SURFACE POINT SOURCE IN A GENERALIZED THERMOELASTIC HALF SPACE

by K. S. HARINATH, *Department of Mathematics, Regional Engineering College, Tiruchirapalli* 620015

(*Received* 8 *September* 1975)

In a generalized thermoelastic half-space a surface source produced by a normal force acting at a point is considered. The displacement components are obtained. Also the potential functions and the temperature deviation near the wave front are approximately evaluated.

INTRODUCTION

We consider a generalized thermoelastic half space in which a normal force acts at a point. The displacement components are calculated and some approximations are given by using the method of steepest descent. This paper is an extension of an earlier paper of the author (Harinath 1975) wherein the surface displacements in a thermoelastic half space with a plane faced free surface produced by unit line sources are derived in the appendix.

BASIC EQUATIONS

The basic equations of generalized thermoelasticity are taken from Puri (1973).

If $\vec{D}$ is the displacement vector, λ, μ are Lame's constants, γ is the coefficient of volume expansion, ρ is the density, C_ϵ is the specific heat at constant strain, k is the coefficient of thermal conductivity and τ is the relaxation time, then the basic equations are given by:

$$\left.\begin{aligned} &\mu\nabla^2\vec{D} + (\lambda + \mu)\nabla(\nabla.\vec{D}) = (3\lambda + 2\mu)\gamma\nabla(\theta + \tau\dot{\theta}) + \rho\ddot{\vec{D}} \\ &k\nabla^2\theta = \rho C_\epsilon(\dot{\theta} + \tau\ddot{\theta}) + \gamma(3\lambda + 2\mu)T_0\{\nabla.(\dot{\vec{D}} + \tau\ddot{\vec{D}})\} \end{aligned}\right\} \quad ...(1)$$

where θ is the temperature deviation and T_0 is the equilibrium temperature prior to the appearance of a disturbance.

Introducing the potentials ϕ, ψ by the equation:

$$\vec{D} = \text{grad}\,\phi + \text{curl}\,(0, -\psi_r, 0) \quad ...(2)$$

Present address: Department of Mathematics, Bangalore University, Bangalore 560001.

in cylindrical polar coordinates, we obtain:

$$u = \phi_r + \psi_{rz};\ v = 0;\ w = \phi_z + \psi_{zz} + \frac{\rho\omega^2}{\mu}\psi \qquad \ldots(3)$$

Assuming a simple harmonic time dependence factor exp $(i\omega t)$, the equations (1) yield a set of differential equations for ϕ, ψ, θ.

If $\alpha = \sqrt{(\lambda + 2\mu)/\rho}$ denotes the isothermal wave velocity, $\beta = \sqrt{\mu/\rho}$ denotes the shear wave velocity, $\xi_j^2 = \eta^2 - \zeta_j^2$, Re $(\xi_j) \geqslant 0$ $(j = 1, 2, 3)$, $\zeta_3^2 = \omega^2/\beta^2$; ζ_1^2, ζ_2^2 are the roots of the equation:

$$\zeta^4 - \zeta^2\left\{\frac{\omega^2}{\alpha^2} + \frac{i\omega\rho C_\epsilon\tau'}{k}(1+\varepsilon\,\tau')\right\} + \frac{i\omega^3\rho C_\epsilon\tau'}{k\,\alpha^2} = 0 \qquad \ldots(4)$$

where $\tau' = 1 - i\omega\tau$ and x_T is the isothermal compressibility, then a set of solutions for the potentials ϕ, ψ and the temperature deviation θ is given by the following equations:

$$\left.\begin{aligned}
\phi &= \int_0^\infty \{A(\eta)e^{-\xi_1 z+i\omega t} + B(\eta)e^{-\xi_2 z+i\omega t}\}\,J_0(r\eta)\eta\,d\eta\\
\psi &= \int_0^\infty C(\eta)e^{-\xi_3 z+i\omega t}\,J_0(r\eta)\eta\,d\eta\\
\theta &= \frac{\rho x_T}{\gamma\tau'}\int_0^\infty \{A(\eta)(\omega^2-\alpha^2\zeta_1^2)e^{-\xi_1 z+i\omega t} + B(\eta)(\omega^2-\alpha^2\zeta_2^2)e^{-\xi_2 z+i\omega t}\}J_0(r\eta)\eta\,d\eta.
\end{aligned}\right\} \qquad \ldots(5)$$

