# UNIVALENT FUNCTIONS WITH BOUNDED DERIVATIVE IN THE UNIT DISC

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Let $G$ be the class of analytic functions in $E=\{z \mid |z|<1\}$ which satisfy the conditions $f(0)=0=f'(0)-1$ and $|f'(z)-1|<1$. For $f' \in G$ the circle in which the values of $f(z)/z$ lie, for fixed value of $f'(z)$ has been obtained and the extreme values of $\text{Re}\,\frac{f(z)}{z\,f'(z)}$, $f \in G$ have been obtained. The radius of starlikeness $\rho_G$ of $G$, has been characterized and it has been shown that $1>\rho_G>0{\cdot}974$. It has also been shown that the class $G_\alpha \subset G$ characterized by the condition $|f'(z)-1|<\alpha$, $0 \leqslant a<1$ is a subset of the class of starlike functions for $0 \leqslant a \leqslant \sqrt{\frac{4}{5}}$.

## INTRODUCTION

Let $S$ denote the class of analytic functions in the unit disc $E=\{z \mid |z|<1\}$, which are univalent and have the normalization $f(0)=0, f'(0)=1, f \in S$. If $S^* \in S$ denotes the class of functions which map $E$ on to a domain starlike with respect to the origin it is known that a necessary and sufficient condition for $f \in S^*$ is that

$$\text{Re}\left\{z\,\frac{f'(z)}{f(z)}\right\}>0,\ z \in E. \qquad \ldots(1)$$

Let $G$ denote the class of analytic functions in $E$ which have the normalization $f(0)=0$, $f'(0)=1$, for $f \in G$ and satisfy the condition

$$|f'(z)-1|<1, z \in E. \qquad \ldots(2)$$

It is well known that $G \in S$. A number $\rho_G$ defined as

$$\rho_G=\sup\left\{|z| \,\middle|\, \min_{f \in G} \text{Re}\,z\,\frac{f'(z)}{f(z)}>0, |z|<1\right\}.$$

is called the radius of starlikeness of the class $G$. MacGregor (1964) has shown that $\rho_G>\sqrt{\frac{4}{5}}$ and it is still not known if $G \in S^*$. In the present paper we prove the following:

*Theorem* 1 — Let $f \in G$, $f'(z)-1=\phi(z)$, $|z|=r$ and let

$$A=A_r(|\phi|)=\tfrac{1}{2}+(r^2-|\phi(z)|^2)\int_0^1 \frac{t\,(1-t)^2\,dt}{(1-tr^2)^2-(1-t)^2|\phi(z)|^2}, \qquad \ldots(3)$$

and

$$B = B_r(|\phi|) = (r^2 - |\phi(z)|^2) \int_0^1 \frac{t(1-t)(1-tr^2)\,dt}{(1-tr^2)^2 - (1-t)^2 |\phi(z)|^2}, \quad \ldots(4)$$

then

$$\operatorname{Re}\left\{\frac{f(z)}{zf'(z)}\right\} \geqslant 1 - \frac{B + A|\phi(z)|}{1+|\phi(z)|} > 0 \text{ if } A(1-|\phi(z)|) > B, \quad \ldots(5)$$

$$\geqslant 1 - \frac{A}{2} - \frac{B^2}{2A(1-|\phi(z)|^2)}$$

$$\text{if } A(1-|\phi(z)|) \leqslant B. \quad \ldots(6)$$

Further

$$\operatorname{Re}\left\{\frac{f(z)}{zf'(z)}\right\} < 1 + \frac{B + A|\phi(z)|}{1-|\phi(z)|} < \frac{2-r}{2(1-r)}. \quad \ldots(7)$$

The above inequalities are sharp.

We note that in view of (2) the function $\phi(z) = f'(z) - 1$, belongs to the class $B_1$ of analytic functions in $E$ which vanish at the origin and satisfy $|\phi(z)| < 1$. By Schwarz Lemma we obtain

$$|\phi(z)| < |z| = r. \quad \ldots(8)$$

Since $\operatorname{Re} \dfrac{f(z)}{zf'(z)} > 0$ if and only if $\operatorname{Re} \dfrac{zf'(z)}{f(z)} > 0$ we establish the following.

