INTERNAL HEAT GENERATION IN AN INFINITE PLATE WITH A TRANSVERSE CIRCULAR CYLINDRICAL HOLE

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A new finite integral transform is introduced to solve the problem of heat conduction in an infinite plate, with a transverse cylindrical hole, subjected to internal heat generation with radiation boundary conditions. Many known cases can be derived by choosing, (i) appropriate values of the constants involved in the integral transform and (ii) form of heat source.

1. INTRODUCTION

Cases of heat production in solids have led to various technical problems in mechanical applications in which heat produced is rapidly sought to be transferred or dissipated. For instance, gas turbine blades, walls of an I.C. engine, outer surface of a space vehicle etc. all depend for their durability on rapid heat transfer from their surfaces.

The problem of flow of heat from buried pipes and cables, cooling of mines has prompted certain investigators to study heat conduction in regions bounded internally by a circular cylinder. Previous analytical studies include the works of Nicholson (1921), Goldstein (1932), Carslaw and Jaeger (1940) and Blackwell (1953). Recently Michalopoulos and Seco (1973) have studied the heat conduction problem for an infinite plate with a transverse cylindrical hole subjected to axisymmetric temperature distribution on the plane surfaces.

The object of this paper is to study the problem of heat transfer in the above mentioned region when heat is generated in the system and is transferred by radiation from the plane surfaces into a medium at constant temperature which may be taken as the zero on the temperature scale. A particular case and a practical example are also considered.

The problem is solved with the help of a new finite integral transform in conjunction with an unconventional Hankel transform. Finite sine and finite cosine transforms are particular cases of the new transform.

2. Required Integral Transforms

(I) An unconventional Hankel transform of an arbitrary function $f(r)$ as given by Seco (1969) is $H\{f(r)\} \equiv \bar{f}(s) = \int_a^\infty t W_0(s, t, a) f(t)\, dt$ ...(2·1)

with the inverse

$$H^{-1}\{\bar{f}(s)\} \equiv f(r) = \int_a^\infty \frac{s \bar{f}(s)\, W_0(s, r, a)\, ds}{J_0^2(sa) + Y_0^2(sa)} \quad ...(2·2)$$

where

$$W_0(s, r, a) = J_0(sr)\, Y_0(sa) - J_0(sa)\, Y_0(sr) \quad ...(2·3)$$

J_0 and Y_0 being zero order Bessel functions of the first and second kind respectively.

(II) To introduce a new integral transform of an arbitrary function which satisfies Dirichlet's conditions in the intervals under consideration, study the solution of ordinary second order differential equation:

$$\frac{d^2 f}{dz^2} + \lambda^2 f = 0 \quad ...(2·4)$$

under the boundary conditions

$$K_1 \frac{df}{dz} - h_1 f = 0 \quad \text{for } z = 0 \quad ...(2·5)$$

$$K_2 \frac{df}{dz} + h_2 f = 0 \quad \text{for } z = L \quad ...(2·6)$$

K_1, K_2, h_1, h_2, being constants.

The general solution of (2·4), for arbitrary constants A and B, is

$$f(z) = A \cos \lambda z + B \sin \lambda z$$

together with the following relations obtained under (2·5) and (2·6)

$$K_1 \cdot B \lambda - h_1 \cdot A = 0 \quad ...(2·7)$$

and

$$K_2 (-A \lambda \sin \lambda L + BL \cos \lambda L) + h_2 (A \cos \lambda L + B \sin \lambda L) = 0 \quad ...(2·8)$$

Thus the condition for existence of the solutions is

$$(K_1 K_2 \lambda^2 - h_1 h_2) \sin \lambda L = \lambda (K_1 h_2 + K_2 h_1) \cos \lambda L. \quad ...(2·9)$$

Let $\{\lambda_n\}$ be the set of positive roots of this equation, then,

$$f(z) = C \left(K_1 \cos \lambda_n z + \frac{h_1}{\lambda_n} \sin \lambda_n z\right) \quad ...(2·10)$$

where $A \,/\, K_1 = C$.