The stresses σ_{zz}, σ_{rz} are given by:

$$\left.\begin{aligned}
\sigma_{zz} &= \lambda\nabla^2\phi + 2\mu\frac{\partial w}{\partial z} - \frac{\gamma}{x_T}(\theta + \tau\dot\theta)\\
\sigma_{rz} &= 2\mu\left(2\phi_{zr} - 2\psi_{zzr} - \frac{\omega^2}{\beta^2}\psi_r\right).
\end{aligned}\right\} \qquad \ldots(6)$$

Boundary Conditions

The plane surface $z=0$ is assumed to have only the normal stress and to satisfy a linearized form of the radiative condition.

If $\delta(r)$ denotes the Dirac δ-function, we specify:

$$\sigma_{zz} = \frac{P}{2\pi r}\delta(r);\ \sigma_{rz} = 0;\ -\theta_z + h\theta = 0 \text{ on } z = 0. \qquad \ldots(7)$$

By making use of the Fourier-Bessel integral, equations (5), (6), (7) yield the following three equations which determine the functions $A(\eta)$, $B(\eta)$, $C(\eta)$:

$$\left.\begin{aligned}
&(2\eta^2\beta^2 - \omega^2)\,A(\eta) + (2\eta^2\beta^2 - \omega^2)\,B(\eta) - 2\xi_3\eta^2\beta^2 C(\eta) = \frac{P}{2\pi\rho}\\
&2\xi_1\beta^2\,A(\eta) + 2\xi_2\beta^2 B(\eta) + (2\eta^2\beta^2 - \omega^2)\,C(\eta) = 0\\
&(h + \xi_1)\,(\omega^2 - \alpha^2\zeta_1^2)\,A(\eta) + (h + \xi_2)\,(\omega^2 - \alpha^2\zeta_2^2)\,B(\eta) = 0.
\end{aligned}\right\} \quad \ldots(8)$$

Solving (8) we get

$$\left.\begin{aligned}
A(\eta) &= \frac{P}{2\pi\rho\triangle(\eta)}(2\eta^2\beta^2 - \omega^2)\,(\omega^2 - \alpha^2\zeta_2^2)\,(h + \xi_2)\\
B(\eta) &= \frac{-P}{2\pi\rho\triangle(\eta)}(2\eta^2\beta^2 - \omega^2)\,(\omega^2 - \alpha^2\zeta_1^2)\,(h + \xi_1)\\
C(\eta) &= \frac{-P\beta^2}{\pi\rho\triangle(\eta)}\{h\omega^2(\xi_1 - \xi_2) - h\alpha^2(\xi_1\zeta_2^2 - \xi_2\zeta_1^2) + \xi_1\xi_2\alpha^2(\zeta_1^2 - \zeta_2^2)\}
\end{aligned}\right] \quad \ldots(9)$$

where

$$\triangle(\eta) \equiv \left\{\begin{aligned}
&(2\eta^2\beta^2 - \rho\omega^2)\,(2\eta^2\beta^2 - \omega^2)\;\;[h\alpha^2\,(\zeta_1^2 - \zeta_2^2) - \omega^2\,(\xi_1 - \xi_2)\\
&+ \alpha^2\,(\xi_1\zeta_1^2 - \xi_2\zeta_2^2)] + 4\eta^2\xi_3\beta^4\;[h\omega^2\,(\xi_1 - \xi_2) - h\alpha^2\,(\xi_1\zeta_2^2\\
&- \xi_2\zeta_1^2) + \xi_1\xi_2\alpha^2\,(\zeta_1^2 - \zeta_2^2)]
\end{aligned}\right\} \quad \ldots(10)$$

From equations (9) and (10) we can observe that the functions $A(\eta)$, $B(\eta)$, $C(\eta)$, $\triangle(\eta)$ are symmetric in η.