*Theorem* 2—The number $\rho_G$ is the largest value of $r$ for which

$$\min_{\phi \in B_1} \left\{ 1 - \frac{A}{2} - \frac{B^2}{2A(1-|\phi(z)|^2)} \right\} > 0, \quad \ldots(9)$$

under the condition

$$A(1-|\phi(z)|) \leqslant B. \quad \ldots(10)$$

In particular $1 > \rho_G > 0.974$.

Since $G$ is not a subset of $S^*$, we consider the class $G_\alpha \subset G$ which satisfies

$$|f'(z) - 1| \leqslant \alpha, \; 0 \leqslant \alpha < 1 \quad \ldots(11)$$

and prove the following:

*Theorem* 3 — If $0 \leqslant \alpha \leqslant \sqrt{4/5}$ then $G_\alpha \subset S^*$.

## 1. Some Auxiliary Results

*Lemma* 1—If $\phi \in B_1$, $0 \leqslant t \leqslant 1$, $|z| = r$, $|\phi(z)| = x$, then

$$\left|\phi(tz) - \frac{\phi(z)\,t(1-r^2)(1-r^2t^2)}{(1-r^2t)^2 - (1-t)^2x^2}\right| \leqslant \frac{t(1-t)(1-tr^2)(r^2-x^2)}{(1-r^2t)^2 - (1-t)^2x^2}, \quad x \geqslant r.$$

$$\ldots(12)$$

This inequality is sharp.

PROOF: If $\phi \in B_1$, and $z,\ \xi \in E$ then

$$F(z,\ \xi) = \frac{\phi(z) - \phi(\xi)}{1 - \overline{\phi(\xi)}\,\phi(z)}\ \frac{1 - \xi z}{z - \xi} \qquad ...(13)$$

is analytic in $E$, $|F(z,\ \xi)| < 1$ and $F(0,\ \xi) = \phi(\xi)/\xi$.

Therefore, the function

$$\frac{F(z,\ \xi) - F(0,\ \xi)}{1 - \overline{F(0,\ \xi)}\,F(z,\ \xi)} \qquad ...(14)$$

belongs to $B_1$. An application of Schwarz' lemma toth is function when $z = t\xi$, $0 \leqslant t \leqslant 1$ yields (12), if $\xi$ is replaced by $z$. Equality in (12) will hold if the function in (14) is the identity function. With the help of (13) and (14) one can then find the form of $\phi(z)$ for which equality holds in (12).

*Lemma* 2—Let $A$ and $B$ be defined by (3) and (4). Then

$$0 \leqslant B \leqslant \tfrac{1}{2}\ \frac{r^2 - x^2}{1 - x^2} \leqslant \tfrac{1}{2} \qquad ...(15)$$

and

$$\tfrac{1}{2} \leqslant \tfrac{1}{2} + \frac{r^2 - x^2}{1 - x^2}\,\frac{1}{12} \leqslant A \leqslant \tfrac{1}{2} + \frac{r^2 - x^2}{1 - x^2}\ \tfrac{1}{2} \leqslant 1. \qquad ...(16)$$

Further,

$$B < \tfrac{1}{2}(A - \tfrac{1}{2})(1 + x^2) + \frac{r^2 - x^2}{4}. \qquad ...(17)$$

PROOF: The inequalities (15) follow from (8) and

$$(1 - tr^2)^2 - (1 - t)^2x^2 \geqslant (1 - tr^2)^2(1 - x^2) > (1 - t)(1 - tr^2)(1 - x^2); \quad ...(18)$$

and the inequalities in (16) follow from (8) and

$$(1 - tr^2)^2 - (1 - t)^2 x^2 \leqslant (1 - t^2r^2)(1 - x^2), \qquad ...(19)$$

on writing

$$A = 1 - (1 - r^2) \int_0^1 \frac{t(1 - t^2r^2)\,dt}{(1 - r^2t)^2 - (1 - t)^2x^2}. \qquad ...(20)$$

