STRESS DISTRIBUTION DUE TO A GRIFFITH CRACK AT THE INTERFACE OF AN ELASTIC LAYER BONDED TO A HALF-PLANE

by S. N. SRIVASTAVA, *Department of Mechanical Engineering, Maulana Azad College of Technology, Bhopal*

and

G. K. DHAWAN, *Department of Mathematics, Maulana Azad College of Technology, Bhopal*

(*Received* 8 *September* 1975)

The plane problem for a bonded medium composed of two different materials is considered. It is assumed that there is a flaw at the interface of an elastic layer and a half-plane which may be idealized as a crack. The free edge is assumed to be stress-free. In the case of an axisymmetric loading the problem is first reduced to a system of simultaneous dual integral equations involving trigonometric kernels and then to singular integral equations of the second kind. The singularity of the system is removed and the equations are solved by taking advantage of the fact that the fundamental function of the integral equations is the weight function of Chebyshev polynomial. Numerical results involving the stress-intensity factors and the strain energy release rate are presented.

1. INTRODUCTION

Multilayered bonded plates and shells will perhaps be one of the most common and basic structural materials in the design and construction of aerospace and hydrospace vehicles in the coming decades. The basic appeal of the idea lies in the great flexibility they offer to design engineers. For example, they provide such properties as high modulus, yield and ultimate strength with high toughness through a combination of layers having various properties, directional strengthening of the structure by means of fibre or filament reinforced layers, obtaining a desirable damping characteristic by including a suitably selected viscoelastic layer, etc.

In studying the mechanical response of the layered composites, generally one may differentiate two groups of problems; the first relates to the bulk response of the composite and usually consists of problems concerning the determination of the mechanical properties of and the overall stress distribution in the medium which is assumed to be free from local imperfections. The second group of problems concerns the micro-mechanics of the medium, in which one is particularly interested in the response of the medium in the neighbourhood of localized imperfections. Common form of these imperfections are broken bonds on the interfaces, voids, inclusions and dislocations in the layers. For mathematical convenience, these imperfections may be all classified as singular surfaces across which the displacement or the stress vector suffer a discontinuity. The importance of these problems lines in their application to the fracture of the composites, for it is reasonable to expect (and the practical evidence indicates) that these imperfections will generally form the nucleus of the fracture initiation and propagation in the medium.

In this paper, we will consider the problem for an elastic layer bonded to a half-plane with different mechanical properties having a Griffith crack at the interface. In the formulation of the problem we shall follow the method suggested in the work of Kuz'min and Ufland (1965). The problem is first reduced to a system of simultaneous dual integral equations involving trigonometric kernels and then to singular integral equations. These equations are solved numerically and results for stress intensity factors and the strain energy release rate, in the case when the shearing stress on the surface of the crack is zero and the normal stress is constant, are derived.

The basic mathematical techniques used in the solution of the problem considered in this paper have been described by Muskhelishvili (1953) and Erdogan (1968, 1969). Solutions for some special cases dealing with the plane and anti-plane problems for bonded semi-infinite planes are given by Erdogan (1963, 1965), England (1965) and Rice and Sih (1965).

2. Formulation of the Problem

We consider an elastic layer bonded to a half-plane. The medium contains a crack at the interface and it is assumed that there is symmetry about y-axis. For a symmetric deformation, the displacement vector U be taken to have components $(u^{(1)}, v^{(1)}, 0)$, $(u^{(2)}, v^{(2)}, 0)$ for the respective medium and the components of stress tensor will be represented by $\sigma_{xx}^{(1)}$, $\sigma_{yy}^{(1)}$, $\tau_{xy}^{(1)}$, $\sigma_{yy}^{(2)}$, $\sigma_{xx}^{(2)}$ and $\tau_{xy}^{(2)}$.

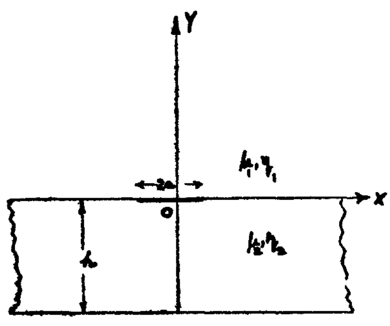

FIG. 1. Griffith crack at the interface parallel to the free boundary.

Let us assume that the body is divided into two domains:

(1) the layer $-h < y < 0$ where h is measured in terms of the distance of the crack from origin which is taken to be the unit of measurement with elastic properties μ_2, η_2 and (2) the half-plane $0 < y < \infty$ with different elastic properties μ_1, η_1.

Let the stresses and components of displacement be denoted by indices 1 and 2 in the two layers.

Boundary Conditions

The free boundary is assumed stress-free and stresses on the surfaces of the crack are prescribed. The conditions can be written as:

$$\left.\begin{aligned} \sigma^{(2)}_{yy}(x, -h) &= 0 \\ \tau^{(2)}_{xy}(x, -h) &= 0 \\ \sigma^{(1)}_{yy}(x, \overset{+}{0}) &= \sigma_1(x) \end{aligned}\right\} \text{ for all } x \qquad \ldots(2{\cdot}1)$$

$$\left.\begin{aligned} \sigma^{(2)}_{yy}(x, \overset{+}{0} &= \sigma_2(x) \\ \tau^{(1)}_{xy}(x, \overset{+}{0}) &= \sigma_1(x) \\ \tau^{(2)}_{xy}(x, \overline{0}) &= \tau_2(x) \end{aligned}\right\} \text{ for } -1 < x < 1. \qquad \ldots(2{\cdot}2)$$

