ON THE GRAPHS HAVING UNIQUE MAXIMAL-CONNECTING PARTITION GRAPHS

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For a graph $G=(V, E)$ a partition P of V is called a maximal-connecting partition (or briefly an *mc*-partition) of G if $|P| = p-k+1$ where $p=|V|$ and k is the number of components of G. The corresponding partition graph $P(G)$ of G is called an *mc*-partition graph of G. In this paper, graphs G whose *mc*-partition graphs are all isomorphic are determined. This problem is seen to be closely related with the dot-composability of finite connected graphs.

INTRODUCTION

All graphs considered in this paper are finite undirected graphs without loops and multiple lines. Recently, the notion of partition graphs of a given graph was introduced by Sampathkumar and Bhave (1973*a*) as follows: Let $G=(V, E)$ be a graph with point set $V(G)=V$ and line-set $E(G)=E$. Let $\mathcal{P}_G$ denote the set of all partitions of the set $V(G)$. If $P \in \mathcal{P}_G$ then we call P a 'partition of G'. Let $P=\{V_1, V_2, \ldots, V_t\} \in \mathcal{P}_G$. Then the partition graph $P(G)$ of G relative to P is a graph having point-set P and the line-set $E(P(G)) = \{(V_i, V_j) \in P \times P / (u, v) \in E(G)$ for some $u \in V_i$ and $v \in V_j$ and $i \neq j\}$. Clearly, each $P \in \mathcal{P}_G$ defines a graph $P(G)$. A graph H is said to be a partition graph if there exists a graph G and a $P \in \mathcal{P}_G$ such that $P(G)$ is isomorphic to H (for any two graphs G_1 and G_2 we write $G_1 \cong G_2$ whenever G_1 is isomorphic to G_2). The notion of partition graphs of a graph generalizes, in a natural way, the notions of homomorphisms and contractions of the graph. Indeed, it was observed (Sampathkumar and Bhave 1973*a*) that a homomorphism of a given graph G is nothing but a partition $P = \{V_1, V_2, \ldots, V_t\} \in \mathcal{P}_G$ in which each V_i is an independent set of G and a contraction of G is nothing but a partition $Q=\{U_1, U_2, \ldots, U_r\} \in \mathcal{P}_G$ such that each subgraph $\langle U_i \rangle$, induced by U_i, is connected in G. Due to this generalization, a number of existing results in graph theory find best proofs in a most general set up. Toward a good appreciation of partition-graph techniques of proof and allied problems the reader is referred to the expository papers by several authors, viz. Acharya (1975), Acharya and Vartak (1975*a,b*) and Sampathkumar and Bhave (1973*a–e*). In this paper, we are concerned with a special class of partitions of a graph called

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maximal-connecting partitions (or briefly, *mc*-partitions). For standard graph theoretic terminology we follow Harary (1969).

mc-PARTITIONS OF A GRAPH

Let G be any graph and $P \in \mathscr{P}_G$. Then P is called a connecting partition of G if $P(G)$ is connected. A connecting partition π of G is called a maximal connecting partition (or, an mc-partition) if for every connecting partition P of G $|P| \leqslant |\pi|$ holds, where for any set S, $|S|$ denotes the cardinality of S. Let π^c_G denote the set of connecting partitions of G and π_G denote the set of *mc*-partitions of G. Clearly, $\pi_G \subseteq \pi^c_G$. If $P \in \pi_G$ then $P(G)$ is called an *mc*-partition graph of G.

For a graph G, the functions $p(G)$ and $q(G)$ will denote respectively the cardinalities of $V(G)$ and $E(G)$. If a single graph is under reference in a statement we use just p and q instead of $p(\,.\,)$ and $q(\,.\,)$. If a graph G has n connected components (we use, henceforth, only 'components' for 'connected components') then the rank $r(G)$ of G is the number given by $r(G) = p - n$. Note that G is a null-graph if and only if $r(G) > p - 2$. Hence, G is a non-null disconnected graph if and only if $0 < r(G) \leqslant p - 2$. The following result was proved by E. Sampathkumar and V. N. Bhave.

Theorem 1·1 (Sampathkumar and Bhave 1973*a*)—The following statements are equivalent for any $P \in \pi^c_G$ where G is any graph having k components.

(*A*) $P \in \pi_G$.

(*B*) The following hold for P:

(i) every $V_i \in P$ has atmost one point from each of the components of G;

(ii) for every component G_i there exists atleast one component G_j, $i = j$, such that for some $V_r \in P$, V_r contains a point of G_i and a point of G_j;

(iii) the number of blocks in $P(G)$ is the same as the number of blocks in G.