Assume that $f(z)$ can be developed in an infinite series of the form

$$f(z) = C_1 Z_1 + C_2 Z_2 + C_3 Z_3 + \ldots$$

$$= \sum_{n=1}^{\infty} C_n Z_n \qquad \ldots(2\cdot11)$$

where

$$Z_n = K_1 \cos \lambda_n z + \frac{h_1}{\lambda_n} \sin \lambda_n z. \qquad \ldots(2\cdot12)$$

Normalization gives the value of C_n as

$$C_n = \frac{\hat{f}(n)}{\int_0^L Z_n^2 \, dZ} \qquad \ldots(2\cdot13)$$

where

$$\hat{f}(n) = F\,[f(z)] = \int_0^L f(z) \left(k_1 \cos \lambda_n z + \frac{h_1}{\lambda_n} \sin \lambda_n z\right) dz \qquad \ldots(2\cdot14)$$

defines the finite integral transform to be used in the subsequent problem. Since

$$\frac{d^2 Z_n}{dZ^2} + \lambda_n^2 Z_n = 0$$

we have

$$\lambda_n^2 \int_0^L Z_n^2 \, dZ = - \int_0^L Z_n \frac{d^2 Z_n}{dZ^2} \, dZ$$

$$= - \left[Z_n \frac{dZ_n}{dZ} \right]_0^L + \int_0^L \left(\frac{dZ_n}{dZ} \right)^2 dZ.$$

Now using boundary conditions (2·5), (2·6) and the expression for Z_n, (2·12), we find

$$\left(Z_n \frac{dZ_n}{dZ} \right)_0^L = - \frac{h_2}{K_2} \left[\frac{K_2^2 (K \lambda_n^2 + h^2)}{\lambda_n^2 K_2^2 + h_2^2} \right] - \frac{h_1}{K_1} \cdot K_1^2$$

and

$$\int_0^L \left(\frac{dZ_n}{dZ} \right)^2 dZ = - \lambda_n^2 \int_0^L Z_n^2 \, dZ + [K_1^2 \lambda_n^2 + h_1^2] \, L$$

so that

$$\int_0^L Z_n^2 \, dZ = \frac{\tfrac{1}{2} [(K_1^2 \lambda_n^2 + h_1^2) \{L (K_2^2 \lambda_n^2 + h_2^2) + K_2 h_2\} + K_1 h_1 (K_2^2 \lambda_n^2 + h^2)]}{\lambda_n^2 (K_2^2 \lambda_n^2 + h_2^2)} \qquad \ldots(2\cdot15)$$

$$= \tfrac{1}{2} M_n \text{ (say)}.$$

Thus the inverse of the above transform is given by

$$f(z) = 2 \sum_{n=1}^{\infty} \frac{\hat{f}(n)}{M_n} \left(K_1 \cos \lambda_n z + \frac{h_1}{\lambda_n} \sin \lambda_n z\right). \qquad \ldots(2{\cdot}16)$$

It is easily seen on integration by parts that

$$F\left(\frac{d^2 f}{dz^2}\right) = \frac{1}{K_2}\left[K_1 \cos \lambda_n L + \frac{h_1}{\lambda_n} \sin \lambda_n L\right]\left(K_2 \frac{df}{dz} + h_2 f\right)_{z=L} - \left(K_1 \frac{df}{dz} - h_1 f\right)_{z=0} - \lambda_n^2 \hat{f}(n). \qquad \ldots(2{\cdot}17)$$

The above relation can also be expressed in an alternative form as

$$F\left(\frac{d^2 f}{dz^2}\right) = \frac{\lambda_n}{h_2}\left[K_1 \sin \lambda_n L - \frac{h_1}{\lambda_n} \cos \lambda_n L\right]\left(K_2 \frac{df}{dz} + h_2 f\right)_{z=L} - \left(K_1 \frac{df}{dz} - h_1 f\right)_{z=0} - \lambda_n^2 \hat{f}(n). \qquad \ldots(2{\cdot}18)$$

Specializing the parameters, the above transform easily yields the finite sine and finite cosine transforms.