In addition, the continuity conditions on $y = 0$ are

$$\left.\begin{aligned} u^{(1)}(x, 0) &= u^{(2)}(x, 0) \\ v^{(1)}(x, 0) &= v^{(2)}(x, 0) \\ \sigma^{(1)}(x, 0) &= \sigma^{(2)}_{yy}(x, 0) \\ \tau^{(1)}_{xy}(x, 0) &= \tau^{(2)}_{xy}(x, 0). \end{aligned}\right\} \qquad \ldots(2{\cdot}3)$$

3. Equations of Equilibrium for the Elastic Field

The basic equations of equilibrium for an elastic medium in two dimensions in the absence of body forces are:

$$\frac{k_i + 1}{2} \frac{\partial^2 u^{(i)}}{\partial x^2} + \frac{k_i + 1}{2} \frac{\partial^2 u^{(i)}}{\partial y^2} + \frac{\partial^2 u^{(i)}}{\partial x\, \partial y} = 0 \qquad \ldots(3{\cdot}1)$$

$$\frac{k_i - 1}{2} \frac{\partial^2 u^{(i)}}{\partial x^2} + \frac{k_i + 1}{2} \frac{\partial^2 u^{(i)}}{\partial y^2} + \frac{\partial^2 u^{(i)}}{\partial x\, \partial y} = 0 \qquad \ldots(3{\cdot}2)$$

$$\tau^{(i)}_{xy} = \mu_i \left(\frac{\partial u^{(i)}}{\partial y} + \frac{\partial v^{(i)}}{\partial x} \right) \qquad \ldots(3{\cdot}3)$$

$$\sigma^{(i)}_{yy} = \frac{4\mu_i}{k_i - 1} \left\{ \frac{k_i + 1}{4} \frac{\partial u^{(i)}}{\partial y} + \frac{3 - 4k_i}{4} \frac{\partial u^{(i)}}{\partial x} \right\} \qquad \ldots(3{\cdot}4)$$

where $k_i = 3 - 4\eta_i$, $i = 1, 2$ for respective medium and $\mu_i = \dfrac{E_i}{2(1 + \eta_i)}$, η_i is the Poisson's ratio and E_i Young's modulus of the elastic materials.

For the solution of these partial differential equations (3·1) and (3·2), we introduce Fourier sine and cosine transforms of $u^{(i)}(x, y)$ and $v^{(i)}(x, y)$.

We define

$$u^{-(i)}(\xi, y) = \mathcal{J}_s\, [u^{(i)}(x, y),\ x \to \xi]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty u^{(i)}(x, y) \sin(\xi x)\, dx \qquad \ldots(3{\cdot}5)$$

$$\bar{v}^{(i)}(\xi, y) = \mathcal{J}_c\left[v^{(i)}(x, y), x \rightarrow \xi\right]$$
$$= \sqrt{\frac{2}{\pi}} \int_0^\infty v^{(i)}(x, y) \cos(\xi x)\, dx \qquad \ldots(3\cdot 6)$$

Multiply (3·1) by sin (ξx) and (3·2) by cos (ξx) and integrate with respect to x from 0 to infinity to get

$$\left[\frac{k_i - 1}{2} \frac{d^2}{dy^2} - \frac{k_i + 1}{2} \xi^2\right] \bar{u}^{(i)} - \xi \frac{d\bar{v}^{(i)}}{dy} = 0 \qquad \ldots(3\cdot 7)$$

$$\left[\frac{k_i + 1}{2} \frac{d^2}{dy^2} - \frac{k_i - 1}{2} \xi^2\right] \bar{v}^{(i)} - \xi \frac{d\bar{u}^{(i)}}{dy} = 0. \qquad \ldots(3\cdot 8)$$

From (3·3) and (3·4), we obtain

$$\bar{\tau}^{(i)}_{xy} = \mathcal{J}_s\left[\tau^{(i)}_{xy}(x, y), x \rightarrow \xi\right]$$
$$= \mu_i \left[\frac{d\bar{u}^{(i)}}{dy} - \xi\, \bar{v}^{(i)}\right] \qquad \ldots(3\cdot 9)$$

$$\bar{\sigma}^{(i)}_{y} = \mathcal{J}_c\left[\sigma^{(i)}_{yy}(x, y),\ x \rightarrow \xi\right]$$
$$= \frac{4\mu_i}{k_i - 1}\left[\frac{k_i + 1}{4} \frac{d\bar{v}^{(i)}}{dy} + \frac{3 - 4k_i}{4} \xi\, \bar{u}^{(i)}\right]. \qquad \ldots(3\cdot 10)$$

4. Solutions for Semi-infinite Plane

In the case when the medium is assumed free from disturbances at infinity, we are interested in the solutions which tend to zero as $y \rightarrow \infty$. Hence the solutions of (3·7) and (3·8) are

$$\left.\begin{aligned} \bar{u}^{(1)} &= [A(\xi) + B(\xi)\, \xi y]\, e^{-\xi y} \\ \bar{v}^{(1)} &= [A_1(\xi) + B_1(\xi)\, \xi y]\, e^{-\xi y} \end{aligned}\right\} \qquad \ldots(4\cdot 1)$$

where $B(\xi) = B_1(\xi)$, $A_1(\xi) = A(\xi) + k_1 B(\xi)$.

