a^5-1)a_1 + \frac{1}{7}(a^7-1)a_2$$

$$F = \frac{1}{2}(a^2-1)b_0 + \frac{1}{4}(a^4-1)b_1 + \frac{1}{6}(a^6-1)b_2$$

$$G = \frac{1}{4}(a^4-1)b_0 + \frac{1}{6}(a^6-1)b_1 + \frac{1}{8}(a^8-1)b_2.$$

(II) *Shearing Stress*

The left-hand side of (6.4) is the value of $\tau_{rz}(r, 0)$. Putting the values of $M(\xi)$, $N(\xi)$ and in turn that of $n(t)$, $m(t)$ and after calculation we get

$$\frac{\pi}{2K}\left[\tau_{rz}(r,0)\right]_{0<r<1} = \frac{b_0+b_1+b_2}{\sqrt{1-r^2}} - \frac{1}{h^3}\left[\frac{3K_0 r}{\sqrt{1-r^2}}\right]$$

$$-\frac{K_1 r}{\sqrt{1-r^2}\,h^5}\left[I - \frac{r^2}{2}\quad H\left(\frac{\sqrt{1-r^2}}{r^2} + \log\{1+\sqrt{r^2-1}\}\right)\right]$$

$$+\frac{1}{h^2}\frac{d}{dr}\left[L_0 D + \frac{2L_1}{h^2}\left\{\frac{2}{3}(4-r^2)D + 6E\right\}\right]\sqrt{1-r^2} + O(h^{-6})$$

where

$$H = \frac{1}{3}(a^3 - 1)\, b_0 + \frac{1}{5}(a^5 - 1)\, b_1 + \frac{1}{7}(a^7 - 1)\, b_2$$

$$I = \frac{1}{5}(a^5 - 1)\, b_0 + \frac{1}{7}(a^7 - 1)\, b_1 + \frac{1}{9}(a^9 - 1)\, b_2. \qquad ...(7.12)$$

(III) *Stress Intensity Factors*

Expressions for stress intensity factors are of great importance for workers in fracture mechanics. These expressions are defined by

$$\rho = \lim_{r \to 1+} (1 - r)^{\frac{1}{2}} [\{\sigma_{zz}(r, 0)\}, r < 1] \qquad ...(7{\cdot}13)$$

$$\tau = \lim_{r \to 1+} (1 - r)^{\frac{1}{2}} [\{\tau_{rz}(r, 0)\}, r < 1]. \qquad ...(7{\cdot}14)$$

By substituting the values of $\sigma_z(r, 0)$ and $\tau_{rz}(r, 0)$ in the above equations, one can easily demonstrate that

$$\left.\begin{aligned} \rho &= \frac{-\mu}{2V_2(1-\eta)}\, n(1) + O(1) \\ \tau &= \frac{-\mu}{2V_2(1-\eta)}\, m(1) + O(1) \end{aligned}\right\} \qquad ...(7{\cdot}15)$$

$$\left.\begin{aligned} \rho &= \frac{-2}{\pi}\, p\, [\sqrt{a^2 - 1} + a_0 + a_1 + a_2] + O(h^{-7}) \\ \tau &= \frac{2}{\pi}\, p\, [b_0 + b_1 + b_2] + O(h^{-7}). \end{aligned}\right\} \qquad ...(7{\cdot}16)$$

From the above expressions the stress intensity factors can be calculated for both the problems.

(IV) *Normal Component of Displacement*

From the results given in Section 5 it is easy to derive that

$$2(1 - \eta)\, \omega(r, 0) = \int_0^\infty [N(1 - \eta) + \{(1 - \eta)(1 + 2\xi h) + \xi^2 h^2\} e^{-2\xi h})$$

$$+ M(\tfrac{1}{2} - \eta - \{(\tfrac{1}{2} - \eta)(1 - 2\xi h) - \xi^2 h^2\} e^{-2\xi h}]\, \xi J_0(\xi r)\, d\xi. \qquad ...(7{\cdot}17)$$

