THE DISCRETE FINITE HILBERT TRANSFORM

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The note presents some new results on discrete and finite Hilbert transforms. In particular the difference between the infinite and the finite transforms is clarified. The results on the finite Hilbert transform are also given in matrix form.

1. INTRODUCTION

Discrete Hilbert transforms have been considered by various authors like Riesz, Varsavsky, Calderon and Zygmund, and Duffin (e.g. see Duffin 1956). Recently their importance to signal processing has caused them to be discussed independently by Kak (1970, 1972, 1973), Cizek (1970) and Gold *et al.* (1969). However, there appears to be a diflerence in the definition of the finite form as given by Kak on one hand and Cizek and Gold on the other. The purpose of this paper is three-fold: (*a*) discuss the exact significance of this difference, (*b*) obtain an elegant form for the finite transform, and (*c*), point out that results on discrete transformations can also be obtained using the theory of discrete analytic functions. The result on the finite transform has also been given in a matrix form.

2. DISCRETE AND CONTINUOUS ANALYTIC FUNCTIONS

Duffin (1956) through an ' a priori ' definition of discrete analytic functions has derived formulae for discrete Hilbert transform. Let $f(z)$ be discrete analytic in the upper half plane and let $f(z) \to 0$ uniformly as $|z| \to \infty$ in the upper plane. Then

$$f(z) = \sum_m f(m)\, \theta(z-m), \quad \text{Im}\{z\} \geqslant 0$$

$$\theta(z) = O\,|z|^{-1}, \quad f(z)\,\theta(z) = O\,|z|^{-1} \qquad \dots (1)$$

Further

$$\theta(z) = q(-z) + iq(1-z)$$

and

$$Lq(z) = 0, \quad z \neq 0$$

$$Lq(0) = 1.$$

We follow Duffin's notation, where L denotes the line integral.

With these constraints it can be shown that

$$v(n) = \sum_{m} u(m)\,[1 - (-1)^{n-m}] \,/\, \pi(n-m) \qquad \ldots (2a)$$

$$u(n) = -\sum_{m} v(m)\,[1 - (-1)^{n-m}] \,/\, \pi(n-m) \qquad \ldots (2b)$$

and $f(z) = 2 \sum_{m} u(m)\, \theta(z-m), \quad \operatorname{Im} z \geqslant 0.$

The same formulae are obtained if we adopt the following different approach. We generate a continuous, well-behaved function $u(x)$ with appropriate values at the lattice points, which represent the given sequence $u(n)$. This function is taken as the real part $u(x)$ of a complex function $f(x)$, whose imaginary part $v(x)$ is the Hilbert transform of the real part. Replacing the parameter x by z we get a function $f(z)$ analytic in the upper half plane. Further the value of $v(x)$ on the real line lattice points is related to the given sequence through the formulas (2*a*) and (2*b*). The continuous well-behaved function has been constructed from the given sequence with the help of the cardinal series expansion (Kak 1970), whereby a sequence $u(n)$ is related to a continuous function $u(x)$ through

$$u(x) = \sin x\pi \sum_{n=-\infty}^{+\infty} \frac{(-1)^n\, u(n)}{\pi(x-n)}. \qquad \ldots (3)$$

This suggests that the approach of considering a discrete sequence and relating it to a continuous analytic function through the above mentioned procedure and doing the required operations on the latter to finally obtain the transformed sequence is equivalent to considering a discrete analytic function and performing the appropriate operations straightway.

A little effort will show that this equivalence can be made to bear upon other results of discrete analytic function theory which could then be employed in the design of digital processing techniques.