Hence the components of displacement vector and stress tensor are

$$u^{(1)}(x, y) = \mathcal{J}_s\left[\{A(\xi) + B(\xi)\, \xi\, y\}\, e^{-\xi y},\ \xi \rightarrow x\right] \qquad \ldots(4\cdot 2)$$

$$v^{(1)}(x, y) = \mathcal{J}\left[\{A_1(\xi) + B(\xi)\, \xi\, y\}\, e^{-\xi},\ \xi \rightarrow x\right] \qquad \ldots(4\cdot 3)$$

$$\sigma^{(1)}_{yy}(x, y) = 2\mu_1 \mathcal{J}_c\left[\left\{A(\xi) + \frac{k_1 + 1}{2} B(\xi) + B(\xi)\, \xi\, y\right\} e^{-\xi y}\, \xi,\ \xi \rightarrow x\right] \qquad \ldots(4\cdot 4)$$

$$\tau^{(1)}_{xy}(x, y) = -2\mu_1 \mathcal{J}_s\left[\left\{A(\xi) + \frac{k_1 - 1}{2} B(\xi) + B(\xi)\, \xi\, y\right\} e^{-\xi y}\, \xi,\ \xi \rightarrow x\right] \qquad \ldots(4\cdot 5)$$

5. Solutions for the Layer $-h < y < 0$

The appropriate solutions in this case are:

$$\bar{u}^{(2)} = [\{C+D\,\xi\,(y+h)\}\cosh\xi\,(y+h) + \{E+F\,\xi\,(y+h)\}\sinh\xi\,(y+h)]\,/\,\sinh\xi h \quad \ldots(5{\cdot}1)$$

$$\bar{v}^{(2)} = [\{C_1+D_1\,\xi\,(y+h)\}\cosh\xi\,(y+h)+\{E_1+F_1\,\xi\,(y+h)\}\sinh\xi\,(y+h)]\,/\,\sinh\xi h. \quad \ldots(5{\cdot}2)$$

where C, D, E......etc. are functions of ξ and

$$\left.\begin{aligned} D &= -F_1, \quad C + E_1 + k_2 D_1 = 0 \\ D_1 &= -F_1, \quad C_1 + E - k_2 D = 0. \end{aligned}\right\} \quad \ldots(5{\cdot}3)$$

Now when the free boundary is stressfree we have to satisfy the conditions $\overset{(2)}{\sigma}_{yy} = 0$ and $\overset{(2)}{\tau}_{xy} = 0$ for $y = -h$. From (3·9) and (3·10) we have

$$\bar{\sigma}^{(2)}_{xy} = \frac{\mu_2\xi}{\sinh\xi h}\Bigg[\left\{\left(\frac{k_2+1}{4}\right)(C_1 + D_1\xi\,(y+h) + F_1) + \frac{3-k_2}{4}(E + F(y+h)\,\xi)\right\}$$

$$\times \sinh\xi\,(y+h) + \left\{\frac{k_2+1}{4}(E_1 + F_1\xi\,(y+h) + D_1) + \frac{3-k_2}{4}(C + d\xi\,(y+h)\right\}$$

$$\times \cosh\xi\,(y+h)\Bigg]. \quad \ldots(5{\cdot}4)$$

$$\bar{\tau}^{(2)}_{xy} = \frac{\mu_2\xi}{\sinh\xi\,\mathrm{h}}\,[\,\{\,C + D\;\xi\;(y+h) + F - E_1 - F_1\;\xi\,(y+h)\,\}\,\sinh\;\xi\,(y+h)$$

$$+ \{E + F\xi\,(y+h) + D - C_1 - D_1\,\xi\,(y+h)\,\}\cosh\xi\,(y+h)\,]. \quad \ldots(5{\cdot}5)$$

These equations imply that

$$E + D - C_1 = 0 \quad \ldots(5{\cdot}6)$$

$$\frac{k_2+1}{4}\,(E_1 + D_1) + \frac{3-k_2}{4}\,C = 0. \quad \ldots(5{\cdot}7)$$

From (5·3), (5·6) and (5·7) we have

$$\left.\begin{aligned} E &= \frac{k_2-1}{2}\,D, \quad C_1 = \frac{k_2+1}{2}\,D \\ E_1 &= \frac{-k_2-1}{2}\,D_1, \quad C = \frac{-k_2+1}{2}\,D. \end{aligned}\right\} \quad \ldots(5{\cdot}8)$$

Hence the expressions for the components of displacement vector and stress tensor are:

$$\overset{(2)}{u}(x, y) = \mathcal{F}_s\Big[\Big(\Big\{ D\,\xi\,(y+h) - \frac{k_2+1}{2} D_1 \Big\} \cosh \xi\,(y+h) + \Big\{ \frac{k_2-1}{2} D - D_1\,\xi\,(y+h) \Big\} \sinh \xi\,(y+h) \Big) \Big/ \sinh \xi h;\ \xi \to x \Big] \quad \ldots(5{\cdot}9)$$

$$\overset{(2)}{v}(x, y) = \mathcal{F}_c\Big[\Big(\Big\{ \frac{k_2+1}{2} D + D_1\,\xi\,(y+h) \Big\} \cosh \xi\,(y+h) - \Big\{ \frac{k_2-1}{2} D_1 + D\,\xi\,(y+h) \Big\} \sinh \xi\,(y+h) / \sinh \xi h;\ \xi \to x \Big] \quad \ldots(5{\cdot}10)$$

$$\sigma_{yy}^{(2)}(x, y) = 2\,\mu_2\,\mathcal{F}_c\,[(\{ D + D_1\xi\,(y+h)\} \sinh (y+h) - D\,\xi\,(y+h) \cosh \xi\,(y+h))/ \sinh \xi h;\ \xi \to x] \quad \ldots(5{\cdot}11)$$

$$\tau_{xy}^{(2)}(x, y) = -2\,\mu_2\,\mathcal{F}_s\,[(\{ D_1 - D\,\xi\,(y+h)\} \sinh \xi\,(y+h) + D_1\,\xi\,(y+h) \times \cosh \xi\,(y+h))/ \sinh h;\ \xi \to x]. \quad \ldots(5{\cdot}12)$$