Substituting the values of M and N from (6·5) and (6·6) for $\eta = \frac{1}{3}$ we have

$$8\omega(r, 0) = 4\int_1^r \frac{n(t)\, dt}{\sqrt{r^2 - t^2}}$$

$$+ \int_0^\infty (6\mu^2 + 2\mu - 1)\, e^{-2\mu}\, J_0(\xi r) \sin \xi t\, d\xi$$

$$+ \int_1^\infty n(t)\, dt \int_0^\infty (6\mu^2 + 4\mu + 2)\, e^{-2}\quad J_0(\xi r) \cos \xi t\, d\xi \qquad ...(7{\cdot}18)$$

for $r > a$

$$\frac{8\pi}{K}\,\omega(r,0) = \int_1^a \left[\sqrt{\left(\frac{a^2-t^2}{r^2-t^2}\right)} + \frac{a_0 + a_1 t^2 + a_2 t^4}{\sqrt{r^2-t^2}}\right] dt$$

$$+ \frac{1}{h}\int_1^a m(t) \int_0^\infty (6\mu^2 + 2\mu - 1)\, e^{-2\mu}\, J_0\left(\frac{\mu r}{h}\right) \sin \frac{\mu t}{h}\, d\mu\, dt$$

$$+ \frac{1}{h}\int_1^a n(t) \int_0^\infty (6\mu^2 + 4\mu + 2)\, e^{-2\mu}\, J_0\left(\frac{\mu r}{h}\right) \cos \frac{\mu t}{h}\, d\mu\, dt$$

$$= \frac{3}{2} L_1 L_2 + \frac{3}{2} A_1 - \frac{5}{4h^2}\left(r^2 A_1 + 2A_2 + r^2 L + 2M\right)$$

$$+ \frac{19}{1536h^4}\left(3r^4 L + 24r^2 M + 8N\right) + O(h^{-6}) \qquad \ldots(7\cdot19)$$

and for $1 < r < a$

$$\frac{8\pi}{K}\,\omega(r,0) = \int_1^r \left[\sqrt{\left(\frac{a^2-t^2}{r^2-t^2}\right)} + \frac{a_0 + a_1 t^2 + a_2 t^4}{\sqrt{r^2-t^2}}\right] dt$$

$$+ \frac{1}{h}\int_1^r m(t) \int_0^\infty (6\mu^2 + 2\mu - 1)\, e^{-2\mu}\, J_0\left(\frac{\mu r}{h}\right) \sin\left(\frac{\mu t}{h}\right) d\mu\, dt$$

$$+ \frac{1}{h}\int_1^r n(t) \int_0^\infty (6\mu^2 + 4\mu + 2)\, e^{-2\mu}\, J_0\left(\frac{\mu r}{h}\right) \cos\left(\frac{\mu t}{h}\right) d\mu\, dt$$

$$= \frac{3}{2} L_1' L_2' + \frac{3}{2} A_1 - \frac{5}{4h^2}\left(r^2 A_1 + 2A_2 + r^2 L' + 2M'\right)$$

$$+ \frac{19}{1536h^4}\left(3r^4 L' + 24r^2 M' + 8N'\right) + O(h^{-6}) \qquad \ldots(7\cdot20)$$

where

$$L_1 = a_0 + \frac{1}{2} a_1 r^2 + \frac{3}{8} a_2 r^4$$

$$L_2 = \sin^{-1}\left(\frac{a}{r}\right) - \sin^{-1}\left(\frac{1}{r}\right)$$

$$L_3 = a\sqrt{r^2-a^2} - \sqrt{r^2-1}$$

$$L_4 = \sqrt{r^2 - a^2}\,(r^2 - 2a^2) - \sqrt{r^2 - 1}\,(r^2 - 2)$$

$$L_5 = \sqrt{r^2 - a^2}\,\{a^2(r^2 - a^2)^2 - 20a^3(r^2 - a^2) + 6a^5\} - \sqrt{r^2 - 1}\,\{(r^2 - 1)^2 - 20\,(r^2 - 1) + 6\}$$

$$L_6 = [a\,(r^2 - 2a^2)\sqrt{r^2 - a^2}\,\{r^4 - 2\,(r^2 - 2a^2)^2\} - (r^2 - 2)\sqrt{r^2 - 1} \times \{r^4 - 2\,(r^2 - 2)^2\}]$$