Substituting equations (9) in (5) we get:

$$\left.\begin{aligned}
\phi &= \frac{P}{2\pi\rho}\int_0^\infty \left\{\begin{matrix}(h + \xi_2)\,(\omega^2 - \alpha^2\zeta_2^2)\,e^{-\xi_1 z + i\omega t}\\ -(h + \xi_1)\,(\omega^2 - \alpha^2\zeta_1^2)\,e^{-\xi_2 z + i\omega t}\end{matrix}\right\}(2\eta^2\beta^2 - \omega^2)\frac{J_0(r\eta)}{\triangle(\eta)}\,\eta\,d\eta\\
\psi &= \frac{-P\beta^2}{\pi\rho}\int_0^\infty \left\{\begin{matrix}h\omega^2\,(\xi_1 - \xi_2) - h\alpha^2\,(\xi_1\zeta_2^2 - \xi_2\zeta_1^2)\\ + \xi_1\xi_2\alpha^2\,(\zeta_1^2 - \zeta_2^2)\end{matrix}\right\} e^{-\xi_3 z + i\omega t}\frac{J_0(r\eta)}{\triangle(\eta)}\,\eta\,d\eta\\
\theta &= \frac{Px_r}{2\pi\gamma\tau'}\int_0^\infty (\omega^2 - \alpha^2\zeta_1^2)\,(\omega - \alpha^2\zeta_2^2)\left\{\begin{matrix}(h + \xi_2)\,e^{-\xi_1 z + i\omega t}\\ -(h + \xi_1)\,e^{-\xi_2 z + i\omega t}\end{matrix}\right\}\\
&\qquad\qquad \times (2\eta^2\beta^2 - \omega^2)\,\frac{J_0(r\eta)}{\triangle(\eta)}\,\eta\,d\eta.
\end{aligned}\right] \quad \ldots(11)$$

To obtain the approximate values of ϕ, ψ, θ near the wave front, first we have to convert the Bessel functions of the first kind in equations (11) into Hankel functions of the second kind and then use the integral representations for the latter. Hence equations (11) take the form:

$$
\left.
\begin{aligned}
\phi &= \frac{Pi}{4\pi^2 \rho} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty - \pi i} \left\{ \begin{array}{l} (h+\xi_2)(\omega^2 - \alpha^2 \zeta_2^2)\, e^{-\xi_1 z + i\omega t} \\ -(h+\xi_1)(\omega^2 - \alpha^2 \zeta_1^2)\, e^{-\xi_2 z + i\omega t} \end{array} \right\} \\
&\qquad \times (2\eta^2 \beta^2 - \omega^2) \frac{e^{r\eta \sinh x}}{\triangle(\eta)} \cdot \eta\, d\eta\, dx \\
\psi &= -\frac{Pi\,\beta^2}{2\pi^2 \rho} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty - \pi i} \left\{ \begin{array}{l} h\,\omega^2 (\xi_1 - \xi_2) - h\,\alpha^2 (\xi_1 \zeta_2^2 - \xi_2 \zeta_1^2) \\ + \xi_1 \xi_2 \alpha^2 (\zeta_1^2 - \zeta_2^2) \end{array} \right\} \\
&\qquad \times \frac{e^{-\xi_3 z + i\omega t - r\eta \sinh x}}{\triangle(\eta)}\, \eta\, d\eta\, dx \\
\theta &= -\frac{Pi\, x_T}{4\pi^2 \gamma\, \tau'} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty - \pi i} (\omega^2 - \alpha^2 \zeta_1^2)(\omega^2 - \alpha^2 \zeta_2^2) \left\{ \begin{array}{l} (h+\xi_2)\, e^{-\xi_1 z + i\omega t} \\ -(h+\xi_1)\, e^{-\xi_2 z + i\omega t} \end{array} \right\} \\
&\qquad \times (2\eta^2 \beta^2 - \omega^2) \frac{e^{r\eta \sinh x}}{\triangle(\eta)} \cdot \eta\, d\eta\, dx.
\end{aligned}
\right] \quad \ldots(12)
$$

The integrals in equations (12) may be approximated by using the method of steepest descent near the wave front (*i.e.* for ω large and at a large distance from the source). The saddle points are found to be $\eta_j = r\,\zeta_j/R$, $j = 1, 2, 3$ where $R = (r^2 + z^2)^{\frac{1}{2}}$ is the distance of the point from the source, provided ω, R are large.

Thus for $\omega >> 1$ and $R >> 1$ we have the expressions given by equations (13), which show that as $R \to \infty$, ϕ, ψ, $\theta \to 0$ as required.