6. Reduction of the Problem to a System of Simultaneous Dual Integral Equations

We still have to satisfy the boundary conditions for $y = 0$. For the sake of simplicity we assume that

$$\sigma_1(x) = \sigma_2(x) = -p(x)$$

$$\tau_1(x) = \tau_2(x) = \tau_1(x).$$

Hence for $0 < x < 1$, we have

$$\mu_2 \int_0^\infty [D_1\,\xi h + D - D\xi h \coth \xi h]\,\xi \cos (\xi x)\, d\xi$$

$$= -\mu_1 \int_0^\infty \Big[A + \frac{k_1+1}{2} B \Big] \xi \cos (\xi x)\, d\xi$$

$$= -\tfrac{1}{4}\pi P(x) \quad \ldots(6{\cdot}1)$$

$$\mu_2 \int_0^\infty [D_1 - D\xi h + D_1 \xi h \coth \xi h]\,\xi \sin (\xi x)\, d\xi$$

$$= \mu_1 \int_0^\infty \Big[A + \frac{k_1-1}{2} \beta \Big] \xi \sin (\xi x)\, d\xi$$

$$= -\tfrac{1}{4}\pi T(x) \quad \ldots(6{\cdot}2)$$

where

$$P(x) = \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} p(x), \quad T(x) = \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} \tau(x)$$

and for $1 < x < \infty$, we have

$$\int_0^\infty \left[\mu_2 \left\{ D_1 \xi h + D - D \xi h \coth \xi h \right\} + \mu_1 \left\{ A + \frac{k_1+1}{2} B \right\} \right] \times \xi \cos (\xi x) \, d\xi = 0 \qquad \ldots(6{\cdot}3)$$

$$\int_0^\infty \left[\mu_2 \left\{ D_1 - D \xi h + D_1 \xi h \coth \xi h \right\} - \mu_1 \left\{ A + \frac{k_1+1}{2} B \right\} \right] \xi \sin \xi x \, d\xi = 0 \qquad \ldots(6.4)$$

$$\int_0^\infty \left[\left\{ D \xi h - D_1 \frac{k_2+1}{2} \right\} \coth \xi h + \frac{k_1+1}{2} B - D_1 \xi h - A \right] \sin \xi x \, d\xi = 0 \qquad \ldots(6{\cdot}5)$$

$$\int_0^\infty \left[\left\{ \frac{k_2+1}{2} + D_1 \xi h \right\} \cosh \xi h - D \xi h - D_1 \frac{k_2-1}{2} - A - k_1 B \right] \cos \xi x \, d\xi = 0. \qquad \ldots(6{\cdot}6)$$

From (6·1) and (6·2), we have

$$X = \mu_2 [D(1 - z \coth z) + D_1 z] = -\mu_1 \left[A + \frac{k_1+1}{2} B \right] \qquad \ldots(6{\cdot}7)$$

$$Y = \mu_2 [-z D_1 + D_1 (1 + z \coth z)] = \mu_1 \left[A + \frac{k_1-1}{2} B \right] \qquad \ldots(6{\cdot}8)$$

where $z = \xi h$.

Let us write

$$M = D \left\{ z \coth z + \frac{k_2-1}{2} \right\} - D_1 \left\{ z + \frac{k_2+1}{2} \coth z \right\} - A \qquad \ldots(6{\cdot}9)$$

$$N = D \left\{ \frac{k_2+1}{2} \coth z - z \right\} + D_1 \left\{ z \coth z - \frac{k_2-1}{2} \right\} - A - k_1 B. \qquad \ldots(6{\cdot}10)$$

From these equations we have a system of simultaneous dual integral equations.

$$\left. \begin{aligned} &\int_0^\infty \xi^{-1} M(\xi) \sin \xi x \, d\xi = 0 \\ &\int_0^\infty \xi^{-1} N(\xi) \cos \xi x \, d\xi \end{aligned} \right\} 1 < x < \infty \qquad \begin{aligned} &\ldots(6{\cdot}11) \\ &\ldots(6{\cdot}12) \end{aligned}$$

$$\int_0^\infty [N(\xi) + L_{11}(z) N(\xi) + L_{12}(z) M(\xi)] \cos \xi x \, d\xi = \frac{-\pi}{2} P(x), \; 0<x<1 \qquad \ldots(6{\cdot}13)$$

$$\int_0^\infty [M(\xi) + L_{21}(z) M(\xi) + L_{22}(z) N(\xi)] \sin \xi x \, d\xi = \frac{\pi}{2} T(x) \qquad \ldots(6{\cdot}14)$$

where

$$L_{11}(z) = -H_3 + (2z-1)H_1 + e^{-2z}\Big[H_1 + (1+2z)H_3 - \frac{1+2z}{\lambda_2}$$
$$\times \{2z(1+\lambda_2) - \lambda_2(1-\lambda_1)\} - \Big(\frac{1+\lambda_2}{\lambda_2}\Big)\Big]$$

$$L_{12}(z) = -H_4 + (2z+1)H_2 + e^{-2z}\Big[H_2 + (1+2z)H_4 + \frac{1+2z}{\lambda_2}$$
$$\times \{2z(1+\lambda_2) + \lambda_2(1-\lambda_1)\} + \Big(\frac{1+\lambda_2}{\lambda_2}\Big)\Big]$$

$$L_{21}(z) = H_3 - (1+2z)H_1 + e^{-2z}\Big[H_1 + (2z-1)H_3 - \frac{2z-1}{\lambda_2}$$
$$\times \{2z(1+\lambda_2) - \lambda_2(1-\lambda_1)\} - \Big(\frac{1+\lambda_2}{\lambda_2}\Big)\Big]$$