(*C*) $|P| = r(G) + 1 = p - k + 1$

(*D*) $r(P(G)) = r(G)$.

The following are some new corollaries to the above theorem:

Corollary 1·1·1—For any graph G,

$$P \in \pi_G \Rightarrow q(P(G)) = q(G). \qquad \ldots(1\cdot1)$$

PROOF: This follows from a remark (cf. Remark 6 of Sampathkumar and Bhave 1973*a*) to Theorem 1·1 that if $G_1, G_2, \ldots, G_k$ are the components of G then an *mc*-partition graph $P(G)$ has each G_i as an induced subgraph and every block of G_i, $1 \leqslant i \leqslant k$, is a block of $P(G)$, and conversely.

Remark 1·1 : Converse of Corollary 1·1·1 is not true. For, consider a disconnected graph G and the partition $P^0 = \Big\{ \{a\} \,/\, a \in V(G) \Big\} \in \mathscr{P}_G$. Then $P^0(G) \cong G$

is disconnected so that $P \notin \pi_G$. A separate paper (Acharya and Vartak 1975*b*) dealing with the properties of the partitions P belonging to the set

$$\phi_G = \left\{ P \in \mathcal{P}_G / q(P(G)) = q(G) \right\}$$

will appear elsewhere.

For any graph G, $\mathcal{P}_G$ will denote the set of homomorphisms of G.

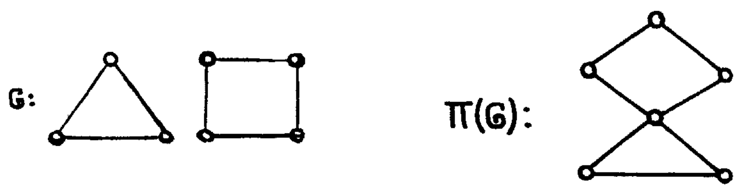
Corollary 1·1·2—For any graph G, $\pi_G \subseteq \mathcal{P}_G$.

PROOF: This follows from the condition (*B*) (i) of Theorem 1·1.

From (*C*) of Theorem 1·1 and Corollary 1·1·1 we have for any $P_1, P_2 \in \pi_G$

$$p(P_1(G)) = p(P_2(G)) \quad \text{and} \quad q(P_1(G)) = q(P_2(G)). \qquad \ldots(1{\cdot}2)$$

This raises possibility of the existence of graphs G for which mc-partition graph is unique (up to isomorphism), that is, graphs G for which all mc-partition graphs $\pi_i(G)$ are isomorphic. In fact, such graphs do exist. One such graph G is shown in Fig. 1(*a*) and the corresponding unique (up to isomorphism) mc-partition graph $\pi(G)$



(*a*) (*b*)

FIG. 1.

is shown in Fig. 1(*b*). However, if G is any connected graph then $\pi_G = \{P^0\}$. In what follows, we proceed to determine the graphs G for which mc-partition graph is unique (up to isomorphism). We need some new terminology. Let $G=(V, E)$ be any graph and $V=\{v_1, v_2, \ldots, v_p\}$. For any $u \in V(G)$, $d_G(u)$ will denote the degree of the point u in G. Write $d_G(v_i)=d_i$, $1 \leq i \leq p$. Then $(d_1, d_2, \ldots, d_p)$ is called the degree-sequence of G. A labelling λ of the points of G is a bijection from V onto $J_p = \{1, 2, \ldots, p\}$. Let $\triangle(G)$ denote the set of all labelings of G and $\lambda \in \triangle(G)$. λ is said to be a normaliser if $\lambda(v_1) \geqslant \lambda(v_2) \geqslant \ldots \geqslant \lambda(v_p)$ implies $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_p$. Note that normalising labellings need not be unique for a graph. Let $\triangle^N(G)$ denote the set of normalising labellings of the graph G. Clearly, a necessary condition for two graphs G and H to be isomorphic is that there exist $\lambda \in \triangle^N(G)$ and $\mu \in \triangle^N(H)$ for which the 'normalised' degree sequences are identical. Let n be any positive integer. Then a sequence $(n_1, n_2, \ldots, n_t)$ of non-negative integers is said to be a t-partition of n if $\sum_{i=1}^{t} n_i = n$. Any two t-partitions of n are said to be t-conjugates of n. From (1·2) it follows that degree-sequences of any two mc-partition graphs of a graph G are $(p-k+1)$-conjugates of $2q$ where k is the number of components of G.