That (20) and (3) are the same follows from the identity

$$(1 - r^2)(1 - r^2t^2) = [(1-r^2t)^2 - (1 - t)^2x^2] - (r^2 - x^2)(1 - t)^2. \qquad ...(21)$$

The inequalities in (18) and (19) are obvious. From (15) and (16) we find that

$$B \leqslant A. \qquad ...(22)$$

In order to prove (17) we use the easily established fact that

$$B = \tfrac{1}{2}(A - \tfrac{1}{2})(1 + x^2) + \frac{r^2 - x^2}{4} - \frac{(r^2 - x^2)(1 - r^2)^2}{2} \int_0^1 \frac{t^3\, dt}{(1 - tr^2)^2 - (1 - t)^2 x^2}. \qquad \text{...(23)}$$

*Lemma* 3—Let $A$ and $B$ be defined by (3) and (4). Then for $0 \leqslant r \leqslant \frac{\sqrt{3}}{2}$,

$$B \leqslant (1 - x)\, A. \qquad \text{...(24)}$$

PROOF: In view of (17) we have

$$B - (1 - x)\, A \leqslant \frac{A}{2}(x^2 + 2x - 1) - \frac{1 - r^2 + 2x^2}{4}. \qquad \text{...(25)}$$

The right-hand side of (25) is negative if $x^2 + 2x - 1 < 0$. If $x^2 + 2x - 1 > 0$, then using (16) in (25) we obtain

$$B - (1 - x)\, A < \tfrac{1}{2}\left\{ -(1 - 2x) - \frac{(1 - r^2)\, x}{1 - x^2} \right\}. \qquad \text{...(26)}$$

The right-hand side of (26) is negative for $r < \frac{1}{\sqrt{3}}$ and for $r > \frac{1}{\sqrt{3}}$, its maximum value is

$$-1 + x\, \frac{7 + r^2 - 2\sqrt{(1 - r^2)(17 - r^2)}}{4} \qquad \text{...(27)}$$

which is certainly negative if

$$7 + r^2 - 2\sqrt{(1 - r^2)(17 - r^2)} < 4.$$

This is true for $0 \leqslant r \leqslant \sqrt{3}/2$.

## 2. Proof of Theorem 1

We note that $f \in G$ if and only if

$$f'(z) = 1 + \phi(z),\ \phi \in B_1. \qquad \text{...(28)}$$

Hence

$$\operatorname{Re} \frac{f(z)}{z f'(z)} = \operatorname{Re} \int_0^1 \frac{1 + \phi(tz)}{1 + \phi(z)}\, dt, \qquad \text{...(29)}$$

and in order to prove Theorem 1 we need to find upper and lower bounds on

$$\operatorname{Re} \int_0^1 \frac{1 + \phi(tz)}{1 + \phi(z)}\, dt,\ \phi \in B_1. \qquad \text{..(30)}$$

From (12) we easily deduce that

$$\left| \int_0^1 \frac{1+\phi(tz)}{1+\phi(z)}\, dt - \frac{A}{1+\phi(z)} - (1-A) \right| \leqslant \frac{B}{|1+\phi(z)|} \qquad \ldots(31)$$

where $A$ and $B$ are defined by (3) and (4). From (31) we obtain

$$\operatorname{Re} \int_0^1 \frac{1+\phi(tz)}{1+\phi(z)}\, dt \geqslant A \operatorname{Re} \frac{1}{1+\phi(z)} + 1 - A - \frac{B}{|1+\phi(x)|}. \qquad \ldots(32)$$

It is readily confirmed that the minimum of the right-hand side of (32) with respect to arg $\phi$, for $\phi \in B_1$, is attained either (i) for $\phi$ real and positive when $A(1-x) >$, $B$ or (ii) $\phi$ satisfying

$$A(1-|\phi(z)|^2) = B\,|1+\phi(z)|. \qquad \ldots(33)$$

For (33) to be true we must have

$$\frac{1}{1+x} \leqslant A/B \leqslant \frac{1}{1-x}. \qquad \ldots(34)$$

Since $A > B$ the first inequality is always true and it is only the second inequality which needs to be satisfied. In case (i) the right-hand side of (32) equals the right-hand side of (5) and in the second case it equals the right-hand side of (6).