$$L_{22}(z) = H_4 - (1+2z)H_2 + e^{-2z}\Big[H_2 + (2z-1)H_4 + \frac{2z-1}{\lambda_2}$$
$$\times \{2z(1+\lambda_2) + \lambda_2(1-\lambda_1)\} + \Big(\frac{1+\lambda_2}{\lambda_2}\Big)\Big]$$

$$H_1(z) = \Big(F_1 + \frac{1+\lambda_2}{\lambda_2}F_4\Big)\Big/(F_4 - \lambda_2)$$

$$H_2(z) = \Big(F_2 - \frac{1+\lambda_2}{\lambda_2}F_4\Big)\Big/(F_4 - \lambda_2)$$

$$H_3(z) = \Big(F_3 + \frac{2z(1+\lambda_2) - \lambda_2(1-\lambda_1)}{\lambda_2}\Big)\Big/(F_4 - \lambda_2)$$

$$H_4(z) = \Big(F_3 - \frac{2z(1+\lambda_2) + \lambda_2(1-\lambda_1)}{\lambda_2}\Big)\Big/(F_4 - \lambda_2)$$

$$F_1(z) = e^{-2z}[2z(1-\lambda_1) - \lambda_1(1+\lambda_2)]$$

$$F_2(z) = e^{-2z}[2z(1-\lambda_1) + \lambda_1(1+\lambda_2)]$$

$$F_3(z) = -e^{-2z}(1-\lambda_1)$$

$$F_4(z) = e^{-2z}[-1 + \lambda_1\lambda_2 - 4z^2 + \lambda_1 e^{-2z}]$$

$$\lambda_1 = \frac{k_1\mu_2 - k_2\mu_1}{\mu_1 + k_1\mu_2}, \qquad \lambda_2 = \frac{\mu_2 + \mu_1 k_2}{\mu_1 - \mu_2}.$$

Note that in (6·13) and (6·14), for $z \to \infty$, $L_{ij} \sim o(e^{-2zh})$ and others the integrals on the left-hand side are convergent; as a result; certain operations such as change of order of integration are permissible.

7. Solution of the Simultaneous Dual Integral Equations

Before finding the solution (6·11)–(6·14), we differentiate (6·11) and (6·12), so that the integral equations become

$$\int_0^\infty M(\xi) \cos \xi x \, d\xi = 0, \quad x > 1 \qquad \ldots(7\cdot1)$$

$$\int_0^\infty W(\xi) \sin \xi x \, d\xi = 0, \quad x > 1 \qquad \ldots(7\cdot2)$$

$$\int_0^\infty [N(\xi)+L_{11}(z) N(\xi)+L_{12}(z) M(\xi)] \cos \xi x \, d\xi = \frac{-\pi}{2} P(x), \; 0<x<1 \qquad \ldots(7\cdot3)$$

$$\int_0^\infty [M(\xi)+L_{21}(z) M(\xi)+L_{22}(z) N(\xi)] \sin \xi x \, d\xi = \frac{\pi}{2} T(x), \; 0<x<1. \qquad \ldots(7\cdot4)$$

Let

$$\int_0^\infty M(\xi) \cos \xi x \, d\xi = \begin{cases} m(x), & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \qquad \ldots(7\cdot5)$$

with $m(0) = 0$. Then integrating by parts

$$M(\xi) = \xi^{-1} \int_0^\infty m'(\xi) \sin \xi x \, d\xi . \qquad \ldots(7\cdot6)$$

Now using the result (10)

$$\int_0^\infty \xi^{-1} \sin \xi t \, . \sin \xi x \, d\xi = \tfrac{1}{2} \log \left| \frac{x+t}{x-t} \right| \qquad \ldots(7\cdot7)$$

we see that

$$\int_0^\infty M(\xi) \sin \xi x \, d\xi = \int_0^1 m(t) \left[\frac{1}{x+t} + \frac{1}{x-t} \right] dt. \qquad \ldots(7\cdot8)$$

Since $m(x)$ is an even function, we define for $-1 \leqslant x \leqslant 0$, $m(x) = m(-x)$.

Then

$$\int_0^1 m(t) \left[\frac{1}{x+t} + \frac{1}{x-t} \right] dt = \int_{-1}^1 \frac{m(t)}{x-t} \, dt \qquad \ldots(7\cdot9)$$

and

$$\int_0^\infty M(\xi)\sin \xi x\, d\xi = \int_{-1}^{1} \frac{m(t)}{x-t}\, dt$$

In the like manner, we let

$$\int_0^\infty N(\xi)\sin \xi x\, d\xi = \begin{cases} n(x), & x \in (0, 1) \\ 0 & , x \in (1, \infty) \end{cases} \qquad \text{...(7·10)}$$

with $n(1) = 0$.

The Fourier cosine transform of N can easily be calculated to yield

$$\int_0^\infty N(\xi)\cos \xi x\, d\xi = \frac{d}{dt}\int_0^\infty \xi^{-1} N(\xi)\sin \xi x\, d\xi$$

$$= \int_0^1 n(t)\left[\frac{1}{x+t} - \frac{1}{x-t}\right] dt. \qquad \text{...(7·11)}$$

Since it is clear from (7·10) that $n(x)$ is an odd function, we define $n(x) = -n(-x)$ for $-1 \leqslant x < 0$ and observe that

$$\int_0^\infty N(\xi)\cos \xi x\, d\xi = -\int_{-1}^{1} \frac{n(t)}{x-t}\, dt \qquad \text{...(7·12)}$$