3. Formulation of the Problem

Consider an infinite plate of thickness L with a transverse circular cylindrical hole of radius 'a' and define a cylindrical system of coordinates (r, θ, z) such that z-axis is coincident with the axis of the hole. Consider the diffusion of heat in the said region when there are sources of heat within it which lead to an axially symmetric temperature distribution. Assume that the rate of generation of heat is independent of temperature. If we set

$T = T(r, z, t)$, the temperature distribution in the plate

$\phi = \phi(r, z, t)$, internal heat source

K = thermal conductivity of the material

α = thermal diffusivity

H_s = Surface conductance

t = time

$h = H_s/K$

the fundamental equation governing the flow of heat is of the form

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{\phi}{K} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \ldots(3{\cdot}1)$$

with boundary and initial conditions which we take as

$$\left.\begin{aligned} &\text{(i)} \quad T(a, z, t) = 0 \\ &\text{(ii)} \quad \lim_{r\to\infty} T(r, z, t) = 0 \\ &\text{(iii)} \quad \frac{\partial T}{\partial z} - hT = 0, \text{ at } z = 0, \quad t > 0, a < r < \infty \\ &\text{(iv)} \quad \frac{\partial T}{\partial z} + hT = 0, \text{ at } z = L, t > 0, a < r < \infty \\ &\text{(v)} \quad T(r, z, 0) = 0. \end{aligned}\right\} \qquad \ldots(3\cdot2)$$

4. Solution of the Problem

The transform (2·1) with the help of (i) and (ii) of (3·2) transforms the differential equation (3·1) to

$$-S^2 \bar{T}(s, z, t) + \frac{\partial^2 \bar{T}}{\partial z^2}(s, z, t) + \frac{\bar{\phi}(s, z, t)}{K} = \frac{1}{\alpha} \frac{\partial \bar{T}(s, z, t)}{\partial t}. \qquad \ldots(4\cdot1)$$

Further, the transformed conditions are

$$\left.\begin{aligned} \frac{\partial \bar{T}}{\partial z} - h\bar{T} &= 0, \quad \text{for } z = 0 \\ \frac{\partial \bar{T}}{\partial z} + h\bar{T} &= 0, \quad \text{for } z = L \end{aligned}\right\} \qquad \ldots(4\cdot2)$$

and $\bar{T}(s, z, 0) = 0.$ ...(4·3)

Now employing the new transform (2·14) with $K_1 = K_2 = 1$, and $h_1 = h_2 = h$, we get on using the boundary conditions (4·2)

$$\frac{d\hat{\bar{T}}}{dt} + \alpha (s^2 + \lambda_n^2)\hat{\bar{T}} = \frac{\alpha}{K}\hat{\bar{\phi}} \qquad \ldots(4\cdot4)$$

which gives

$$\hat{\bar{T}} = \frac{\alpha}{K} e^{-\beta t}\int_0^t \hat{\bar{\phi}}(s, n, \tau)\, e^{\beta\tau}\, d\tau \qquad \ldots(4\cdot5)$$

where

$$\beta = \alpha(s^2 + \lambda_n^2). \qquad \ldots(4\cdot6)$$

Inverting (4·5) first by (2·16) and then by (2·2), we get the solution as

$$T(r, z, t) = \frac{2\alpha}{K}\int_0^\infty \left[\left\{\sum_{n=1}^{\infty} \frac{e^{-\beta t}}{M_n}\int_0^t \hat{\bar{\phi}}(s, n, \tau)\, e^{\beta\tau}\, d\tau\right\} \times \left(\cos \lambda_n z + \frac{h}{\lambda_n}\sin \lambda_n z\right)\right] \frac{S W_0(s, r, a)\, ds}{J_0^2(sa) + Y_0^2(sa)} \qquad \ldots(4\cdot7)$$

where M_n is given by (2·15) with the prescribed values of the constants involved.