$$L = L_1 L_2 + \frac{3}{4}(a_1 + r^2 a_2) L_3 + \frac{1}{8} a_2 L_4$$

$$M = a_0\left(\frac{1}{2} r^2 L_2 + L_3\right) + a_1\left(\frac{3}{8} r^4 L_2 + \frac{1}{2} r^2 L_3 + \frac{1}{8} L_4\right) + N_1$$

$$N = a_0 L_3 + a_1 N_1 + a_2 N_2$$

$$N_1 = \frac{1}{32}\left[10 r^6 L_2 - 15 r^4 L_3 + \frac{3}{16} r^2 L_4 - \frac{1}{6} L_5\right]$$

$$N_2 = \frac{1}{128}\left[35 r^8 L_2 - 56 r^6 L_3 + 28 r^4 L_4 - \frac{2}{9} r^2 L_5 + L_6\right]$$

$$L_1' = a_0 + \frac{1}{2} a_1 r^2 + \frac{3}{8} a_2 r^4$$

$$L_2' = \frac{1}{2}\pi - \sin^{-1}\left(\frac{1}{r}\right)$$

$$L' = L_1' L_2' - \frac{3}{4}(a_1 + a_2 r^2)\sqrt{r^2 - 1} - \frac{1}{8} a_2 \sqrt{r^2 - 1}\,(r^2 - 2)$$

$$M' = a_0\left[\frac{1}{2} r^2 L'_2 - \sqrt{r^2 - 1}\right] + a_1\left[\frac{3}{8} r^4 L_2' - \frac{1}{2} r^2 \sqrt{r^2 - 1} - \frac{1}{8}\sqrt{r^2 - 1}\,(r^2 - 2)\right]$$

$$N' = -a_0\sqrt{r^2 - 1} + a_1 N_1' + a_2 N_2'$$

$$N_1' = \frac{1}{32}\left[10 r^6 L_2' + 15 r^4 \sqrt{r^2 - 1} - \frac{3}{16} r^2 (r^2 - 2)\sqrt{r^2 - 1} - \frac{1}{6}\sqrt{r^2 - 1}\left((r^2 - 1)^2 - 20\,(r^2 - 1) + 6\right)\right]$$

$$N_2' = \frac{1}{128}\left[35 r^8 L_2' + 56 r^6 \sqrt{r^2 - 1} - 28 r^4 (r^2 - 2)\sqrt{r^2 - 1} + \frac{2}{9} r^2 \times \left((r^2 - 1)^2 - 20\,(r^2 - 1) + 6\right) - (r^2 - 2)\sqrt{r^2 - 1}\,\{r^4 - 2\,(r^2 - 2)^2\}\right].$$

(V) *Crack Energy*

Another quantity of physical interest is the crack energy W. This is given by

$$W = -2\pi \int_{1}^{\infty} r\, \sigma_{zz}(r, 0)\, \omega(r, 0)\, dr. \qquad \ldots(7{\cdot}21)$$

In the present case $\sigma_{zz}(r, 0) = p$. Hence

$$W = -\frac{\pi p K}{1-\eta}\Bigg[a\int_{1}^{a} m(t) \int_{0}^{\infty} \xi^{-1} \Big[\tfrac{1}{2} - \eta - \{(\tfrac{1}{2} - \eta)(1 - 2\xi h) - \xi^2 h^2\} e^{-2\xi h}$$

$$\times J_1(a\xi) \sin \xi t \, d\xi dt + a\int_{1}^{a} n(t) \int_{0}^{\infty} \xi^{-1} \Big[1 - \eta + \{(1-\eta)(1+2\xi h) + \xi^2 h^2\} e^{-2\xi h}$$

$$\times J_1(a\xi) \cos \xi t \, d\xi dt - \int_{1}^{a} m(t) \int_{0}^{\infty} \xi^{-1} \Big[\tfrac{1}{2} - \eta - \{(\tfrac{1}{2} - \eta)(1 - 2\xi h) - \xi^2 h^2\} e^{-2\xi h}$$