Let G and H be two graphs (finite or infinite) on disjoint sets of points and u, v be arbitrary points of G and H, respectively. Define $D_{u,v}(G, H)$ to be the graph obtained from G and H by identifying the points u and v. Let $\mathcal{G}_2$ denote the set of graphs G which have exactly two non-trivial components G^1 and G^2; and $\mathcal{G}_c$ denote the set of connected graphs. Let $G \in \mathcal{G}_2$. Then for any $P \in \pi_G$, $P(G) = D_{u,v}(G^1, G^2)$ for some $u \in V(G^1)$ and $v \in V(G^2)$ where $G^1, G^2 \in \mathcal{G}_C$. Conversely, let $G^1, G^2 \in \mathcal{G}_C$ and $u \in V(G^1)$ and $v \in V(G^2)$. Then $H = G^1 \cup G^2 \in \mathcal{G}_2$ and furthermore $D_{u,v}(G^1, G^2) \cong P(H)$ where P is an elementary homomorphism identifying u and v; in fact, $P \in \pi_H$. Thus, for any two connected graphs G and H we may regard $D_{u,v}(G, H)$, $u \in V(G)$, $v \in V(H)$, as an mc-partition graph of $K = G \cup H$, and conversely.

Two graphs G and H are said to be dot-composable (see Rao 1975 and Zelinka 1975) if $D_{u,v}(G, H)$ is independent of the choice of u, v ; in other words, G and H are dot-composable if all $p(G)$. $p(H)$ 'dot composed' graphs $D_{u,v}(G, H)$ are isomorphic. If G and H are dot-composable then the resulting unique (upto isomorphism) graph will be denoted by $G \circ H$. For a full appreciation of this topic the reader is referred to Rao (1975) and Zelinka (1975). However, we need only the following result:

Theorem 1·2 (Zelinka 1975)—If G and H are finite graphs then G and H are dot-composable if and only if G and H are point-symmetric.

Indeed, we use the following version of Theorem 1·2.

Lemma 1·1—If G and H are finite connected graphs then G and H are dot-composable if and only if G and H are point-symmetric.

We now prove the main result of this paper.

Theorem 1·3—Let G be a finite disconnected graph having k components. Then all mc-partition graphs of G are isomorphic if, and only if, G satisfies the following two conditions:

(I) Atmost two components of G are non-trivial.

(II) If there are two non-trivial components then they must be point-symmetric.

PROOF: *Necessity*—The proof of this part is by contraposition. Suppose (I) does not hold. Then G must have at least three non-trivial components, say, G_1, G_2 and G_3. Let other components (if any) be labelled $G_4, G_5, \ldots, G_k$. We shall now show the existence of two mc-partitions π_1 and π_2 for which $\pi_1(G) \ncong \pi_2(G)$. Let $a_1, a_2, \ldots, a_k$ be points of $G_1, G_2, \ldots, G_k$, respectively, such that a_i is a point of maximum degree in G_i, $1 \leq i \leq k$. Let b_1 be a point of G_2 adjacent to a_2 and let other points of G be labelled $b_2, b_3, \ldots, b_{p-k}$. Clearly, one has

$$d_G(b_j) \leq d_G(a_i) \text{ if } b_j \in V(G_i),\ 1 \leq i \leq k. \quad \ldots(1\cdot3)$$

Let $A=\{a_i/1\leqslant i\leqslant k\}$, $B=\{a_1, a_2\}$ and $C=\{b_1, a_3, a_4, \ldots, a_k\}$. Further, let $V_j=\{b_j\}$. $1\leqslant j\leqslant p-k$. Now define partitions π_1 and π_2 of G as follows:

$$\pi_1=\{A\}\cup\{V_j\,/\,1\leqslant j\leqslant p-k\}$$

$$\pi_2=\{B, C\}\cup\{V_i\,/\,2\leqslant j\leqslant p-k\}.$$

Since $|\pi_1|=|\pi_2|=p-k+1$, by Theorem 1·1 (C) we have $\pi_1, \pi_2\in\pi_G$. Furthermore, since

$$d_{\pi_1(G)}(A)=\sum_{i=1}^{k}d_G(a_i),$$

$$d_{\pi_1(G)}(V_j)=d_G(b_j),\ 1\leqslant j\leqslant p-k.$$

$$d_{\pi_2(G)}(B)=d_G(a_1)+d_G(a_2),$$

$$d_{\pi_2(G)}(C)=d_G(b_1)+\sum_{i=3}^{k}d_G(a_i).$$

$$d_{\pi_2(G)}(V_j)=d_G(b_j),\ 2\leqslant j\leqslant p-k.$$

(see Acharya and Vartak 1975*b*) degree-sequences $D(\pi_1)$ and $D(\pi_2)$ of the graphs $\pi_1(G)$ and $\pi_2(G)$, respectively, are given by

$$D(\pi_1)=\Big(\sum_{i=1}^{k}d_G(a_i),\ d_G(b_1),\ d_G(b_2),\ldots,\ d_G(b_{p-k})\Big)$$

$$D(\pi_2)=\Big(d_G(a_1)+d_G(a_2),\ d_G(b_1)+\sum_{i=3}^{k}d_G(a_i),\ d_G(b_2),\ldots\ldots,\ d_G(b_{p-k})\Big).$$