If we substitute (7·5), (7·9), (7·10) and (7·12) into the set of equations (6·11)–(6·14), we see that (6·11) and (6·12) are automatically satisfied while the equations (6·13) and (6·14) imply that $m(t)$ and $n(t)$ must be solutions to the system of singular integral equations:

$$\frac{1}{\pi}\int_{-1}^{1} \frac{n(t)}{(x-t)}\, dt - \frac{1}{\pi}\int_{-1}^{1} [K_{11}(x, t)\, n(t) + K_{12}(x, t)\, m(t)]\, dt = P(x), -1 < x < 1 \qquad \text{...(7·13)}$$

$$\frac{1}{\pi}\int_{-1}^{1} \frac{m(t)}{(x-t)}\, dt - \frac{1}{\pi}\int_{-1}^{1} [K_{21}(x, t)\, m(t) + K_{22}(x, t)\, n(t)]\, dt = T(x), -1 < x < 1 \qquad \text{...(7·14)}$$

with

$$K_{11}(x, t) = \int_0^\infty \mathrm{L}_{11}(\xi h)\cos \xi x \sin \xi t\, d\xi,$$

$$K_{12}(x, t) = \int_0^\infty \mathrm{L}_{12}(\xi h)\cos \xi x \cos \xi t\, d\xi,$$

$$K_{21}(x, t) = \int_0^\infty L_{21}(\xi h) \sin \xi x \cos \xi t \, d\xi,$$

$$K_{22}(x, t) = \int_0^\infty L_{22}(\xi h) \sin \xi x \sin \xi t \, d\xi.$$

Equations (7·13) and (7·14) provide the system of integral equations to determine the unknown functions $m(t)$ and $n(t)$. Once these functions are obtained, all the desired field quantities in the medium can be expressed as and evaluated from definite integrals involving them and appropriate kernels. Note that continuity of the displacements along the bonded portion of the interface, *i.e.*

$$u^{(1)} - u^{(2)} = 0, \; v^{(1)} - v^{(2)} = 0, \; |x| > 1, y = 0$$

requires that, in addition to $m(t) = 0 = n(t)$ for $|x| > 1$, which is used to derive the integral equations, $m(t)$ and $n(t)$ must satisfy the following conditions:

$$\left.\begin{aligned} &\int_{-1}^{1} m(x)\, dx = 0 \\ &\int_{-1}^{1} n(x)\, dx = 0. \end{aligned}\right\} \qquad \ldots(7{\cdot}15)$$

From the expressions for L_{ij} it is seen that as h goes to infinity L_{ij} and as a result k_{ij} go to zero and (7·13), (7·14) reduce to its dominant system representing two bonded half-planes with an interface crack (Erdogan 1965). In addition to $h \to \infty$ if we also let $\mu_1 = \mu_2$, $k_1 = k_2$ we recover the simple (uncoupled) singular integral equation, for a homogeneous infinite plane with a crack (Erdogan and Gupta 1971).

Again if we only take $\mu_1 = \mu_2$, $k_1 = k_2$ we recover a system of simultaneous ntegral equations for a homogeneous semi-infinite medium with a Griffith crack parallel to the free surface (Srivastava and Kumar 1969).

The Solution of the Integral Equations

To solve the system of singular integral equations (7·13) and (7·14), use will be made of the method described by Erdogan *et al.* (1972). The fundamental function for equations is

$$w(t) = (1 - t^2)^{-\frac{1}{2}} \qquad \ldots(7{\cdot}16)$$

Hence, the related orthogonal polynomials are the Chebyshev polynomials of first kind $Tn(t)$. From symmetry it is seen that

$$n(x) = n(-x)$$
$$m(x) = -m(-x).$$

Then, the unknown functions may be expressed as follows:

$$\left.\begin{aligned} n(x) &= (1-t^2)^{-\frac{1}{2}} \sum_{0}^{\infty} A_n\, T_{2n}(x) \\ m(x) &= (1-t^2)^{-\frac{1}{2}} \sum_{0}^{\infty} B_n\, T_{2n-1}(x). \end{aligned}\right\} \qquad \ldots(7{\cdot}17)$$

With equations (7·17) and from the orthogonality condition (Erdelyi 1953, Szego 1939)

$$\frac{1}{\pi}\int_{-1}^{1} T_m(t)\, T_n(t)\, (1-t^2)^{-\frac{1}{2}} \quad = \quad \begin{cases} 0, & m \neq n \\ 1, & m = n = 0 \\ \frac{1}{2}, & m = n > 0 \end{cases}$$

it follows that the condition ...(7·18)

$$\int_{-1}^{1} m(x)\, dx = 0$$

is satisfied and

$$\int_{-1}^{1} n(x)\, dx = 0$$

gives $A_0 = 0$.

Substituting from (7·17) into (7·13) and (7·14) and using the following relations (Erdelyi 1953, Szego 1939).

$$\frac{1}{\pi}\int_{-1}^{1} T_n(t)\, (1-t^2)^{-\frac{1}{2}} \; \frac{dt}{t-x} = \begin{cases} \mathrm{u}_{n-1}(x), & |\, x\,| < 1 \\ G_n(x), & |\, x\,| > 1 \end{cases} \qquad \ldots(7{\cdot}19)$$

where

$$G_n(x) = \frac{[\,(x^2-1)^{\frac{1}{2}} - x\,]^n}{(-1)^{n+1}\,(x^2-1)^{\frac{1}{2}}}$$

it follows that

$$\left.\begin{aligned} \sum_{1}^{\infty}\left[A_n\; U_{2n-1}(x) + A_n\, \overset{11}{H_n}(x) + B_n\, \overset{12}{H_n}(x)\right] &= P(x), -1 < x < 1 \\ \sum_{1}^{\infty}\left[B_n\; U_{2n-1}(x) + A_n\, \overset{21}{H_n}(x) + B_n\, \overset{22}{H_n}(x)\right] &= T(x), -1 < x < 1 \end{aligned}\right\} \qquad \ldots(7{\cdot}20)$$