$$\times J_1(\xi) \sin \xi t \, d\xi dt - \int_{1}^{a} n(t) \int_{0}^{\infty} \xi^{-1} \Big[1 - \eta + \{(1-\eta)(1+2\xi h) + \xi^2 h^2\} e^{-2\xi h}$$

$$\times J_1(\xi) \cos \xi t \, d\xi dt \Bigg]. \qquad \ldots(7{\cdot}22)$$

For $\eta = \frac{1}{3}$, we have

$$W = \frac{8(1-\eta^2) p^2}{3E}\, \omega\left(\frac{1}{h}\right) \qquad \ldots(7{\cdot}23)$$

where

$$\omega\left(\frac{1}{h}\right) = I_1 + I_2 + I_3 + I_4 - I_5 - I_6 - I_7 - I_8$$

$$I_1 = -\frac{3}{4} a \int_{1}^{a} m(t) \int_{0}^{\infty} \xi^{-1} J_1(a\xi) \sin \xi t \, d\xi dt$$

$$I_2 = -\frac{3}{4} a \int_{1}^{a} m(t) \int_{0}^{\infty} \xi^{-1} (6\xi^2 h^2 + 2\xi h - 1) e^{-2\xi h} J_1(a\xi) \sin \xi t \, d\xi dt$$

$$I_3 = -\frac{3}{2} a \int_{1}^{a} n(t) \int_{0}^{\infty} \xi^{-1} J_1(a\xi) \cos \xi t \, d\xi dt$$

$$I_4 = -\frac{3}{2} a \int_{1}^{a} n(t) \int_{0}^{\infty} \xi^{-1} (3\xi^2 h^2 + 4\xi h + 2) e^{-2\xi h} J_1(a\xi) \cos \xi t \, d\xi dt.$$

The integrals I_5–I_8 can be obtained by putting $a = 1$ in the above integrals.

For the numerical computation of the inner integrals in I_1–I_4, we shall use the following results given in Sneddon (1951).

$$\int_0^\infty \xi^{-1} \sin \xi t \, J_1 (\xi a) \, d\xi = a / [t + (t^2 - a^2)^{\frac{1}{2}}], \quad t > a$$

$$= t / a, \; t < a$$

$$\int_0^\infty \xi^{-1} e^{-2\xi h} J_1 (a \xi) \sin \xi t \, d\xi = \frac{1}{a} \left(t - R \sin \frac{\phi}{2} \right)$$

$$\int_0^\infty e^{-2\xi h} J_1 (a \xi) \sin \xi t \, d\xi = \frac{1}{aR} \left[t \cos \frac{\phi}{2} - 2h \sin \frac{\phi}{2} \right]$$

$$\int_0^\infty \xi J_1 (a \xi) \sin \xi t \, e^{-2\xi h} \, d\xi = \frac{a}{R^3} \sin \frac{3\phi}{2}$$

$$\int_0^\infty \xi^{-1} e^{-2\xi h} J_1 (a \xi) \cos \xi t \, d\xi = \frac{1}{a} (R \cos \tfrac{1}{2} \phi - 2h)$$

$$\int_0^\infty e^{-2\xi h} J_1 (a \xi) \cos \xi t \, d\xi = \frac{1}{a} - \frac{1}{aR} \left(t \sin \frac{\phi}{2} - 2h \cos \frac{\phi}{2} \right)$$

$$\int_0^\infty \xi e^{-2\xi h} J_1 (a \xi) \cos \xi t \, d\xi = \frac{a}{R^3} \cos \frac{3\phi}{2}$$

where

$$R^4 = (a^2 + 4h^2 - t^2)^2 + 16a^2h^2t^2$$

$$a^2 + 4h^2 - t^2 = R^2 \cos \phi$$

$$4ath = R^2 \sin \phi, \tan \phi = \frac{4ath}{a^2 + 4h^2 - t^2} \cdot$$

8. Numerical Solution of Integral Equations

The formulae derived in the last section are of value only if $1/h$ is very small. When it is only slightly less than unity the simultaneous Fredholm integral equations derived in Section 7 have to be solved numerically.