5. A Particular Case

Suppose $\phi\,(r, z, t)$ can be expressed as the product of space dependent and time dependent functions, then

$$\phi\,(r, z, t) = \phi_1(r) \cdot \phi_2(z) \cdot \phi_3(t) \quad \ldots(5\cdot 1)$$

and we easily see that

$$\bar{\phi}\,(s, z, t) = \bar{\phi}_1(s) \cdot \phi_2(z) \cdot \phi_3(t)$$

also

$$\hat{\bar{\phi}}\,(s, n, t) = \bar{\phi}_1(s) \cdot \hat{\phi}_2(n) \cdot \phi_3(t).$$

The solution (4·7) then takes the form

$$T\,(r, z, t) = \frac{2\alpha}{K} \int_0^\infty \bar{\phi}_1(s) \left[\left\{ \sum_{n=1}^{\infty} \frac{\hat{\phi}_2(n)}{M_n} e^{-\beta t} \int_0^t \phi_3(\tau)\, e^{\beta\tau}\, d\tau \right\} \left(\cos \lambda_n z + \frac{h}{\lambda_n} \sin \lambda_n z\right)\right] \frac{S\, W_0(s, r, a)\, ds}{J_0^2(sa) + Y_0^2(sa)}. \quad \ldots(5\cdot 2)$$

Example—Let

$$\phi_1(r) = \begin{cases} T_0 & a \leqslant r \leqslant b \\ 0, & b < r \end{cases}$$ which defines annular region surrounding the hole, b being *a* constant.

$\phi_2(z) = e^{-\mu z}$, $\mu > 0$ and constant

$\phi_3(t) = \phi_0$, a constant

Then

$$\bar{\phi}_1(s) = \int_a^b r\, W_0(s, r, a)\, T_0\, dr$$

$$= T_0 \left[\frac{b\, W_1(s, b, a)}{s} - \frac{2}{\pi\, s^2}\right]$$

$$= Y_1 \quad \text{(say)}$$ [see Michalopoulos and Seco (1973)]

$$\hat{\phi}_2(n) = \int_0^L e^{-\mu z}\left(\cos \lambda_n z + \frac{h}{\lambda_n} \sin \lambda_n z\right) dz$$

$$= \frac{e^{-\mu L}}{\lambda_n^2 + \mu^2}\left[\frac{(\lambda_n^2 - \mu h)}{\lambda_n} \sin \lambda_n L - (\mu + h)(\cos \lambda_n L - e^{\mu L})\right]$$

$$= Y_2 \quad \text{(say)}$$

and

$$\int_0^t \phi_0\, e^{\beta\tau}\, d\tau = \frac{\phi_0}{\beta}\,(e^{\beta t} - 1).$$

Substituting the above values in (5·2), the temperature distribution in the plate is found to be

$$T(r, z, t) = \frac{2\alpha}{K} \int_0^\infty Y_1 \left\{ \sum_{n=1}^{\infty} \frac{Y_2 \phi_0}{M_n \beta} (1 - e^{-\beta t}) \left(\cos \lambda_n z + \frac{h}{\lambda_n} \sin \lambda_n z\right) \right\} \times \frac{S\, W_0(s, r, a)\, ds}{J^2(sa) + Y_0^2(sa)}. \quad \ldots(5\cdot3)$$

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REFERENCES

Blackwell, J. H. (1953). Radial-axial heat flow in regions bounded internally by circular cylinders. *Can. J. Phys.*, **31**, 472.

Carslaw, H. S., and Jaeger, J. C. (1940). Some two-dimensional problems in conduction of heat with circular symmetry. *Proc. Lond. math. Soc.*, **46**, series 2, 361.

Goldstein, S. (1932). Some two-dimensional problems with circular symmetry. *Proc. Lond. math. Soc.*, **34**, series 2, 51.

Michalopoulos, C. D., and Seco, J. J. (1973). Transient heat conduction in an infinite plate with a transverse circular cylindrical hole. *J. Heat Transfer, Trans. ASME*, **95**, series C, 414.

Nicholson, J. W. (1921). A problem in the theory of heat conduction. *Proc. R. Soc.*, 226.

Seco, J. J. (1969). Use of generalized Hankel transforms in the solution of some axially-symmetric problems in heat conduction. M.S. thesis, University of Houston, Texas.