where

$$\left.\begin{aligned}
H_n^{11}(x) &= \frac{1}{\pi}\int_{-1}^{1} K_{11}(x,t)\,T_{2n}(t)\,(1-t^2)^{-\frac{1}{2}}\,dt\\
H_n^{12}(x) &= \frac{1}{\pi}\int_{-1}^{1} K_{12}(x,t)\,T_{2n-1}(t)\,(1-t^2)^{-\frac{1}{2}}\,dt\\
H_n^{21}(x) &= \frac{1}{\pi}\int_{-1}^{1} K_{21}(x,t)\,T_{2n}(t)\,(1-t^2)^{-\frac{1}{2}}\,dt\\
H_n^{22}(x) &= \frac{1}{\pi}\int_{-1}^{1} K_{22}(x,t)\,T_{2n-1}(t)\,(1-t^2)^{-\frac{1}{2}}\,dt.
\end{aligned}\right\} \quad \ldots(7\cdot21)$$

In order to solve the functional equations (7·20), a weighted residual method will be used (simply both sides will be expanded into series of Chebyshev polynomials and the coefficients will be compounded) to reduce them to a system of algebraic equations. Thus multiplying the first equation by $U_{2k-1}(x)\,(1-x^2)^{\frac{1}{2}}$, the second by $U_{2k-2}(x) \times (1-x^2)^{\frac{1}{2}}$, truncating the series at the nth term and integrating in the interval, we obtain

$$\left.\begin{aligned}
\frac{\pi}{2}A_k + \sum_1^N (a_{kn}A_n + b_{kn}B_n) &= F_{1k}\\
\frac{\pi}{2}B_k + \sum_1^N (c_{kn}A_n + d_{kn}B_n) &= F_{2k}
\end{aligned}\right\} \quad \ldots(7\cdot22)$$

where $k = 1, 2, \ldots, N$ and

$$\left.\begin{aligned}
a_{kn} &= \int_{-1}^{1} U_{2k-1}(x)\,H_n^{11}(x)\,(1-x^2)^{\frac{1}{2}}\,dx\\
b_{kn} &= \int_{-1}^{1} U_{2k-1}(x)\,H_n^{12}(x)\,(1-x^2)^{\frac{1}{2}}\,dx\\
c_{kn} &= \int_{-1}^{1} U_{2k-2}(x)\,H_n^{21}(x)\,(1-x^2)^{\frac{1}{2}}\,dx\\
d_{kn} &= \int_{-1}^{1} U_{2k-2}(x)\,H_n^{22}(x)\,(1-x^2)^{\frac{1}{2}}\,dx\\
F_{1k} &= P(x), \quad F_{2k} = T(x).
\end{aligned}\right\} \quad \ldots(7\cdot23)$$

After obtaining the constants A_k and B_k from (7·22), any desired field quantity may by calculated by evaluating the related definite integrals.

Now (7·13) and (7·14) give stresses on $y = 0$ for $|x| > 1$ as well as $|x| < 1$ Using (7·15), the second of relation (7·19) and (7·21) it is found that

$$\left.\begin{aligned} \frac{1+k_i}{2\mu_i}\,\tau_{xy}(x,0) &= \sum_1^{\infty}\left[A_n G_{2n}(x) + A_n \overset{11}{H_n}(x) + B_n \overset{12}{H_n}(x)\right], \quad |x| < 1 \\ \frac{1+k_i}{2\mu_1}\,\sigma_{yy}(x,0) &= \sum_1^{\infty}\left[B_n G_{2n-1}(x) + A_n \overset{21}{H_n}(x) + B_n \overset{22}{H_n}(x)\right], \quad |x| > 1. \end{aligned}\right\} \quad \ldots(7\cdot24)$$

Since $H_n^{ij}(x)$, $(i, j=1, 2)$ are bounded functions, from equations (7·19) and (7·24), the stress intensity factors may be obtained as

$$\left.\begin{aligned} k_1 &= \lim_{x\to 1}\,(x^2-1)^{\frac{1}{2}}\,\sigma_{yy}(x,0) = \frac{-2\mu_i}{1+k_i}\sum_1^{\infty} B_n \\ k_2 &= \lim_{x\to 1}\,(x^2-1)^{\frac{1}{2}}\,\tau_{xy}(x,0) = \frac{-2\mu_i}{1+k_i}\sum_1^{\infty} A_n. \end{aligned}\right\} \quad \ldots(7\cdot25)$$

8. Numerical Results and Discussion

In the numerical examples the main questions which require careful consideration are the evaluations of the infinite integrals in the kernels k_{ij} (x, t), the evaluations of the functions $H_n^{ij}(x)$ and a_{kn}, b_{kn}, c_{kn}, d_{kn}. The functions a_{kn}, b_{kn}, c_{kn}, d_{kn}, and $H_n^{ij}(x)$ are evaluated by using Gauss-Chebyshev integration formulas. Hence, the accuracy in calculating those functions could easily be controlled by adjusting the number of terms in the Gauss-Chebyshev sums.

The results are calculated on taking $\eta_1 = 0{\cdot}35$, $E_1 = 4{\cdot}5 \times 10^5$ p.s.i.; $\eta_2 = 0{\cdot}3$, $E_2 = 10^7$ p.s.i. and $p(x) = \sigma_0$, a constant and $\tau_1(x) = 0$. The results are shown in Figs. 2 and 3.