Computations were carried out for values of $h = 1{\cdot}05$, $1{\cdot}1$, $1{\cdot}20$, $1{\cdot}30$, $1{\cdot}6667$ and $a = 1{\cdot}20$, $1{\cdot}60$ and $2{\cdot}0$. The kernels are computed using Gauss-Laguerre quadrature formula. The integral equations were solved using the method of Fox and Goodwin

(1953). Using these values of $m(t)$ and $n(t)$, we can calculate the normal component of displacement. The variation of w with h and a are shown graphically in Figs. 2–4.

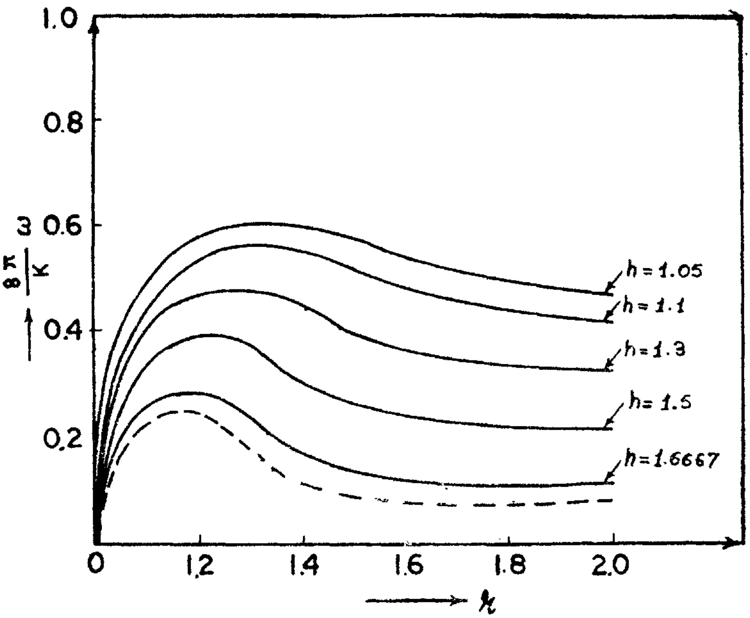

FIG. 2. Variation of $\omega(r, 0)$ with r and h for $a = 1{\cdot}2$

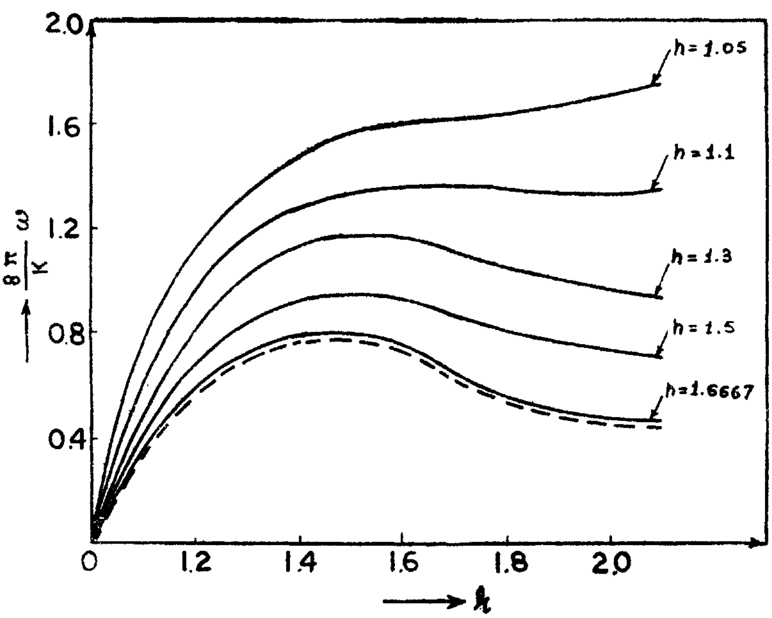

FIG. 3. Variation of $\omega(r, 0)$ with r and h for $a = 1{\cdot}6$.