3. Discrete Finite Hilbert Transform

When the data sequence $u(n)$ is periodic with a period N, the formula (2*a*) can be simplified to the following form (Kak 1972):

$$v(n) = \begin{cases} \dfrac{2}{N} \displaystyle\sum_{k \text{ odd}} u(k) \cot \dfrac{\pi}{N}\,(n-k); & n \text{ even} \\[2ex] \dfrac{2}{N} \displaystyle\sum_{k \text{ even}} u(k) \cot \dfrac{\pi}{N}\,(n-k); & n \text{ odd}. \end{cases} \qquad \ldots (4)$$

The inverse formulae are skew-symmetrical. The above result can be written in a more compact form as:

$$v(n) = \frac{1}{N} \sum_{k=0}^{N-1} u(k) \left[1 - (-1)^{n-k}\right] \cot \frac{\pi}{N} (n-k). \qquad \ldots (5)$$

The same result was obtained by Cizek (1970) and Gold *et al.* (1969) using the cotangent form of the Hilbert transform. However for periodic sequences, the equivalence given by (3) does not hold. It can be shown that if (3) were used for this case, the identity of the end data points will not be maintained, producing aliasing error. This is demonstrated by considering the Fourier transform of the sequence and its analytic continuant (Linden 1959). This will also be proved by Hilbert transforming and inverse Hilbert transforming a short test sequence using formula (4); this also shows the approximation inherent in this formula.

For $u(n)$, with period N, where N is odd, we can use the Fourier-series kernel to construct a continuous function

$$u(x) = \sum_{k=0}^{N-1} u(n) \frac{\sin\left[\pi(x-k)\right]}{N \sin \frac{\pi}{N}(x-k)}. \qquad \ldots (6)$$

Hilbert transforming $u(x)$, we can obtain the transformed sequence $v(n)$ with the help of (6) (for odd N):

$$v(n) = \begin{cases} \sum\limits_{k \text{ even}} \dfrac{u(k)}{N} \cot \dfrac{\pi(n-k)}{2N} - \sum\limits_{k \text{ odd}} \dfrac{u(k)}{N} \tan \dfrac{\pi(n-k)}{2N}, & n \text{ odd} \\ \sum\limits_{k \text{ odd}} \dfrac{u(k)}{N} \cot \dfrac{\pi(n-k)}{2N} - \sum\limits_{k \text{ even}} \dfrac{u(k)}{N} \tan \dfrac{\pi(n-k)}{2N}, & n \text{ even.} \end{cases} \qquad \ldots (7)$$

The inverse formulae would be skew-symmetrical.

Formulae (7) and their inverses can also be more compactly written down as

$$v(n) = u(n) * \left\{ \frac{1-(-1)^n}{2N} \cot \frac{\pi n}{2N} - \frac{1+(-1)^n}{2N} \tan \frac{\pi n}{2N} \right\} \qquad \ldots (8a)$$

$$u(n) = -v(n) * \left\{ \frac{1-(-1)^n}{2N} \cot \frac{\pi n}{2N} - \frac{1+(-1)^n}{2N} \tan \frac{\pi n}{2N} \right\} \qquad \ldots (8b)$$

These results do not apply to even units long sequences since the relationship (6) is invalid for such sequences. To apply these formulas to such sequences one would have to add one point of zero strength.

Example 1—Let N equal 3 with $u(n)=0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}$ which represent the three independent samples of a sine wave. Application of (8) gives the transformed sequence

$v(n)$ as -1, 0·5, 0·5. This can be easily identified as the corresponding independent samples of $-\cos t$. Now since Hilbert transforming $\sin t$ yields $-\cos t$, the analogy emphasized between the continuous Hilbert transformation and the discrete Hilbert transformation seems to be justified.

Example 2—Let $N=3$ and $u(n) = 1, 2, 3$. Here $v(n)$ turns out to be $\frac{1}{\sqrt{3}}$, $-\frac{2}{\sqrt{3}}$ and $+\frac{1}{\sqrt{3}}$. Transforming $v(n)$ again with the help of the inverse of (8) yields $-1, 0, 1$, which is different fror the $u(n)$ we started with. The reason for this lies in the fact that $u(n)$ as originally given in this example corresponds to a zero-mean periodic function added to a constant. Now since the Hilbert transform of a constant equals zero, on inverse transformation we get a zero-mean sequence. Therefore only sequence for which $\sum u(n)=0$ transform back exactly.