Thus the inequalities (5) and (6) are established and to complete the proof of Theorem 1 we need only prove the inequality (7). From (31) we get

$$\operatorname{Re} \int_0^1 \frac{1+\phi(tz)}{1+\phi(z)}\, dt \geqslant A \operatorname{Re} \frac{1}{1+\phi(z)} + 1 - A + \frac{B}{|1+\phi(z)|}. \qquad \ldots(35)$$

The maximum of the right-hand side of (35) with respect to arg $\phi$, for $\phi \in B$, is attained for $\phi$ real and negative and we obtain

$$A \operatorname{Re} \frac{1}{1+\phi(z)} + 1 - A + \frac{B}{|1+\phi(z)|} \leqslant 1 + \frac{B+Ax}{1-x}. \qquad \ldots(36)$$

The right-hand side of (36) is a monotane increasing function of $|\phi| = x \leqslant r$ and its maximum is attained for $|\phi| = r$. This completes the proof of Theorem 1.

The extremal function for (7) is the function $f(z) = z + \frac{1}{2}z^2$ and equality is attained at $z = -r$. The sharpness of (5) and (6) follows from the fact that there exist functions $\phi \in B_1$, $f(z) - 1 = \phi(z)$, for which equality is attained.

In view of Lemma 2 it is easily confirmed that

$$(B + Ax) < 1 + x$$

and hence the right-hand side of (5) is always positive. Further, because

$$1 - \frac{A}{2} - \frac{B^2}{2A(1-x^2)} = 1 - \frac{B+xA}{1+x} - \frac{\{B - A(1-x)\}^2}{2A(1-x^2)},$$

it follows that the right-hand side of (6) is smaller than the right-hand side of (5). Moreover, both these values are equal for $B = A(1-x)$.

The inequalities (5), (6) and (7) give the extreme values of

$$\mathrm{Re}\ \frac{f(z)}{2f'(z)} \text{ for } f \in G \text{ depending upon } |f'(z) - 1|.$$

It may be remarked here that the inequality (31) can be written in the form

$$\left| \frac{f(z)}{z} - 1 - (1-A)(f'(z) - 1) \right| \leqslant B, \qquad \ldots(37)$$

and therefore it gives the region of values of $f(z)/z$ for a fixed value of $f'(z)$. In fact, Lemma 1 enables us to solve a general class of extremum problems as below:

Let $\phi \in B_1$ and let the real function $\psi\left(\phi(z), \int_0^1 \phi(tz)\, dt\right)$ be well defined and finite for every $\phi \in B_1$. Moreover, if $\psi$ satisfies the condition that for any fixed value $\phi(z)$, $|\phi(z)| < 1$, it reaches a certain specific limit on the circumference of the circle

$$\left| \int_0^1 \phi(zt)\, dt - \phi(z)(1-A) \right| \leqslant B,$$

then denoting this exact limit by " Extr " we can assert that

$$\underset{\phi \in B_1}{\mathrm{Extr}} \ \underset{|z| = r < 1}{\mathrm{Extr}} \ \psi\left(\phi(z), \int_0^1 \phi(tz)\, dt\right)$$

$$= \underset{|\phi| \leqslant r}{\mathrm{Extr}} \ \underset{\theta \in [0, 2\pi]}{\mathrm{Extr}} \ \psi\left(\phi(z), \phi(z)(1 - A_r(|\phi|)) + B_r(|\phi|)\, e^{i\theta}\right),$$

where $A_r(|\phi|)$ and $B_r(|\phi|)$ are defined by (3) and (4), respectively.

Thus the problem of finding the extreme values of $\psi\left(\phi(z), \int_0^1 \phi(tz)\, dt\right)$, $\phi \in B_1$ reduces to the determination of the extremum of a function of three variables, one of which lies on the unit circumference and the other two lie in the circle $|w| \leqslant r$. In most interesting cases the extremum on the unit circumference is easy to determine and the problem then reduces to finding the extremum of a function of two variables which lie in the circle $|w| \leqslant r$.