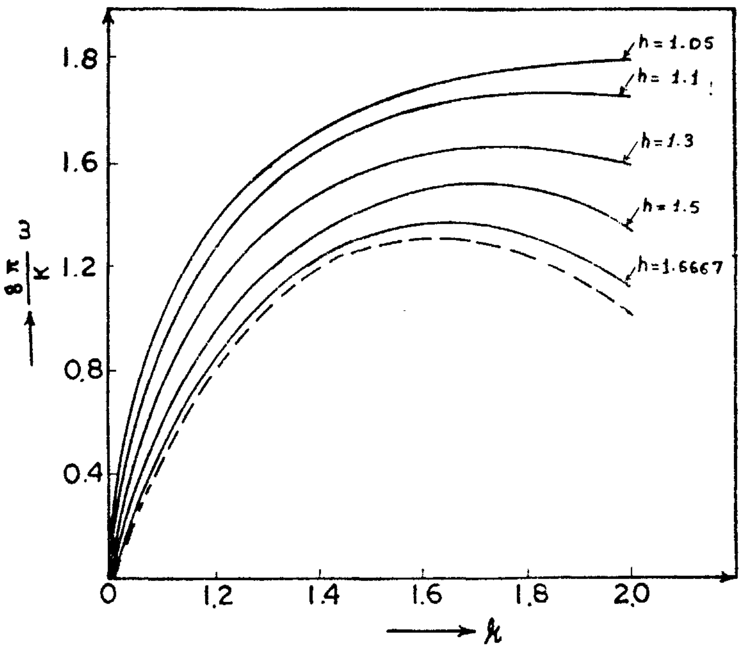

FIG. 4. Variation of $\omega(r, 0)$ with r and h for $a = 2{\cdot}00$.

The results regarding the variation of stress intensity factor are shown in Fig. 5 The results are compared with those of Dhawan (1973). We find the following:

TABLE I

($a = 1{\cdot}6$)

h	1·05	1·1	1·2	1·3	1·5	1·666	2·4	2·4
For crack in plate	7·3743	6·7108	4·4786	3·6980	2·6902	2·1241	1·6404	1·6204
For crack in semi-infinite solid	7·0231	6·3912	4·2653	3·5219	2·5621	2·0239	1·5623	1·5432

TABLE II

($a = 2{\cdot}0$)

h	1·05	1·1	1·2	1·3	1·5	1·6667	2·0	2·4
For crack in plate	8·4677	7·5507	4·8568	4·0575	3·1963	2·6882	2·1535	2·6034
For crack in semi-infinite solid	7·9231	7·0567	4·5391	3·7921	2·9872	2·5123	2·0126	1·8721