We now claim that $\sum_{i=1}^{k}d_G(a_i)$ is not equal to any term of the sequence $D(\pi_2)$. Clearly, $d_G(a_1)+d_G(a_2)<\sum_{i=1}^{k}d_G(a_i)$ because, G_3 being nontrivial, $d_G(a_3)\geqslant 1$. Next, consider $\sum_{i=1}^{k}d_G(a_i)-\Big(d_G(b_1)+\sum_{i=3}^{k}d_G(a_i)\Big)=d_G(a_1)+d_G(a_2)-d_G(b_1)$. Since $d_G(b_1)\leqslant d_G(a_2)$ we get $d_G(a_2)-d_G(b_1)\geqslant 0$, and hence $d_G(a_1)+d_G(a_2)-d_G(b_1)\geqslant d_G(a_1)\geqslant 1$ as G_1 is non-trivial. Thus we get $d_G(b_1)+\sum_{i=3}^{k}d_G(a_i)<\sum_{i=1}^{k}d\,G(a_i)$.

Next, suppose that $\sum_{i=1}^{k}d_G(a_i)=d_G(b_j)$ for some j, $2\leqslant j\leqslant p-k$. Since G_1, G_2 and G_3 are non-trivial components of G $\sum_{i=1}^{k}d_G(a_i)=d_G(b_j)\geqslant 3>0$. This implies that b_j must belong to a non-trivial component of G, say $b_j\in V(G_r)$, $1\leqslant r\leqslant k$. Then $d_G(a_r)\geqslant 1$, and hence $d_G(b_j)=\sum_{i=1}^{k}d_G(a_i)\leqslant d_G(a_r)$ as a_r is a point of maximum degree in G_r. This implies that

$$\sum_{\substack{i=1\\ k\neq r}}^{k}d_G(a_i)\leqslant 0$$

and since each term of this summation is a non-negative integer it follows that $d_G(a_i)=0$ for all i, $1 \leq i \neq r \leq k$. This contradicts our supposition that G has at least three non-trivial components. Thus we have proved that one term of $D(\pi_1)$, viz. $\sum_{i=1}^{k} d_G(a_i)$ is different from any term of $D(\pi_2)$ which shows that for no normalisers of $\pi_1(G)$ and $\pi_2(G)$, $D(\pi_1)$ and $D(\pi_2)$ are identical. This proves that any isomorphism between $\pi_1(G)$ and $\pi_2(G)$ is impossible, and hence that (I) must hold.

Next, if (II) does not hold then G must have exactly two non-trivial components G_1 and G_2, and other components (if any) must be trivial; this claim is a consequence of the fact that (I) holds. Furthermore, atleast one of G_1 and G_2 must be non-point-symmetric. Now, let $\pi \in \pi_G$. Then $\pi(G)$ can be regarded as the graph $D_{a,b}(G_1, G_2)$ for some $a \in V(G_1)$ and $b \in V(G_2)$. But since G is finite it follows from Lemma 1·1 that there must exist a pair (a', b'), $a' \in V(G_1)$ and $b' \in V(G_2)$ such that $D_{a',b'}(G_1, G_2) \ncong D_{a,b}(G_1, G_2) \cong \pi(G)$; that is, there must exist $\pi' \in \pi_G$ (for which $\pi'(G) \cong D_{a',b'}(G_1, G_2)$) such that $\pi'(G) \ncong \pi(G)$. Thus, condition (II) is also necessary.

Sufficiency—If all the components of G are trivial then all mc-partition graphs of G are trivial (in fact, there exists only one *mc*-partition of G in this case) and the result holds trivially. If exactly one component of G, say G_1, is non-trivial then all other components of G being trivial any mc-partition graph of G is isomorphic to G_1. Lastly, if G has two non-trivial components G_1 and G_2 then by (II) they must be point-symmetric, and since all other components of G are trivial it follows from Lemma 1·1 that any *mc*-partition graph $\pi(G)$ of G is isomorphic to $G_1 \, o \, G_2$. Because of (I) the proof of this part is seen to be complete.

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