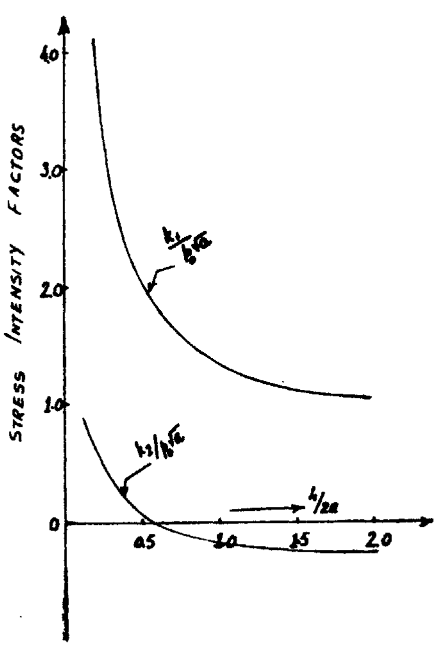

FIG. 2. Stress intensity factors versus *h*.

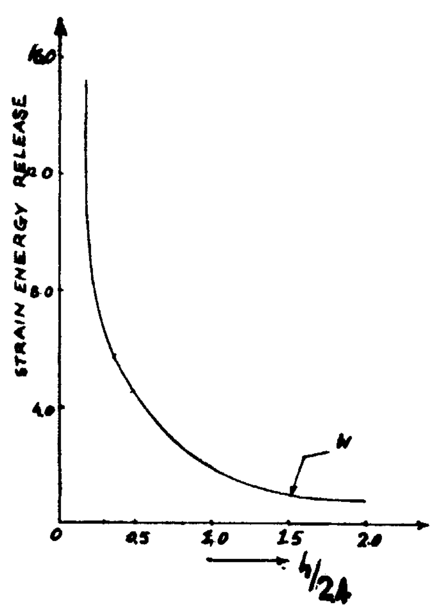

FIG. 3. Strain energy release rate versus *h*

It is seen that as the relative layer thickness approaches zero, the stress intensity factors and the strain energy release rate go to infinity. The results for the asymptotic case $h \to \infty$ are given in Table I.

TABLE I

Stress Intensity factors for H $\to \infty$

Materials η_1	Materials η_2	$\frac{k_1}{\sigma_0}$	$\frac{k_2}{\sigma_0}$	$W = \frac{k_1^2 + k_2^2}{\sigma_0^2}$
0·35	0·3	1	−0·1342	1·0180

It should be noted that in the problem discussed in the paper, the stiffness of the material on the $y > 0$ side of the crack is greater than that of $y > 0$ side. Hence the interface shear round the crack tip $x = +1$ is always politive. On other hand, the second component k_2 of the stress intensity factor is calculated to be negative. This should cause no confusion, since, unlike the homogeneous material, in the nonhomogeneous case the factors k_1 and k_2 are not directly identified with normal and shear stresses on the plane of the crack. This can be seen by expressing the expressions for intensity factors around $x = 1$ as

$$\sigma_{yy} + i\,\tau_{xy} = \frac{k_1 + i\,k_2}{\sqrt{x^2 - 1}} \left[\cos\left(\log \frac{x+1}{x-1}\right) + i \sin\left(\log \frac{x+1}{x-1}\right)\right]$$

where the results indicate that the dominant term for the shear stress is $k_1 \sin\left(\log \frac{x+1}{x-1}\right)$ rather than $k_2 \cos\left(\log \frac{x+1}{x-1}\right)$.

To conclude we see that in addition to its elastostatic structural applications, the solution may be useful as an approximation to the delamination problem caused by the reflected stress waves in layered materials.

REFERENCES

England, A. H. (1965). A crack between dissimilar media. *J. appl. Mech.*, **32**, Series E, 400.

Erdelyi, A. (1953). Higher Transcendental Functions, Vol. 2. McGraw-Hill Book Co., Inc., New York.

Erdogan, F. (1963). Stress distribution in a non-homogeneous elastic plane with cracks. *J. appl. Mech.*, **30**, Series E, 232.

——————(1965). Stress distribution in bonded dissimilar media with cracks. *J. appl. Mech.*, **32**, Series E, 403.

——————(1966). Elastic-plastic anti-plane problems for bonded dissimilar media containing cracks and cavities. *Int. J. Solides. Struct.*, **2**, 447.

——————(1968). Simultaneous dual integral equations with trigonometric and Bessel kernels. *Z. angew Math. Mech.*, **48**, 217.

———(1969). Approximate solution of systems of singular integral equations. *J. appl. Math.*, **17**, 1041.

Erdogan, F., and Gupta, G. (1971). The stress analysis of multi-layered comoisutes with flaw. *Int. J. Solid Struct.*, **7**, 39.

Erdogan, F., Gupta, G. D., and Cook, T. S. (1972). The numerical solutions of singular integral Equations. Methods of Analysis and Solutions to Crack Problems, edited by G.C.S. iH. Noordhoff Publishers.

Gradshteyn, I. S., and Ryzhik, I. M. (1965). Tables of Integrals Series and Products. Academic Press Inc., New York.

Kuz'Min, Iu. N., and Ufland, Ia. (1965). Axisymmetric problem in the theory of elasticity for a half-space weakened by a plane crack. *Priki. Mat. Mekh.*, **29**, 1132.

Muskhelishvili, N. I. (1953). Singular Integral Equations. Noordhoff Publishers.

Rice, J. R., and Sih, G. C. (1965). Plane problems of cracks in dissimilar media. *J. appl. Mech.*, **32**, 418.

Srivastava, K. N., and Kumar, M. (1969). Axisymmetric problems in the theory of elasticity for a semi-infinite two-dimensional medium containing a Griffith crack. *Proc. natn. Inst. Sci. India*, **35**, No. 3, 471.

Szego, G. (1939). Orthogonal Polynomials. Colloquium Publication, Vol. 23, American Mathematical Society.