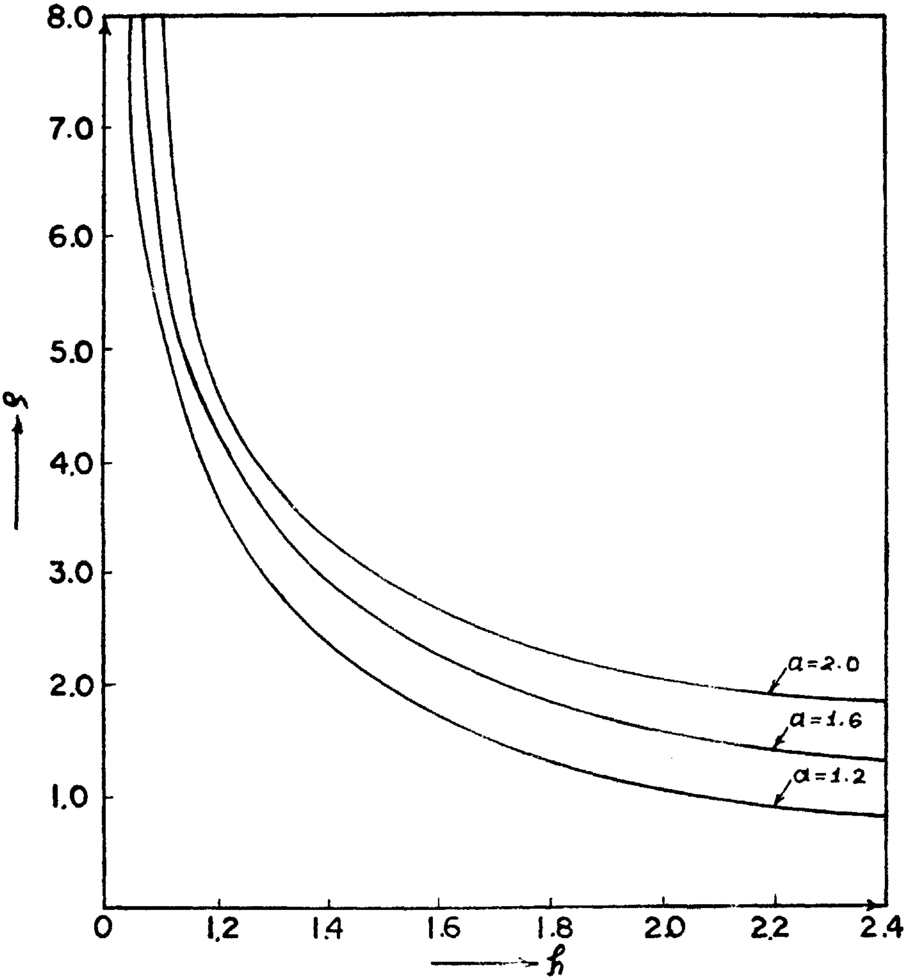

FIG. 5. Variation of normal stress intensity factor with h.

These results seem to indicate that the semi-infinite solid has a little effect on the stress intensity factor. The computations of σ_{zz} employing (7·11) also confirmed the results tabulated above. We found that the magnitude of stress decreases by about 7 per cent for $a = 1{\cdot}6$ and about 5 per cent for $a = 2{\cdot}0$. For values of $a > 4$ the decrease was almost negligible. Further on comparing the results of this paper and that of crack in a plate (for $h = 2{\cdot}5$, $a = 2{\cdot}0$) with those of crack with cylindrical cavity (Srivastava and Lee 1972) and without cavity in an infinite medium are : 1·8723, 2·0034, 1·7562 and 1·7321 respectively.

Using the values of $m(t)$ and $n(t)$ and the result (7·23) we can calculate W. The variation with h and a of W is shown graphically in Fig. 6.

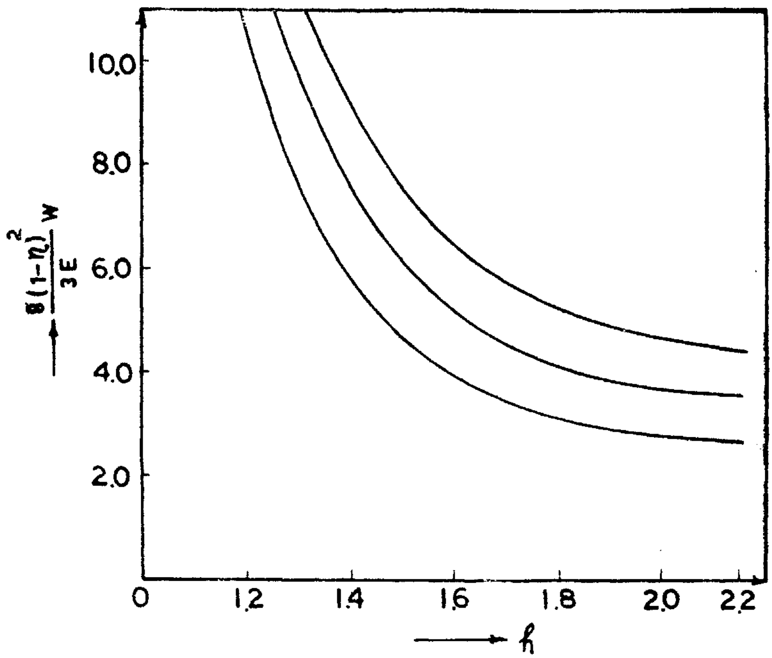

FIG. 6. Variation of crack energy with h and a.

It is of interest to compare the results of this paper with those contained in the paper of Lowengrub and Sneddon (1963) and Dhawan (1973). A comparison of diagrams shows similarity in the results.

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