We will now explicitly show the sensitivity of our formulae to the average value of the sequence. Let us write the sequence $u(n)$ as a vector point in a N-dimensional space:

U^T: $(u(0)\ u(1)\ u(2) \ldots u(N-1))$

Thus U is a column vector.

Similarly the Hilbert transformed sequence $v(n)$ can be written as

V^T: $(v(0)\ v(1)\ v(2) \ldots v(N-1))$

Then the result (8) can be expressed as

$$V = HU \qquad \ldots(9)$$

where H is the Hilbert transformation matrix defined by the following circulant matrix :

$$H = \begin{vmatrix} 0 & -\frac{1}{N}\cot\frac{\pi}{2N}, \frac{1}{N}\tan\frac{2\pi}{2N}, -\frac{1}{N}\cot\frac{3\pi}{2N} & \cdots & \frac{1}{N}\tan\frac{(N-1)\pi}{2N} \\ \frac{1}{N}\tan\frac{(N-1)\pi}{2N}, & 0, \frac{1}{N}\cot\frac{\pi}{2N}, & \cdots\cdots & \\ -\frac{1}{N}\cot\frac{(N-1)}{2N}\pi, & \frac{1}{N}, \tan\frac{(N-1)}{2N}\pi, 0, & \cdots\cdots & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \end{vmatrix} \qquad \ldots(10)$$

This is because $\tan\frac{(N-1)\pi}{2N} = \cot\frac{\pi}{2N}$. The matrix H, therefore, is a skew-symmetric matrix. Also, it is symmetric about its second main diagonal.

Let the elements in the first row of H be taken as $h_0, h_1, \ldots h_{N-1}$. The eigenvalues ψ_k and the eigenvectors y_k of H are the solutions of

$$Hy = \psi y \qquad \ldots(11)$$

or equivalently of the N difference equations

$$\sum_{k=0}^{m-1} h_{N-m+k}\, y_k + \sum_{k=m}^{N-1} h_{k-m}\, y_k = \psi\, y_m, \quad m = 0, 1, .., N-1. \qquad ...(12)$$

It can be easily shown that $y_k = \{\exp(-2\pi\, imk/N)\}$ is a solution, resulting in the eigenvalues:

$$\psi_m = \sum_{k=0}^{N-1} h_k \exp(-2\pi\, imk/N) \qquad ...(13)$$

add the corresponding eigenvectors:

$$Y_m = N^{-\frac{1}{2}}\,(1, \exp(-2\pi\, im/N), .., \exp[-2\pi\, im(N-1)/N]). \qquad ...(14)$$

It can also be shown that

$$HH^T = -I + A \qquad ...(15)$$

where I is the identity matrix and A is the 'averaging' matrix defined by

$$A = [a_{ij}] = [\tfrac{1}{n}] \qquad ...(16)$$

The above relation shows that inverse transforming a Hilbert transformed sequence will yield the original sequence only when the average value of the sequence is zero.

4. Finite Transform Formulae for Large N

When $N \rightarrow \infty$, (8) can be simplified using the following results:

$$\tan \frac{\pi n}{2N} \rightarrow \frac{\pi n}{2N}$$

and

$$\cot \frac{\pi n}{2N} \rightarrow \frac{2N}{\pi n}.$$

Using these results and substituting $N \rightarrow \infty$, we have

$$v(n) = u(n) * \frac{[1-(-1)^n]}{\pi n} \qquad ...(17)$$

which is identical to (2). This is as expected since for $N \rightarrow \infty$, we effectively consider only the given sequence and not its periodic repetitions.

5. Conclusions

Having established discrete finite Hilbert transform relations for periodic sequences, it should be straightforward to design schemes of implementation keeping in view that arbitrary periodic sequences can be expanded in terms of sine and cosine sequences. What one would therefore require would be an all-pass digital network with a 90°

phase shift in the frequency interval of interest. One way of obtaining this is to use the well-known phase splitting networks (Storer 1971, Gold and Rader 1969).

To conclude, the exact form of discrete finite Hilbert transformation is as given by equation (8). The relations (4) which have been claimed by Cizek and Gold to be the finite formulas are approximate in the manner as has been discussed in our paper.

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