## 3. Proof of Theorem 2

It is clear that $\rho_G$ will be the largest value for $r$ for which

$$\min_{|\phi| \leq r} \left\{ 1 - \frac{A}{2} - \frac{B^2}{2A(1-|\phi|^2)} \right\} \geqslant 0, \qquad \ldots (38)$$

under the condition

$$A(1-|\phi|) \leqslant B. \qquad \ldots (39)$$

This minimum does not seem easy to evaluate although the integrals for $A$ and $B$ can be readily evaluated. Further, from (38) it is clear that $\rho_G$ cannot equal one. In fact, for $r=1$, $A=1$, $B=\frac{1}{2}$ and the expression in (38) can become negative because of the presence of the factor $(1-|\phi|^2)$ in the denominator of the last term.

From (6) it follows that $\operatorname{Re} \dfrac{f(z)}{zf'(z)} > 0$ if

$$A(2-A) \geqslant \frac{B^2}{1-x^2}, \; 0 \leqslant x \leqslant r. \qquad \ldots (40)$$

In view of Lemma 2

$$A(2-A) \geqslant 3/4, \qquad \ldots (41)$$

and

$$\frac{B^2}{1-x^2} \leqslant \frac{1}{4} \frac{(r^2-x^2)^2}{(1-x^2)^3}, \; 0 \leqslant x \leqslant r. \qquad \ldots (42)$$

The maximum of $\dfrac{(r^2-x^2)^2}{(1-x^2)^3}$ for $0 \leqslant x \leqslant r$,

occurs for $x^2 = 3r^2 - 2$ because by Lemma 3 in this case $r > \sqrt{3}/2$.

$$\text{Thus} \quad \frac{B^2}{1-x^2} \leqslant \frac{1}{27(1-r^2)}. \qquad \ldots (43)$$

Finally (40) will certainly hold if

$$\frac{3}{4} \geqslant \frac{1}{27(1-r^2)} \quad \text{or } r^2 \leqslant 77/81.$$

## 4. Proof of Theorem 3

From (11) it is evident that $f \in G_\alpha$, if and only if

$$f'(z) = 1 + \alpha\, \phi(z), \quad \phi \in B_1. \qquad \ldots (44)$$

Hence

$$\operatorname{Re} \frac{f(z)}{zf'(z)} = \operatorname{Re} \int_0^1 \frac{1 + \alpha\, \phi(z)}{1 + \alpha\, \phi(z)}\, dt, \qquad \ldots (45)$$

and proceeding as in the proof of Theorem 1 we obtain

$$\operatorname{Re} \int_0^1 \frac{1 + \alpha \phi(tz)}{1 + \alpha \phi(z)} \geqslant 1 - \alpha \frac{B + A \mid \phi \mid}{1 + \alpha \mid \phi \mid} \quad \text{if } A(1 - \alpha \mid \phi \mid \geqslant \alpha B, \qquad \ldots (46)$$

$$\geqslant 1 - \frac{A}{2} - \frac{\alpha^2 B^2}{2A(1 - \alpha^2 \mid \phi \mid^2)} \quad \text{if } A(1 - \alpha \mid \phi \mid \leqslant \alpha B. \qquad \ldots (47)$$

As min Re $\frac{f(z)}{zf'(z)}$ is attained on the boundary and we are interested in finding the range of $\alpha$ for which Re $\frac{f(z)}{zf'(z)} \geqslant 0$, $f \in G_\alpha$ in the whole unit disc, we obtain from (47) on putting $r = 1$ that

$$\operatorname{Re} \int_0^1 \frac{1 + \alpha \phi(tz)}{1 + \alpha \phi(z)}\, dt \geqslant \tfrac{1}{2}\left(1 - \frac{\alpha^2}{4(1 - \alpha^2)}\right).$$

This is non-negative for $\alpha \leqslant \sqrt{4/5}$.

One expects this value of $\alpha$ to be sharp, but we do not have any example to demonstrate it.

## Reference

MacGregor, T. H. (1964). A class of univalent functions. *Proc. Am. math. Soc.*, **15**, 311-17.