For $R >> 1$ and ω large:

$$
\left.
\begin{aligned}
\phi &= -\frac{Piz}{2\pi \rho R^5} \left[\begin{array}{l} \zeta_1 (2\beta^2 \zeta_1^2 r^2 - \omega^2 R^2)(\omega^2 - \alpha^2 \zeta_2^2)(Rh + i\zeta_2 z) \\ \qquad \times \dfrac{\exp(-i\zeta_1 R + i\omega t)}{\triangle(r\zeta_1/R)} \\ -\zeta_2 (2\beta^2 \zeta_2^2 r^2 - \omega^2 R^2)(\omega^2 - \alpha^2 \zeta_1^2)(Rh + i\zeta_1 z) \\ \qquad \times \dfrac{\exp(-i\zeta_2 R + i\omega t)}{\triangle(r\zeta_3/R)} \end{array} \right] \\
\psi &= -\frac{P\beta z^2\, \omega\,(\zeta_1 - \zeta_2)}{\pi \rho R^4 \triangle(r\zeta_3/R)} \{h\,\omega^2 R + h\,\alpha^2 R\,\zeta_1 \zeta_2 + i\,\zeta_1 \zeta_2 (\zeta_1 + \zeta_2)\, z\, \alpha^2\} \\
&\qquad \times \exp(-i\zeta_3 R + i\omega t) \\
\theta &= -\frac{Piz\, x_T}{2\pi \gamma \tau' R^5} (\omega^2 - \alpha^2 \zeta_1^2)(\omega^2 - \alpha^2 \zeta_2^2) \\
&\qquad \times \left[\begin{array}{l} \zeta_1 (2\beta^2 \zeta_1^2 r^2 - \omega^2 R^2)(Rh + i\zeta_2 z) \dfrac{\exp(-i\zeta_1 R + i\omega t)}{\triangle(r\zeta_1/R)} \\ -\zeta_2 (2\beta^2 \zeta_2^2 r^2 - \omega^2 R^2)(Rh + i\zeta_1 z) \dfrac{\exp(-i\zeta_2 R + i\omega t)}{\triangle(r\zeta_2/R)} \end{array} \right]
\end{aligned}
\right] \quad \ldots(13)
$$

In case U_0, W_0 denote the displacements on $z=0$, a ready calculation shows that U_0, W_0 have the forms given by the following equations:

$$\left.\begin{aligned} U_0 &= \frac{P}{2\pi\rho}\int_0^\infty \left\{ \begin{array}{l} (\xi_1-\xi_2)\,[(2\eta^2\beta^2-\omega^2)(\omega^2-\alpha^2\zeta_1^2)-2h\,\xi_3\,\beta^2\omega^2] \\ +2\,\xi_3\,\alpha^2\beta^2\,[h\,(\xi_1\zeta_2^2-\xi_2\zeta_1^2)-\xi_1\xi_2(\zeta_1^2-\zeta_2^2)] \end{array} \right\} \\ &\qquad\qquad \times \frac{J_1(r\eta)}{\triangle(\eta)}\cdot\eta^2\,d\eta \\ W_0 &= \frac{-P}{2\pi\rho}\int_0^\infty \left[\begin{array}{r} h\,\omega^2(\xi_1-\xi_2)-h\,\alpha^2(\xi_1\zeta_2^2-\xi_2\zeta_1^2) \\ +\xi_1\,\xi_2\,\alpha^2(\zeta_1^2-\zeta_2^2) \end{array} \right] (4\eta^2\beta^2-\omega^2) \\ &\qquad\qquad \times \frac{J_0(r\eta)}{\triangle(\eta)}\,\eta\,d\eta \end{aligned}\right] \quad \ldots(14)$$

Equations (14) yield the required surface displacement components.

ACKNOWLEDGEMENT

The author expresses his sincere thanks to Prof. Dr. V. R. Thiruvenkatachar for his initiation and guidance.

REFERENCES

Harinath, K. S. (1975). Rayleigh waves over a thermoelastic halfspace with a slightly curved free surface. *J. Math. Phys. Sci.*, **9**, 93.

Puri, P. (1973). Plane waves in generalized thermoelasticity. *Int. J. Engng. Sci.*, **11**, 735.